AN OVERVIEW OF BOND PERCOLATION

ROHIN GILMAN

Abstract. In this paper, we will discuss bond percolation on the integer lattice $\mathbb{Z}^d$, a simplified model for the potential for water to diffuse in a porous material. We will discuss the subcritical phase, where a submerged object would not be wetted at the center, and the supercritical phase, where a submerged object could be wetted at the center. Finally, we will discuss physically motivated conjectures about the critical point and rigorous results that are known for critical percolation on a binary tree.

Contents

1. Introduction

Suppose we immerse a porous object in water and want to know the probability that the center of the object becomes wet, or that the water percolates. This process can be modeled by the lattice on $\mathbb{Z}^d$ where the the probability that an edge is open (that water can flow between two vertices) is p and the probability that an edge is closed (that water cannot flow between two vertices) is $1-p$. The probability that the center of the object becomes wet is the probability that there exists an infinite open cluster. The fundamental result of percolation theory is that there is a critical probability p_c such that $p > p_c$ implies that there is an infinite subgraph connected by open edges in the lattice (that the center of the object gets wet) while if $p < p_c$, every open subgraph is finite (the center of the object does not get wet).

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We define the graph-theoretic distance $\delta(x,y)$ between two vertices x and y in $\mathbb{Z}^d$ to be

$$\delta(x,y) = \sum_{i=0}^{d} |x_i - y_i| \,,$$

where $x = (x_1, x_2, \ldots, x_d)$ and $y = (y_1, y_2, \ldots, y_d)$, and we define $|x|$ to be the distance $\delta(0,x)$ between the origin and x. We will construct the cubic lattice $\mathbb{L}^d$ by taking the vertices $\mathbb{Z}^d$ and adding the edges

$$\mathbb{E}^d = \left\{ \langle x,y \rangle \mid x,y \in \mathbb{Z}^d \text{ and } \delta(x,y) = 1 \right\}.$$

We let $0 \le p \le 1$ and declare the edges in $\mathbb{E}^d$ to be open with probability p and closed with probability $1-p$.

A path on $\mathbb{L}^d$ is called *open* if all if its edges are open. Consider the random subgraph of $\mathbb{L}^d$ containing the vertices $\mathbb{Z}^d$ and the open edges of $\mathbb{E}^d$. The connected components of this graph are called *open clusters* and we write $C(x)$ for the open cluster containing the vertex x. By the translation invariance of the lattice, the distribution of $C(x)$ is independent of the choice of x. The open cluster $C(0)$ at the origin is typical of such clusters and is represented simply as C. We denote the number of vertices in an open cluster $C(x)$ to be $|C(x)|$.

1.1. **Bond Percolation as a Phase Transition.** One of the main quantities of interest in bond percolation is the percolation probability $\theta(p)$, the probability that a given vertex belongs to an infinite open cluster. No generality is lost by taking this vertex to be the origin, so we define

$$\theta(p) = \mathbb{P}_p(|C| = \infty). \tag{1.1}$$

It is possible to show that there exists a critical value $p_c = p_c(d)$ of p such that

$$\theta(p) \begin{cases} = 0 & \text{if } p < p_c, \\ > 0 & \text{if } p > p_c; \end{cases}$$

$p_c(d)$ is called the *critical probability* and is defined by

$$p_c(d) = \sup \{ p \mid \theta(p) = 0 \}.$$

If $d = 1$ then the critical probability $p_c(1) = 1$ since there exist infinitely many closed edges both to the left and right of the origin when $p < 1$. From this point on, we assume $d \ge 2$.

When $p < p_c$, we call this the *subcritical phase*. In Section 4, we will discuss what happens in this phase, specifically, the behavior of the size of an open cluster.

If A and B are sets of vertices of the lattice, we write $A \leftrightarrow B$ if there exists an open path joining a vertex in A to a vertex in B. If A is a set of vertices in the lattice then we denote the *surface* of A, the vertices in A which are adjacent to a vertex not in A, to be ∂A.

We define another distance function on $\mathbb{Z}^d$

$$\|x\| = \max \{ |x_i| : 1 \le i \le d \}.$$

We denote the *box* with side-length $2n$ and centered at the origin

$$B(n) = \left\{ x \in \mathbb{Z}^d : \|x\| \le n \right\}.$$

We will show that the probability that $0 \leftrightarrow \partial B(n)$ decays exponentially as $n \to \infty$.

When $p > p_c$, we call this the *supercritical phase*. In this case, we know that the probability that there exists an infinite open cluster is strictly positive. In Section 5, we will show that there is exactly one infinite cluster in the supercritical phase.

1.2. **The Critical Phase.** In addition to exploring the subcritical and supercritical phases, it is also interesting to think about what happens at and near the critical probability.

It is believed that $\theta(p)$ decays to 0 as a power of $(p - p_c)$ as $p \downarrow p_c$. We write that as $p \downarrow p_c$,

$$\theta(p) \approx (p - p_c)^{\beta} \quad \text{for some } \beta > 0, \tag{1.2}$$

where β depends on d and the logarithmic relation $\approx$ denotes

$$\lim_{p \downarrow p_c} \frac{\log \theta(p)}{\log(p - p_c)} = \beta. \tag{1.3}$$

This conjecture requires that $\theta(p_c) = 0$, which is also an open question when $3 \leq d \leq 18$.

We define two additional quantities, χ and χ^f. The mean size of an open cluster $\chi(p)$ is defined as

$$\chi(p) = \mathbb{E}_p(|C|). \tag{1.4}$$

It is proven in [2] that $\chi(p) < \infty$ if and only if $p < p_c$. When $p > p_c$, $\chi(p) = \infty$, so we define a second quantity, the mean size of a finite open cluster $\chi^f(p)$ to be

$$\chi^f(p) = \mathbb{E}_p(|C|\,; |C| < \infty). \tag{1.5}$$

Whenever $\theta(p) = 0$, $\chi^f(p) = \chi(p)$.

It is believed that there exist $\gamma, \gamma' > 0$ such that as $p \uparrow p_c$,

$$\chi(p) \approx (p_c - p)^{-\gamma}, \tag{1.6}$$

and as $p \downarrow p_c$,

$$\chi^f(p) \approx (p - p_c)^{-\gamma'}, \tag{1.7}$$

where the asymptotic relationships are interpreted as in (1.3). It is further conjectured that $\gamma = \gamma'$. Since $\chi(p) = \chi^f(p)$ when $p < p_c$, equations (1.6) and (1.7) can be combined. There exists $\gamma > 0$ such that as $p \to p_c$,

$$\chi^f(p) \approx |p - p_c|^{-\gamma}. \tag{1.8}$$

These relations have not been proven on the lattice $\mathbb{L}^d$ for general d; however, in the next sections, we instead examine percolation on a binary tree where these relations have been proven. In Section 6, we use Galton-Watson processes to show that on a binary tree, these conjectures about critical percolation are true.

2. Measure Theory Preliminaries

To begin discussing bond percolation, we will first need to discuss some results from measure theory.

Definition 2.1. A class $\mathscr{A}$ of subsets of a set X is called an *σ-algebra* if

(a) $\varnothing \in \mathscr{A}$.
(b) If $A \in \mathscr{A}$ and $B \in \mathscr{A}$, then $A \setminus B \in \mathscr{A}$.
(c) If $A \in \mathscr{A}$ and $B \in \mathscr{A}$, then $A \cup B \in \mathscr{A}$.
(d) $X \in \mathscr{A}$.

(e) If $A_n \in \mathscr{A}$ for $n \in \mathbb{N}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathscr{A}$.

A set function μ is a function that is defined on a class of sets. We call a set function μ extended real-valued if its values are in $\mathbb{R} \cup \{-\infty, \infty\}$.

Definition 2.2. A *measure* is an extended real-valued set function μ such that

(a) The domain $\mathscr{A}$ of μ is a σ-algebra.
(b) μ is nonnegative on $\mathscr{A}$.
(c) If $\{E_n\}_{n=1}^{\infty}$ are mutually disjoint sets of $\mathscr{A}$ then

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n).$$

(d) $\mu(\varnothing) = 0$.

If X is a space, $\mathscr{A}$ is a σ-algebra of subsets of X, and μ is a measure with domain $\mathscr{A}$ then the triple $(X, \mathscr{A}, \mu)$ is a *measure space.* The sets of $\mathscr{A}$ are *μ-measurable* (or just *measurable*).

If f is an extended real-valued function on a measurable set X_0 of a measure space then f is called a *measurable function* if for any open set $M \subseteq \mathbb{R}$, the inverse image $f^{-1}(M) = \{y \in X_0 \mid f(y) \in M\}$ is a measurable set and $f^{-1}(-\infty)$ and $f^{-1}(\infty)$ are measurable.

Theorem 2.3. *Let f be an extended real-valued function on a measurable set $X_0 \subseteq X$ where X is a measure space. f is measurable if and only if for every real number $c \in \mathbb{R}$, the set $f^{-1}\{(-\infty, c)\}$ is measurable and the sets $f^{-1}(-\infty)$ and $f^{-1}(\infty)$ are measurable.*

A property P on the points of a measure space is said to be true *almost everywhere* (abbreviated *a.e.*) if the set of points for which P is not true has measure 0. If $E = \varnothing$, then P is true everywhere.

3. Measure Theoretic Probability and Bond Percolation

In this paper, we will use the modern language of probability to discuss bond percolation. A *probability space* is a measure space $(\Omega, \mathcal{F}, \mathbb{P})$ where the set Ω is referred to as the sample space, the σ-algebra $\mathcal{F}$ is referred to as the collection of events, and the probability measure $\mathbb{P}$ is a measure on $(\Omega, \mathcal{F})$ such that $\mathbb{P}(\Omega) = 1$.

A random variable X on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a measurable function $X : \Omega \to \mathbb{R}$. Since a function $X : \Omega \to \mathbb{R}$ is measurable if and only if for each $c \in \mathbb{R}$, the set $\{\omega \in \Omega \mid X(\omega) \leq c\}$ is in $\mathcal{F}$, $X : \Omega \to \mathbb{R}$ is a random variable if and only if for each for each $c \in \mathbb{R}$, the set $\{\omega \in \Omega \mid X(\omega) \leq c\}$ is an event. We will write the event $\{\omega \in \Omega \mid X(\omega) \leq c\}$ as $\{X \leq c\}$, omitting ω from the notation.

The expectation $\mathbb{E}(X)$ of a non-negative random variable X is its integral with respect to the probability measure $\mathbb{P}$,

$$\mathbb{E}(X) = \int_{\Omega} X(\omega) d\mathbb{P}(\omega),$$

where the integral is defined in the Lebesgue sense, as in [1]. Observe that by this definition,

$$\mathbb{E}(1_A) = \mathbb{P}(A)$$

for each $A \in \mathcal{F}$, where 1_A is the indicator function for the event $A \in \mathcal{F}$, given by

$$1_A(\omega) = \begin{cases} 1, & \text{if } \omega \in A, \\ 0, & \text{if } \omega \notin A. \end{cases}$$

Two sub-σ-algebras $\mathcal{G}_1$ and $\mathcal{G}_2$ of $\mathcal{F}$ are independent if for each $A \in \mathcal{G}_1$ and for each $B \in \mathcal{G}_2$,

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B).$$

We say that two random variables X_1 and X_2 are independent if the σ-algebras $\sigma(X_1)$ and $\sigma(X_2)$ are independent, where $\sigma(X)$ is the σ-algebra generated by X.

Let $\Omega = \{0,1\}^{\mathbb{E}^d}$ be the set of all functions $\omega : \mathbb{E}^d \to \{0,1\}$, or equivalently, the set of all possible states of our lattice where for any edge $e \in \mathbb{E}^d$, $\omega(e) = 0$ indicates that e is closed and $\omega(e) = 1$ indicates that e is open. For $\omega_1, \omega_2 \in \Omega$ we say that $\omega_1 \leq \omega_2$ if and only if $\omega_1(e) \leq \omega_2(e)$ for all $e \in \mathbb{E}^d$. Let $\mathscr{C}$ be the collection of cylinder sets in 2^Ω and let $\mathcal{F} = \sigma(\mathscr{C})$ be the σ-algebra generated by $\mathscr{C}$.

It is possible to show that there exists a measure $\mathbb{P}_p$ on $\mathcal{F}$ such that for each $n \in \mathbb{N}$, $e_1, \dots, e_n \in \mathbb{E}^d$, and $l_1, \dots, l_n \in \{0,1\}$

$$\mathbb{P}_p \{\omega \in 2^\omega : \omega(e_1) = l_1, \dots, \omega(e_n) = l_n\} = \prod_{i=1}^{n} p^{l_i}(1-p)^{1-l_i},$$

and $(\Omega, \mathcal{F}, \mathbb{P}_p)$ is the probability space used for bond percolation when the probability of an edge being open is p.

4. Radius of an Open Cluster

In the subcritical phase, the probability that there exists an infinite open cluster is zero; however, it is interesting to consider the behavior of the size of a finite open cluster. In this section, we will prove that the radius of an open cluster decays exponentially. To show this, we will require an additional theorem, the BK inequality, which we will introduce first.

Definition 4.1. Let $(\Omega, \mathcal{F}, \mathbb{P}_p)$ be the bond percolation probability space. An event $A \in \mathcal{F}$ is called *increasing* if $1_A(\omega) \leq 1_A(\omega')$ whenever $\omega \leq \omega'$, where 1_A is the indicator function of A.

More generally, a random variable N on $(\Omega, \mathcal{F}, \mathbb{P}_p)$ is called *increasing* if $N(\omega) \leq N(\omega')$ whenever $\omega \leq \omega'$.

Let $e_1, e_2, \dots, e_n$ be n distinct edges of $\mathbb{L}^d$, and let A and B be increasing events that depend on the vector $\omega = (\omega(e_1), \omega(e_2), \dots, \omega(e_n))$ of the states of these edges only. Each ω is uniquely specified by the set $K(\omega) = \{e_i \mid \omega(e_i) = 1\}$. The event $A \circ B$ is defined to be the set of all ω for which there exists a subset H of $K(\omega)$ such that ω', determined by $K(\omega') = H$, belongs to A, and ω'', determined by $K(\omega'') = K(\omega) \setminus H$, belongs to B. Colloquially, $A \circ B$ is the set of configurations for which there exist disjoint sets of open edges such that the first set guarantees A and the second set guarantees B.

Theorem 4.2 (BK Inequality for Bond Percolation)**.** *Consider bond percolation on $\mathbb{L}^d$ with probability p. Let A and B be increasing events defined by the states of only finitely many edges. Then*

$$\mathbb{P}_p(A \circ B) \leq \mathbb{P}_p(A)\mathbb{P}_p(B). \tag{4.3}$$

For a proof of this theorem, see [2].

Recall that the surface of $B(n)$ is defined by

$$\partial B(n) = \left\{x \in \mathbb{Z}^d : \|x\| = n\right\}.$$

For $x \in \mathbb{Z}^d$, we write $B(n,x)$ for the box with side length $2n$ centered about x and $\partial B(n,x)$ for the surface of this box. We write $\mathbb{P}_p(0 \leftrightarrow \partial B(n))$ for the probability that there exists an open path joining the origin to some vertex in $\partial B(n)$.

Theorem 4.4 (Exponential Tail Decay of the Radius of an Open Cluster)**.** *Suppose that $\chi(p) < \infty$. There exists $\sigma(p) > 0$ such that*

$$\mathbb{P}_p(0 \leftrightarrow \partial B(n)) \leq e^{-n\sigma(p)} \quad \text{for all } n. \tag{4.5}$$

Proof. Let $x \in \partial B(n)$. We define by $\tau_p(0,x) = \mathbb{P}_p(0 \leftrightarrow x)$ the probability that there exists an open path joining the origin to x. We define by N_n the number of vertices in $\partial B(n)$ that are connected to the origin. Then the mean value of N_n is given by

$$\mathbb{E}_p(N_n) = \sum_{x \in \partial B(n)} \tau_p(0,x). \tag{4.6}$$

Observe that

$$\begin{aligned}\sum_{n=0}^{\infty} \mathbb{E}_p(N_n) &= \sum_{n=0}^{\infty} \sum_{x \in \partial B(n)} \tau_p(0,x) \\ &= \sum_{x \in \mathbb{Z}^d} \tau_p(0,x) \\ &= \mathbb{E}_p \left|\left\{x \in \mathbb{Z}^d \mid 0 \leftrightarrow x\right\}\right| = \chi(p).\end{aligned}$$

Let $n = m + k$. If there exists an open path from the origin to $y \in \partial B(m+k)$ then there exists $x \in \partial B(m)$ such that x is connected by disjoint paths both to the origin and to a vertex on the surface $\partial B(k,x)$. Then we can apply the BK inequality to obtain

$$\begin{aligned}\mathbb{P}_p(0 \leftrightarrow \partial B(m+k)) &\leq \sum_{x \in \partial B(m)} \mathbb{P}_p(0 \leftrightarrow x)\mathbb{P}_p(x \leftrightarrow \partial B(k,x)) \\ &= \sum_{x \in \partial B(m)} \tau_p(0,x)\mathbb{P}_p(0 \leftrightarrow \partial B(k)).\end{aligned}$$

Thus,

$$\mathbb{P}_p(0 \leftrightarrow \partial B(m+k)) \leq \mathbb{E}_p(N_m)\mathbb{P}_p(0 \leftrightarrow \partial B(k)) \quad \text{for } m,k \geq 1. \tag{4.7}$$

If $n = mr$, we can induct on (4.7) to obtain

$$\mathbb{P}_p(0 \leftrightarrow \partial B(n)) \leq \mathbb{E}_p(N_m)^r. \tag{4.8}$$

Since $\chi(p) < \infty$, $\sum_{m=0}^{\infty} \mathbb{E}_p(N_m) < \infty$, so $\mathbb{E}_p(N_m) \to 0$ as $m \to \infty$. Therefore, there exists m such that $\eta = \mathbb{E}_p(N_m) < 1$.

Let n be a positive integer $mr + s$ where r and s are nonnegative integers and $0 \leq s < m$. Then

$$\mathbb{P}_p(0 \leftrightarrow \partial B(n)) \leq \mathbb{P}(0 \leftrightarrow \partial B(mr))$$

since $n \geq mr$. Using the previous induction, we find that

$$\begin{aligned}\mathbb{P}_p(0 \leftrightarrow \partial B(n)) &\leq \eta^r \\ &\leq \eta^{-1+n/m}\end{aligned}$$

since $n < m(r+1)$. This provides an exponentially decaying bound of the form (4.5) which completes the proof. □

5. Uniqueness of the Infinite Open Cluster

In the supercritical phase, there exists an infinite open cluster with probability one; however, we can prove a slightly stronger result. In this section we prove that when $\theta(p) > 0$, the infinite cluster is unique.

Theorem 5.1 (Uniqueness of the Infinite Open Cluster)**.** *If p is such that $\theta(p) > 0$, then*

$$\mathbb{P}(\textit{there exists exactly one infinite open cluster}) = 1.$$

Proof. If $p = 0$ then there exists no infinite cluster so the claim is satisfied. If $p = 1$ then for each $e \in \mathbb{E}^d$, $\omega(e) = 1$, so the entire lattice $\mathbb{L}^d$ is open and connected. Thus, the infinite open cluster is unique.

Assume $0 < p < 1$. Let N be the number of infinite open clusters. Let B be a finite, connected subgraph, and let E_B be the set of edges in $\mathbb{L}^d$ joining pairs of vertices in B. We denote by $N_B(0)$the number of infinite open clusters when all of the edges of B are declared to be closed, and we denote by $N_B(1)$ the number of infinite open clusters when all of the edges of B are declared to be open. Finally, we denote by M_B the number of infinite open clusters that intersect B.

Since N is a translation-invariant function on the sample space $\Omega = \{0,1\}^{\mathbb{E}^d}$, it is almost surely constant. Then for each p,

$$\text{there exists } k \in \mathbb{N}_0 \cup \{\infty\} \text{ such that } \mathbb{P}_p(N = k) = 1. \tag{5.2}$$

We will now show that $k \in \{0, 1, \infty\}$ by contradiction. Suppose that (5.2) is true with $2 \leq k < \infty$. Let B be a finite set of connected vertices. Since B is finite, every possible configuration of E_B has positive probability. Since declaring all of the edges of B to be closed and declaring all of the edges of B to be open are possible configurations of E_B and N is constant,

$$\mathbb{P}_p(N_B(0) = N_B(1) = k) = 1$$

by the almost sure constantness of N.

We will show that if $k < \infty$, $N_B(0) = N_B(1)$ if and only if $M_B \in \{0, 1\}$. We will prove the contrapositive: if $M_B \geq 2$ then $N_B(0) \neq N_B(1)$. Since B is finite, $N_B(0) \geq N$ because the M_B infinite open clusters which intersect B will not be combined when all of the edges of B are declared to be closed. Since B is connected, $N_B(1) = N - (M_B - 1)$. This is because when all of the edges of B are declared to be open, the M_B infinite open clusters are combined to form only 1 infinite open cluster. Since $M_B \geq 2$, $N_B(1) \leq N - 1$. Therefore, $N_B(0) \neq N_B(1)$.

Thus, $\mathbb{P}_p(M_B \geq 2) = 0$. Intuitively, we see that as B approaches $\mathbb{Z}^d$, the number of infinite clusters intersecting B, M_B approaches the total number of infinite clusters, N. If we let B be the diamond $S(n) = \{x \in \mathbb{Z}^d \mid \delta(0,x) \leq n\}$ and let $n \to \infty$, we obtain

$$0 = \mathbb{P}_p(M_{S(n)} \geq 2) \to \mathbb{P}(N \geq 2), \tag{5.3}$$

so by contradiction, $k \in \{0, 1, \infty\}$.

Now the only remaining case to eliminate is $k = \infty$. Suppose that $k = \infty$. We will derive another contradiction.

We call a vertex x a *trifurcation* if

(a) x lies in an infinite open cluster.
(b) There exist exactly three open edges incident to x.
(c) The deletion of x and its incident edges splits this infinite cluster into exactly three disjoint infinite clusters.

The event that x is a trifurcation is denoted T_x. Since $\mathbb{P}_p(T_x)$ is constant for all x,

$$\frac{1}{|B(n)|}\mathbb{E}_p\left(\sum_{x\in B(n)} 1_{T_x}\right) = \mathbb{P}_p(T_0). \tag{5.4}$$

We would now like to show that $\mathbb{P}_p(T_0) > 0$. Let $M_B(0)$ be the number of infinite open clusters which intersect B when all the edges of $\mathbb{E}_B$ are declared to be closed. Then $M_B(0) \geq M_B$, and by the same arguments used to obtain (5.3), if we let B be the diamond $S(n)$ and let $n \to \infty$, we obtain

$$\mathbb{P}_p(M_{S(n)}(0) \geq 3) \geq \mathbb{P}_p(M_{S(n)} \geq 3) \to \mathbb{P}_p(N \geq 3) = 1 \quad \text{as } n \to \infty,$$

since $k = \infty$. Therefore, there exists N such that

$$\mathbb{P}_p(M_{S(N)}(0) \geq 3) \geq \frac{1}{2}.$$

Observe that the event $\left\{M_{S(n)} \geq 3\right\}$ is independent of the states of the edges in $\mathbb{E}_{S(n)}$ and that if the event $\left\{M_{S(n)} \geq 3\right\}$ occurs then there exist three vertices $x, y, z \in \partial S(n)$ that belong to distinct infinite open clusters of $\mathbb{E}^d \setminus \mathbb{E}_{S(n)}$.

Let $\omega \in \left\{M_{S(n)} \geq 3\right\}$, and choose any distinct $x = x(\omega), y = y(\omega), z = z(\omega) \in \partial S(n)$ that belong to distinct infinite open clusters of $\mathbb{E}^d \setminus \mathbb{E}_{S(n)}$. There exist in $\mathbb{E}_{S(n)}$ three paths joining x, y, and z to the origin such that the origin is the unique vertex common to any two of these paths and that each path touches exactly one vertex lying in $\mathbb{E}_{S(n)}$.

Let $J_{x,y,z}$ be the event that the edges in these paths are open and that all other edges in in $\mathbb{E}_{S(n)}$ are closed. Let $m = \min\{p, 1-p\}$. Since $S(n)$ is finite,

$$\mathbb{P}_p(J_{x,y,z} \mid M_{S(n)}(0) \geq 3) \geq m^R > 0,$$

where R is the number of edges in $\mathbb{E}_{S(n)}$. Now

$$\begin{aligned}\mathbb{P}_p(0 \text{ is a trifurcation}) &\geq \mathbb{P}_p(J_{x,y,z} \mid M_{S(n)}(0) \geq 3)\mathbb{P}_p(M_{S(n)}(0) \geq 3) \\ &\geq \frac{1}{2}m^R > 0\end{aligned}$$

for all $n \geq N$, so $\mathbb{P}_p(T_0) > 0$ from (5.4). From (5.4), we determine that

$$\mathbb{E}_p\left(\sum_{x\in B(n)} 1_{T_x}\right),$$

the mean number of trifurcations inside $B(n)$, grows in the manner of $|B(n)|$ as $n \to \infty$. We will use this to derive a contradiction.

Let Y be a finite set with $|Y| \geq 3$. We define a *3-partition* $\Pi = \{P_1, P_2, P_3\}$ of Y to be a partition of Y into exactly three non-empty sets P_1, P_2, P_3. Given two

3-partitions $\Pi = \{P_1, P_2, P_3\}$ and $\Pi' = \{P_1', P_2', P_3'\}$, Π and Π' are called *compatible* if there exists an ordering of their elements such that $P_1 \supseteq P_2' \cup P_3'$. A collection $\mathscr{P}$ of 3-partitions is said to be compatible if each pair of partitions in $\mathscr{P}$ is compatible.

Lemma 5.5. *If $\mathscr{P}$ is a compatible family of distinct 3-partitions of Y then $|\mathscr{P}| \leq |Y| - 2$.*

Proof. We will do a proof by induction. If $|Y| = 3$ then $|\mathscr{P}| \leq 1$ since there is only one possible partition. Assume $|\mathscr{P}| \leq |Y| - 2$ whenever $|Y| \leq n$ for some fixed $n \geq 3$. Let Y be such that $|Y| = n + 1$. Choose some fixed $y \in Y$ and let $Z = Y \setminus \{y\}$.

Let $\mathscr{P}$ be a family of compatible partitions of Y. A partition $\Pi \in Y$ can be written as $\Pi = \{P_1 \cup \{y\}, P_2, P_3\}$ for some disjoint subsets P_1, P_2, P_3 such that $P_2, P_3 \neq \varnothing$ and $Z = P_1 \cup P_2 \cup P_3$. Let $\mathscr{P}'$ be the set of all such partitions where $P_1 \neq \varnothing$, and let $\mathscr{P}'' = \mathscr{P} \setminus \mathscr{P}'$.

The triples $\{P_1, P_2, P_3\}$ for all $\Pi = \{P_1 \cup \{y\}, P_2, P_3\} \in \mathscr{P}'$ form a compatible family of 3-partitions of Z, so by the induction hypothesis,

$$|\mathscr{P}'| \leq |Z| - 2 = |Y| - 3.$$

It remains to show that $|\mathscr{P}''| \leq 1$. Suppose that there are at least 2 distinct compatible 3-partitions of Y in $\mathscr{P}''$, say $\{\{y\}, A_2, A_3\}$ and $\{\{y\}, B_2, B_3\}$ Since these partitions are compatible, we can write $A_2 \supseteq \{y\} \cup B_2$, which is a contradiction since $y \notin A_2$.

Therefore,

$$|\mathscr{P}| = |\mathscr{P}'| + |\mathscr{P}''| \leq |Y| - 2.$$

□

Returning to the proof of Theorem 5.1, let K be a connected open cluster of $B(n)$. Observe that $B(n) \setminus B(n-1) = \partial B(n)$. If x is a trifurcation in $K \cap B(n-1)$, the removal of x from $K \cap B(n-1)$ induces a partition of $K \cap \partial B(n)$ into the three sets joined by open paths to x by each of the three open edges incident to x. Therefore, x corresponds to a 3-partition $\Pi(x) = \{P_1, P_2, P_3\}$ of $K \cap \partial B(n)$ such that each P_i is nonempty, each P_i is a subset of a connected open subgraph of $B(n) \setminus \{x\}$, and for all $i \neq j$, $P_i \not\leftrightarrow P_j$ in $B(n) \setminus \{x\}$. If x and x' are distinct trifurcations of $K \cap B(n-1)$, then by the definition of trifurcation, $\Pi(x)$ and $\Pi(x')$ are distinct and compatible.

By Lemma 5.5, the number of trifurcations in $K \cap B(n-1)$, $\tau(K)$, satisfies

$$\tau(K) \leq |K \cap \partial B(n)| - 2. \tag{5.6}$$

If we sum (5.6) over all connected clusters K of $B(n)$, we obtain

$$\sum_{x \in B(n-1)} 1_{T_x} = \sum_{K \in B(n)} \tau(K) \leq \sum_{K \in B(n)} (|K \cap \partial B(n)| - 2) \leq |\partial B(n)|.$$

Taking expectations and using 5.4, we obtain

$$\mathbb{E}\left(\sum_{x \in B(n-1)} 1_{T_x}\right) = |B(n-1)| \, \mathbb{P}_p(T_0) \leq |\partial B(n)|. \tag{5.7}$$

Since $|B(n-1)|$ grows as n^d as $n \to \infty$ and $|\partial B(n)|$ grows as n^{d-1} as $n \to \infty$, this is a contradiction, completing the proof. □

6. The Binary Tree and Galton-Watson Processes

Let T be the binary tree defined in the following way. We label the root vertex of T by $\varnothing$, the empty sequence. We label the two children of the vertex labeled $\lambda_1\lambda_2\dots\lambda_n$ by $\lambda_1\lambda_2\dots\lambda_n 1$ and $\lambda_1\lambda_2\dots\lambda_n 2$. Each vertex is connected by an edge to its two children. We say that a vertex is in the nth generation if its label is a sequence of length n.

Let $0 < p < 1$. We declare each of the edges of T to be open with probability p and closed otherwise, independently of all other edges.

In order for an infinite open cluster to exist, at least one edge connecting a vertex to one of its children must be open. Then for each vertex, there exists an infinite open cluster if $2p \geq 1$. Thus, the critical probability of percolation on T is given by $p_c = \frac{1}{2}$.

To prove the relations of the critical exponents on a binary tree, we will use Galton-Watson processes. To describe these, we must first introduce some preliminaries.

Definition 6.1. If μ, ν are two probability measures on the probability space $\left(\mathbb{N}_0, 2^{\mathbb{N}_0}\right)$, their *convolution* is the probability measure on $\left(\mathbb{N}_0, 2^{\mathbb{N}_0}\right)$ uniquely determined by the formula

$$(\mu * \nu)(\{n\}) = \sum_{i=0}^{n} \mu(\{j\})\nu(\{n-j\}) \quad \text{for all } n \in \mathbb{N}_0.$$

If μ is a probability measure on $(\mathbb{N}_0, 2^{\mathbb{N}_0})$ then its *convolution products* $\{\mu^{*n}\}_{n\in\mathbb{N}}$ are defined by

$$\mu^{*1} = \mu, \quad \mu^{*(n+1)} = \mu * \mu^{*n}.$$

Definition 6.2. If X is a real-valued random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then its *distribution* μ_X is the probability measure on $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$ defined by

$$\mu_X(A) = \mathbb{P}\{X \in A\},$$

where $\mathscr{B}_{\mathbb{R}}$ is the Borel σ-algebra of $\mathbb{R}$, the smallest σ-algebra containing all open subsets of $\mathbb{R}$.

Proposition 6.3. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. If X, Y are two independent, real-valued random variables taking values in $\mathbb{N}_0$ then $\mu_{X+Y} = \mu_X * \mu_Y$.*

Definition 6.4. Let $T \subseteq \mathbb{N}$ and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A family $(X_n)_{n\in T}$ of real-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ is called *identically distributed* if there is a probability measure μ on $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$ such that

$$\mu_{X_n} = \mu \quad (\text{for all } n \in T).$$

If a family $(X_n)_{n\in T}$ of real-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ is both mutually independent and identically distributed, it is called *independent and identically distributed (i.i.d)*.

6.1. Galton-Watson Processes.

In a Galton-Watson process, a population of individuals evolves in discrete time ($n = 0, 1, 2, \dots$) according to the following conditions. Each nth generation individual produces a random number of individuals, called offspring, in the $(n+1)$st generation. If $\alpha, \beta, \gamma, \dots$ are nth generation individuals, their offspring counts $\xi_\alpha, \xi_\beta, \xi_\gamma, \dots$ are mutually independent and are independent of the offspring counts of individuals of previous generations. Finally, these offspring counts are identically distributed, with common distribution $\{p_k\}_{k\in\mathbb{N}_0}$. The state Z_n of a Galton-Watson process at time n is the number of individuals in the nth generation.

Definition 6.5. A *Galton-Watson process* $\{Z_n\}_{n\in\mathbb{N}_0}$ with offspring distributions $F = \{p_k\}_{k\in\mathbb{N}_0}$ is a discrete-time Markov chain taking values in the set of nonnegative integers with the following transition probabilities:

$$\mathbb{P}\{Z_{n+1} = k \mid Z_n = m\} = {p_k}^{*m}. \tag{6.6}$$

Then by induction using Proposition 6.3, the conditional distribution of Z_{n+1} given that $Z_n = m$ is the sum of m independent and identically distributed variables, each with distribution F.

Construction 6.7. Let Ω be a probability space that supports an infinite sequence of independent and identically distributed random variables, all with distribution $F = \{p_k\}_{k\in\mathbb{N}_0}$. Assume that these these are arranged in a doubly infinite array,

$$\left\{\xi_i^j\right\}_{i,j\in\mathbb{N}}.$$

Let $Z_0 = 1$, and define

$$Z_{n+1} = \sum_{i=1}^{Z_n} \xi_i^{n+1}. \tag{6.8}$$

Since the random variables ξ_i^n are mutually independent, the sequence $\{Z_n\}_{n\in\mathbb{N}_0}$ has the Markov property and the conditional distribution satisfies (6.6).

For some choices of F, the Galton-Watson process is not very interesting. One example is when F places all of its weight on a single value $k \in \mathbb{N}$. Then $Z_n = k^n$ for every $n \geq 1$. Another example is when F is such that $p_0p_1 > 0$ and $p_0 + p_1 = 1$. Then the population remains at its initial size, $Z_0 = 1$, until it eventually becomes 0 with probability 1, where it stays.

To avoid these cases, we will make the assumption henceforth that F is not a point mass and that there exists $k \geq 2$ such that $p_k > 0$. We also assume that F has finite mean $M > 0$ and finite variance $\sigma^2 > 0$.

We denote the generating function of the offspring distribution $F = \{p_k\}_{k\in\mathbb{N}_0}$ of a Galton-Watson process to be

$$\varphi(t) = \sum_{k=0}^{\infty} p_k t^k. \tag{6.9}$$

Observe that bond percolation on a tree is a Galton-Watson process in which $Z_0 = 1$, and the generating function is given by

$$\begin{aligned}\varphi(t) &= (1-p)^2 + 2p(1-p)t + p^2t^2 \\ &= \left((1-p) + pt\right)^2.\end{aligned}$$

6.2. **Extinction of a Galton-Watson Process.** We say that a Galton-Watson process goes *extinct* if $Z_n = 0$ for all but finitely many n.

Proposition 6.10. *The probability ζ of extinction of a Galton-Watson process is the smallest nonnegative root $t = \zeta$ of the equation*

$$\varphi(t) = t. \tag{6.11}$$

Proof. For a Galton-Watson process to go extinct, one of two things must happen.

(a) The single individual present in the 0th generation has no offspring, or
(b) Each of the offspring of the single individual yield a Galton-Watson process that goes extinct.

The probability of (a) is p_0, and if $Z_1 = k \geq 1$, the probability of (b) is ζ^k since each vertex in the first generation must produce a Galton-Watson process that reaches extinction. Thus,

$$\zeta = p_0 + \sum_{k=1}^{\infty} p_k \zeta^k = \varphi(\zeta).$$

Therefore, the extinction probability ζ is a root of (6.11). □

We can condition a bond percolation process on the binary tree on extinction. Let Z_n be the number of nth generation descendants of $\varnothing$ in the binary tree. The sequence $Z = (Z_n \mid n \in \mathbb{N})$ is a Markov chain as defined previously with transition probabilities

$$\mathbb{P}_p(Z_{n+1} = j \mid Z_n = i) = \binom{2i}{j} p^j (1-p)^{2i-j}. \tag{6.12}$$

Let $\overline{Z}_n$ be the number of nth generation descendants of $\varnothing$ in the binary tree conditioned on the event E that the process becomes extinct.

Proposition 6.13. *$\overline{Z}$ is a Markov chain with transition probabilities*

$$\mathbb{P}_p(\overline{Z}_{n+1} = j \mid \overline{Z}_n = i) = \mathbb{P}_p(Z_{n+1} = j \mid Z_n = i)\mathbb{P}_p(E)^{j-i} \quad \textit{for } i, j \in \mathbb{N}_0. \tag{6.14}$$

Proof. $\overline{Z}$ has the following transition probabilities:

$$\begin{aligned}
\mathbb{P}_p(\overline{Z}_{n+1} = j \mid \overline{Z}_n = i) &= \frac{\mathbb{P}_p(\overline{Z}_{n+1} = j; \overline{Z}_n = i)}{\mathbb{P}_p(\overline{Z}_n = i)} \\
&= \frac{\mathbb{P}_p(Z_{n+1} = j; Z_n = i \mid E)}{\mathbb{P}_p(Z_n = i \mid E)} \\
&= \frac{\mathbb{P}_p(Z_{n+1} = j; Z_n = i; E)/\mathbb{P}_p(E)}{\mathbb{P}_p(Z_n = i; E)/\mathbb{P}_p(E)} \\
&= \frac{\mathbb{P}_p(Z_{n+1} = j; Z_n = i; E)}{\mathbb{P}_p(Z_n = i; E)} \\
&= \frac{\mathbb{P}_p(Z_{n+1} = j; E \mid Z_n = i)/\mathbb{P}_p(Z_n = i)}{\mathbb{P}_p(E \mid Z_n = i)/\mathbb{P}_p(Z_n = i)} \\
&= \frac{\mathbb{P}_p(Z_{n+1} = j; E \mid Z_n = i)}{\mathbb{P}_p(E \mid Z_n = i)} \\
&= \frac{\mathbb{P}_p(Z_{n+1} = j \mid Z_n = i)\mathbb{P}_p(E)^j}{\mathbb{P}_p(E)^i} \\
&= \mathbb{P}_p(Z_{n+1} = j \mid Z_n = i)\mathbb{P}_p(E)^{j-i} \quad \text{for } i, j \in \mathbb{N}_0.
\end{aligned}$$

□

Substituting from (6.12) and using $\mathbb{P}_p(E) = 1 - \theta(p) = \left(\frac{1-p}{p}\right)^2$, we obtain

$$\begin{aligned}\mathbb{P}_p(\overline{Z}_{n+1} = j \mid \overline{Z}_n = i) &= \binom{2i}{j} p^j (1-p)^{2i-j} \left(\frac{1-p}{p}\right)^{2(j-i)} \\ &= \binom{2i}{j} (1-p)^j p^{2i-j} \\ &= \mathbb{P}_{1-p}(Z_{n+1} = j \mid Z_n = i).\end{aligned}$$

We have shown that the distribution of the branching process conditioned on extinction, is identical to that of a subcritical branching process with parameter $1-p$.

7. Critical Exponents on a Binary Tree

As in bond percolation, we would like to know that there is a $\beta > 0$ such that as $p \downarrow p_c$,

$$\theta(p) \approx (p - p_c)^{\beta} \quad \text{for some } \beta > 0. \tag{7.1}$$

Proposition 7.2. *For percolation on the binary tree, the critical exponent β is equal to* 1.

Proof. Let C be the open cluster containing the root $\varnothing$. This is a Galton-Watson process where $Z_0 = 1$, and the generating function is given by

$$\begin{aligned}\varphi(t) &= (1-p)^2 + 2p(1-p)t + p^2t^2 \\ &= ((1-p) + pt)^2 .\end{aligned}$$

The cluster C is finite if and only if this branching process comes extinct, and the extinction probability is the smallest nonnegative root of the equation $\varphi(t) = t$. Using the quadratic formula, we see that if $p \leq \frac{1}{2}$, this root is equal to 1, and if $p \geq \frac{1}{2}$, this root is equal to $\left(\frac{1-p}{p}\right)^2$. Thus,

$$\theta(p) = \begin{cases} 0 & \text{if } p \leq \frac{1}{2}; \\ 1 - \left(\frac{1-p}{p}\right)^2 & \text{if } p \geq \frac{1}{2}. \end{cases} \tag{7.3}$$

If we take the derivative of $1 - \left(\frac{1-p}{p}\right)^2$ at $p = \frac{1}{2}$, we obtain the linear approximation $y = 8\left(p - \frac{1}{2}\right)$ and thus, the relation

$$\theta(p) \sim 8\left(p - \frac{1}{2}\right) \quad \text{as } p \downarrow \frac{1}{2}. \tag{7.4}$$

□

As in bond percolation, we would like to know that there exists $\gamma > 0$ such that as $p \to p_c$,

$$\chi^f(p) \approx |p - p_c|^{-\gamma} . \tag{7.5}$$

Proposition 7.6. *For percolation on the binary tree, the critical exponent γ is equal to* 1.

Proof. Suppose that $p < p_c = \frac{1}{2}$. Recall that $\chi(p) = \mathbb{E}_p(|C|)$. A vertex $x \in T$ lies in C if an only if every edge is open in the unique path of the tree joining $\varnothing$ to x. There are 2^n vertices in the nth generation, so

$$\begin{aligned}\chi(p) &= \sum_{n=0}^{\infty} 2^n p^n = \sum_{n=0}^{\infty} (2p)^n \\ &= (1-2p)^{-1} \quad \text{if } p < \frac{1}{2} \\ &= \frac{1}{2}\left(\frac{1}{2} - p\right)^{-1}.\end{aligned}$$

Thus $\chi(p) \approx \left(\frac{1}{2} - p\right)^{-1}$, which implies $\gamma = 1$.

It now remains to show that $\chi^f(p) \approx |p - p_c|^{-\gamma}$ as $p \to p_c$. Assume that $p > p_c$. Recall that

$$\chi^f(p) = \mathbb{E}_p(|C|\,;|C| < \infty).$$

Then

$$\begin{aligned}\chi^f(p) &= \mathbb{E}_p(|C|\,;|C| < \infty) \\ &= \mathbb{E}_p(|C| \mid |C| < \infty)\mathbb{P}_p(|C| < \infty) \\ &= \chi(1-p)(1-\theta(p)) \quad \text{(by Proposition 6.13)} \\ &= \frac{1}{2}\left(\frac{1-p}{p}\right)^2 (1-p)^{-1}.\end{aligned}$$

Thus, $\chi^f(p) \approx |p - p_c|^{-1}$ when $p > p_c$. □

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