THREE SIDEWAYS REFLECTIONS ON RAMSEY THEORY

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Abstract. In this paper, we approach foundational results in Ramsey Theory in unorthodox ways: in particular, we use set theoretic and model theoretic constructions (ultrafilters and ultraproducts, respectively) to prove both the finite and infinite cases of Ramsey's Theorem. We then examine the infinite case of this theorem more carefully, proving a pair of results about specific infinite ordinals.

Contents

1. Ramsey Theory 1
1.1. The Ultrafilter 2
1.2. Infinite Ramsey's Theorem 4
2. Model Theory 5
2.1. The Ultraproduct 6
2.2. Finite Ramsey's Theorem 7
3. Extension to Higher Ordinals 8
3.1. The Erdős–Rado Theorem 10
Acknowledgements 12
References 12

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1. Ramsey Theory

In general terms, Ramsey theory is the study of the emergence of pattern in large structures. Questions in Ramsey theory often take the form "How large does an arbitrarily generated structure need to be in order to ensure that it contains a certain substructure?" One classic example is as follows:

> *How many people must be at a party to ensure there exists either a group of n mutual acquaintances or of n mutual strangers?*

If we formulate this question in terms of graph theory, representing people as vertices and relations between them (acquaintances/strangers) as coloured edges (red/blue), it becomes:

> *What is the smallest integer m such that every bichromatic colouring of the complete graph on m vertices contains a monochromatic subgraph on n vertices?*

It is not even readily apparent that this question always has an answer, i.e., that for a sufficiently large n, one could not two-colour arbitrarily large complete graphs in such a way as to avoid monochromatic subgraphs on n vertices. However, the fact that this question indeed always has an answer is the essence of the following theorem:

Theorem 1.1 (Finite Ramsey's Theorem)**.** *For every $n < \omega$, there exists $R_k(n) < \omega$ such that every k-colouring of the complete graph on $R_k(n)$ vertices contains a monochromatic subgraph on n vertices.*

This theorem, in turn, is a finite version of the following theorem first published in 1930 by Frank Ramsey [1]. Note that we use $[X]^n$ to denote the set of all n-element subsets of X:

Theorem 1.2 (Ramsey's Theorem)**.** *For any infinite set I together with a k-colouring of the elements of $[I]^n$, there exists an infinite set $M \subseteq I$ such that $[M]^n$ is monochromatic.*

Setting $n = k = 2$, we see that this theorem implies that any two-colouring of a complete infinite graph contains a monochromatic infinite subgraph. Setting $k > 2$ shows us that the same is true for any finite colouring, while setting $n > 2$ yields similar results about hypergraphs.

Our proof of Ramsey's Theorem (Theorem 1.2) will make use of a set theoretic structure called the *ultrafilter*. We will continue to make use of this structure as we enter into the topic of model theory; in the second section of this paper, we will use an intimately related construction called the *ultraproduct* to prove Finite Ramsey's Theorem (Theorem 1.1).

1.1. **The Ultrafilter.**

Definition 1.3 (Filter)**.** Let I be a non-empty set. A *filter* is a collection D of subsets of I such that:

(i) $D \neq \emptyset$
(ii) $\emptyset \notin D$
(iii) $X \in D$ and $X \subset Y \Rightarrow Y \in D$
(iv) $X \in D$ and $Y \in D \Rightarrow X \cap Y \in D$

Definition 1.4 (Ultrafilter)**.** A filter D on I is called an *ultrafilter* if for every $X \subseteq I$, $X \in D$ if and only if $I \setminus X \notin D$.

It may be helpful to think of an ultrafilter as a designation of which subsets of I are "large." The above criteria then state that some subsets must be large, the empty set is not large, a set containing a large set is itself large, the intersection of large sets is likewise large, and finally that the complement of a large set is small (and vice versa).

Observe that if we simply pick any $i \in I$ and define $D = \{E \subseteq I \mid i \in E\}$, D will satisfy all the above criteria, and is thus an ultrafilter. We call ultrafilters of this form *principal* ultrafilters, and all other ultrafilters *non-principal* ultrafilters. It turns out that any ultrafilter containing a finite set is a principal ultrafilter:

Proposition 1.5. *Suppose D is an ultrafilter over an infinite set I, and D contains some finite set $F \subset I$. Then D is a principal ultrafilter.*

Proof. We will show that if D contains a finite set of size $n > 1$, then D must also contain a finite set of size $n - 1$. This will imply that D contains a singleton set, from which it follows that D is a principal ultrafilter.

Given a finite set $F \subset D$ of size $|F| = n$, assign an arbitrary ordering to the elements of F, and let F_i denote the set F with its ith element removed. Because $n > 1$, all sets F_i are non-empty.

We now claim that at least one of the sets $F_1 \ldots F_n$ belongs to the ultrafilter D. Suppose for a contradiction that none of them do: it then follows that $I \setminus F_1 \ldots I \setminus F_n$ all belong to D, and so does their intersection, $I \setminus F_1 \ \cap \ \ldots \ \cap \ I \setminus F_n$. But this latter set is simply $I \setminus F$, which by assumption does not belong to D. From this contradiction, it follows that at least one of the sets $F_1 \ldots F_n$ must belong to D, i.e., D contains a set of size $n - 1$.

It then follows that D contains a singleton set, i.e. that for some $i \in I$, $\{i\} \in D$. By the properties of an ultrafilter (Definitions 1.3 and 1.4), we then have $i \in F \Leftrightarrow F \in D$, i.e. D is a principal ultrafilter.

□

This fact in turn implies the following property of non-principal ultrafilters, which will be useful in our proof of Ramsey's Theorem:

Proposition 1.6. *Let D be a non-principal ultrafilter over an infinite set I, and let $A \subset I$ and $B \subset I$ be complementary with respect to a cofinite subset $X \subset I$. Then exactly one of the sets A and B is in D, i.e., $A \in D \Leftrightarrow B \notin D$.*

Proof. Let F denote the finite set $I \setminus X$. Consider the sets $A + F$ and B. These are complementary with respect to I, and therefore exactly one of them is in D.

Consider first the case in which $A + F \notin D$ and $B \in D$. Suppose we also had $A \in D$. This would violate the property that ultrafilters are closed under supersets (property (iii) of Definition 1.3). It therefore follows that $A \notin D$, while $B \in D$.

Now consider the case that $A + F \in D$ and $B \notin D$. Suppose for a contradiction we had $A \notin D$, which would imply $I \setminus A \in D$. We would then have $I \setminus A \ \cap \ A + F \in D$. But this intersection is simply F, whence $F \in D$. By Proposition 1.5, this implies that D is a principal ultrafilter, contrary to our assumption. It therefore follows that $A \in D$ while $B \notin D$.

□

It is evidently simple to find a principal ultrafilter over any set. Our next task, however, is to show that a *non-principal* ultrafilter can be found on any *infinite* set. This result will follow from the following two propositions:

Proposition 1.7. *For any infinite set I, the collection of all its cofinite subsets*

$$F = \{X \subseteq I : \mid I \setminus X \mid < \infty\}$$

is a filter on I.

This type of filter is called a *Fréchet filter*. One can easily verify that it is indeed a filter by checking that it satisfies the criteria of Definition 1.3.

Proposition 1.8. *Any filter F on a set I can be extended to an ultrafilter D on I such that $F \subseteq D$.*

This result is a consequence of Zorn's Lemma, which we will not prove in this paper; a proof of it can be found in [2].

Observe that if we extend the Fréchet filter on any infinite set to an ultrafilter, the resulting ultrafilter is necessarily non-principal. It thus follows that every infinite set admits a non-principal ultrafilter. Using this fact, we can now prove Ramsey's Theorem (Theorem 1.2) in the case where $n = k = 2$:

1.2. Infinite Ramsey's Theorem.

Theorem 1.9 (Ramsey's Theorem, $n = k = 2$.)**.** *Every bichromatic colouring of the complete graph on an infinite set I contains an infinite monochromatic subgraph.*

Proof. (based on [2])
Let D be a non-principal ultrafilter on I, and let $\{i_0, i_1, i_2, \ldots\}$ be an ordering of the vertices of I. Let R and B ("red" and "blue") be the two colours with which the edges of I are coloured, so that for every $i_j \neq i_k$, the edge $(i_j, i_k) \in R$ or $(i_j, i_k) \in B$ (and not both).

For every $i_k \in I$, let R_k and B_k be the sets of vertices indexed greater than i_k which are connected to i_k with red or blue edges, respectively:

$$R_k = \{i_j \in I \mid j > k,\ (i_j, i_k) \in R\}$$
$$B_k = \{i_j \in I \mid j > k,\ (i_j, i_k) \in B\}$$

For every $i_k \in I$, colour the *vertex* i_k red if R_k is in D, and blue if B_k is in D. Note that for any $i_k \in I$, R_k and B_k complement one another with respect to the set of points in I greater than i_k, which is a cofinite subset of I. Therefore, by Proposition 1.6, exactly one of the sets R_k and B_k will always be in D.

Now let R' and B' denote the sets of red and blue coloured vertices, respectively. Because R' and B' complement one another with respect to I, it follows that exactly one belongs to the ultrafilter D. Suppose without loss of generality that $R' \in D$. We then construct an infinite subsequence i_{n_k} of $\{i_0, i_1, i_2, \ldots\}$ inductively, in the following way:

Let i_{n_0} be the smallest indexed element of R'. Having already chosen $\{i_{n_0}, i_{n_1}, \ldots, i_{n_k}\}$, let the next element $i_{n_{k+1}}$ be the smallest indexed element of the set

$$R' \cap R_{n_0} \cap \ \ldots\ \cap R_{n_k}.$$

Because each of the intersected sets is in D, the intersection itself must be in D, and is therefore non-empty, so this selection can indeed repeat *ad infinitum.* The elements of the subsequence i_{n_k} thus form an infinite subset of the vertices of I, all of which are connected by edges of the same colour.

□

Note that this result can easily be extended to prove Ramsey's theorem for any k, i.e., in cases where the infinite graph is coloured by more than two colours. Given a k-coloured infinite graph, let one of the colours be called "red"; we can then use the above process to show the existence of either an infinite subgraph entirely in red, or an infinite subgraph in which only the $(k-1)$ colours other than red appear. We then repeat this process recursively until we find an infinite monochromatic subgraph.

Our next task is to extend this result to prove Finite Ramsey's Theorem (Theorem 1.1). We will take a model theoretic approach to this proof: in particular, we will use a construction called the *ultraproduct* to demonstrate that Finite Ramsey's Theorem follows from Ramsey's Theorem. Let us therefore introduce the foundations of model theory:

2. Model Theory

Model theory is, broadly speaking, the study of mathematical structure through the lens of mathematical logic. The fundamental concepts in model theory are those of a formal *language* and its various potential *models*, i.e., ways of interpreting that language:

Definition 2.1 (Language)**.** A formal *language* is a set of symbols, which we further classify as *relation symbols*, *function symbols*, and *constant symbols*.

Definition 2.2 (Model)**.** A *model* of a given language is an *interpretation* of its symbols, given by the *interpretation function* I, together with a set A (called the "universe") on which this interpretation is defined.

In particular, I maps n-placed relation symbols in the language to n-placed relations on A, n-placed function symbols to n-placed functions on A, and constant symbols to elements of A. We will use the notation $\langle A, I\rangle$ to refer to the model with universe A and interpretation function I.

In this paper, we will work primarily with the language of bichromatic graphs, which consists only of two two-placed relation symbols, say R and B ("red" and "blue"). Any bichromatic graph is a model of this language: its universe A is the set of vertices of the graph, $R(x, y)$ holds if the edge between vertices x and y is red, as does $B(x, y)$ if this edge is blue.

We also wish to be able to speak of models that *satisfy* sentences in a given language. We will not go through the labour of formally defining satisfaction (or sentences); it will suffice to say that a model M satisfies a sentence ϕ in a language L, denoted

$$M \vDash \phi,$$

if, when that sentence is interpreted according to the model, it is a true statement. More formal definitions can be found in [2].

We now introduce the *ultraproduct*, a model which functions as a sort of averaging together of a sequence of models:

2.1. **The Ultraproduct.**

Suppose L is a language, I is a nonempty set, and D is an ultrafilter over I. For each $i \in I$, let M_i be a model of L with universe A_i. Now let C be the Cartesian product of these universes:

$$C = \prod_{i \in I} A_i$$

We can think of C as the set of all functions with domain I such that for each $i \in I$, $f(i) \in A_i$. Let us now define the relation $=_D$ on functions $f, g \in C$ by

$$f =_D g \Leftrightarrow \{i \in I \mid f(i) = g(i)\} \in D.$$

The relation $=_D$ is reflexive (because $I \in D$), symmetric (for the same reason), and transitive (because D is closed under intersection), and is therefore an equivalence relation. We can therefore speak of the equivalence class f_D of any $f \in C$.

We can now define the *ultraproduct* of the models M_i over the ultrafilter D, which we denote as:

$$U = \prod_D A_i.$$

The ultraproduct is itself a model of the language L, and its universe is the set of all equivalence classes of $=_D$, denoted in the following way:

$$\prod_D A_i = \{f_D \mid f \in \prod_{i \in I} A_i\}.$$

The interpretation function of the ultraproduct is then defined as follows:

(i) Let R be an n-placed relation symbol in L. The interpretation of R is the relation S such that:

$$S(f_D^0 \dots f_D^{n-1}) \Leftrightarrow \{i \in I \mid R(f^0(i) \dots f^n(i)\} \in D.$$

(ii) Let F be an n-placed function in L, and let F_i be its interpretation in the model M_i. The interpretation of F is the function H such that:

$$H(f_D^0 \dots f_D^{n-1}) = \langle F_i(f^0(i) \dots f^{n-1}(i)) \mid i \in I \rangle_D.$$

(iii) Let c be a constant symbol in L. Then the interpretation of c is the constant b such that:

$$b = \langle a_i \mid i \in I \rangle_D.$$

The ultraproduct functions as a sort of limit of the models M_i: in particular, a first-order property is satisfied in the ultraproduct if and only if the set of models M_i in which it is satisfied is "large," i.e., it belongs to the ultrafilter D. This is indeed the essence of Łoś's Theorem:

Theorem 2.3 (Łoś's Theorem). *Let ϕ be a statement in a language L. Let D be an ultrafilter over a set I, and let A_i be a collection of models of L. Then:*

$$\prod_D A_i \vDash \phi(f_0, \ldots, f_{n-1}) \Leftrightarrow \{i \in I \mid A_i \vDash \phi(f_0(i), \ldots, f_{n-1}(i))\} \in D.$$

A proof of this theorem, which is also called the "fundamental theorem of ultraproducts," can be found in [2].

We are now ready to prove Finite Ramsey's Theorem using the ultraproduct construction. Let us first restate the theorem at hand:

2.2. **Finite Ramsey's Theorem.**

Theorem 1.1 (Finite Ramsey's Theorem) *For every $n < \omega$, there exists $R_k(n) < \omega$ such that every k-colouring of the complete graph on $R_k(n)$ vertices contains a monochromatic subgraph on n vertices.*

Proof. We will proceed by contrapositive, i.e., we will show that if finite Ramsey's Theorem is not true, this implies that (infinite) Ramsey's Theorem is also not true. Recall that we have proven the latter in Section 1.1.

Supposing finite Ramsey's Theorem is not true, there exists some $n < \omega$ such that for every $i < \omega$, we can two-colour the complete graph on i vertices in such a way as to avoid any monochromatic subgraphs on n vertices. In model theoretic terms, we can state that for every $i < \omega$, there exists a model $M_i = \langle A_i, I_i \rangle$ of the language of bichromatic graphs $L = \langle R, B \rangle$ such that:

(i) $(\forall x \in A_i)(\forall y \in A_i)((R_i(x, y) \vee B_i(x, y)) \wedge \neg(R_i(x, y) \wedge B_i(x, y)))$,
i.e., M_i is a complete graph

(ii) $\neg((\exists x_0 \in A_i) \ldots (\exists x_{n-1} \in A_i)((\bigwedge_{0 \leq i < j \leq n-1} R_i(x_i, x_j)) \vee (\bigwedge_{0 \leq i < j \leq n-1} B_i(x_i, x_j)))$,
i.e., M_i does not contain a monochromatic subgraph of size n.

(iii) $(\exists x_0 \in A_i) \ldots (\exists x_{i-1} \in A_i)(\bigwedge_{0 \leq i < j \leq i-1} x_i \neq x_j)$,
i.e., M_i contains at least i vertices

where R_i and B_i denote the interpretations of R and B in the model M_i, i.e., $R_i = I_i(R)$ and $B_i = I_i(B)$.

Now let D be any non-principal ultrafilter, and consider the ultraproduct $\prod_D M_i$. Because properties (i) and (ii) above hold for every M_i, they also hold in the ultraproduct, i.e., $\prod_D M_i$ is a complete bichromatic graph with no monochromatic subgraph of size n.

To complete the proof, we need to show that $\prod_D M_i$ is an infinite graph. To this end, observe that for any $i < \omega$, the property "M_k has fewer than i vertices" holds for finitely many models M_k. i.e.,

$$\{M_k \mid |A_k| < i\} \text{ is finite.}$$

Because D is a non-principal ultrafilter, by Proposition 1.5, it does not contain any finite sets. Therefore,

$$\{M_k \mid |A_k| < i\} \notin D$$

from which it follows that

$$\{M_k \mid |A_k| \geq i\} \in D.$$

That is, for every $i < \omega$, the set of models which have i or more vertices belongs to the ultrafilter D. It then follows that for every $i < \omega$, the ultraproduct $\prod_D M_i$ has at least i elements, which implies that $\prod_D M_i$ is indeed infinite.

The ultraproduct $\prod_D M_i$ is therefore a complete infinite bichromatic graph with no monochromatic subgraph of size n, which contradicts Ramsey's Theorem (Theorem 1.2).

□

3. Extension to Higher Ordinals

In Section 1.1, we proved that every bichromatic colouring of a complete infinite graph contains an infinite monochromatic subgraph. But as we know from the work of Cantor, "infinity" comes in many different sizes (infintely many, in fact). What if we refine our notion of infinity, then, and ask questions such as "Does every 2-colouring of a complete uncountable graph contain an uncountable monochromatic subgraph?"

To investigate such questions, let us introduce the following notation:

Definition 3.1 (Arrow Notation)**.** When κ, λ and m are cardinals and n is a natural number, we write

$$\kappa \to (\lambda)^n_m$$

to mean that every partition of $[\kappa]^n$ into m pieces contains a subset of size λ of pieces all of which are in the same paritition (called a *homogeneous* subset). In the case where $n = 2$, then, $\kappa \to (\lambda)^2_m$ means that every m-colouring of the complete graph on κ vertices contains a monochromatic subgraph of size λ.

Observe that if $\kappa \to (\lambda)^n_m$ holds true, it will continue to hold true if κ is made larger, or if λ, m, or n are made smaller.

Using this notation, we can interpret Ramsey's Theorem (Theorem 1.2) as stating that for any finite n and m,

$$\aleph_0 \to (\aleph_0)^n_m.$$

The question “does every bichromatic colouring of a complete uncountable graph contain an uncountable monochromatic subgraph?” can thus be notated “does $\aleph_1 \to (\aleph_1)^2_2$?”

Our next task will be to show that the answer to this question is, perhaps surprisingly, no. The proof of this result will hinge upon the following proposition:

Proposition 3.2. *The lexicographically ordered set* $\{0,1\}^\kappa$ *does not have a monotonic sequence of order type* κ^+.

We will prove this proposition in the case where $\kappa = \aleph_0$, as this is the case which is relevant to the above question. The proof for general κ, which works largely in the same way, can be found in [4].

Proof. Suppose there indeed exists a monotonic sequence $\{f_\alpha : \alpha < \kappa^+\} \subset \{0,1\}^\kappa$ of order type $\aleph_1$, and assume without loss of generality that this sequence is monotonically increasing. For each $\alpha < \aleph_1$, let ξ_α be the point up to which f_α is identical to its successor $f_{\alpha+1}$. To state this more formally, let us write $f_a \restriction n$ to denote the sequence f_a truncated after $n-1$ elements, and $f_a(n)$ to denote the nth element of f_a. We then let ξ_α be the greatest ordinal such that $f_\alpha \restriction \xi_\alpha = f_{\alpha+1} \restriction \xi_\alpha$, but $f_\alpha(\xi_\alpha) = 0$ while $f_{\alpha+1}(\xi_\alpha) = 1$.

Because each sequence f_α is countable, for every α, ξ_α is finite. Then by the pigeonhole principle, there must exist some finite ξ such that $\xi_\alpha = \xi$ for $\aleph_1$-many values of α. Let A be the set of sequences f_α such that $\xi_\alpha = \xi$, so that $|A| = \aleph_1$. The following claim, however, will lead to a contradiction of the latter equation:

Claim. Suppose f_α and f_β are elements of $\{f_\alpha : \alpha < \kappa^+\}$ such that $\xi_\alpha = \xi_\beta$, and furthermore $f_\alpha \restriction \xi_\alpha = f_\beta \restriction \xi_\alpha$. Then $f_\alpha = f_\beta$.

Proof of Claim. By assumption, $f_\alpha < f_{\beta+1}$, from which it follows that $f_\alpha \leq f_\beta$. Likewise, $f_\beta < f_{\alpha+1}$, and so $f_\beta \leq f_\alpha$. Therefore, $f_\alpha = f_\beta$.

Given any $\xi < \aleph_0$ then, there exist only finitely many distinct sequences f_α such that $\xi_\alpha = \xi$. In other words, the set A as previously defined must be finite. This contradicts the previous result that $|A| = \aleph_1$, thus concluding the proof.

□

We will now prove that $\aleph_1 \not\to (\aleph_1)^2_2$ by showing that if the opposite were true, the above proposition would be violated. We will even be able to prove the following more general theorem:

Theorem 3.3. *For every* κ,

$$2^\kappa \not\to (\kappa^+)^2_2.$$

Proof. (based on [4])
Let $\lambda = 2^\kappa$, and let $\{f_\alpha : \alpha \in \lambda\}$ be an enumeration of $\{0,1\}^\kappa$ of order type λ. Define an ordering $\prec$ on λ in the following way: given $\alpha, \beta \in \lambda$, set $\alpha \prec \beta$ when $f_\alpha < f_\beta$, where $<$ represents the lexicographic ordering on $\{0,1\}^\kappa$, and vice versa.

We then two-colour the complete graph on λ as follows: given $\alpha, \beta \in \lambda$, colour the edge (α, β) green if the ordering $\prec$ of $\{\alpha, \beta\}$ agrees with the natural ordering of λ; otherwise, colour this edge red. We claim that this colouring can not contain a monochromatic subgraph of cardinality κ^+.

Suppose such a subgraph does exist, and call the set of its vertices H, so that $|H| = \kappa^+$. Because H is a subset of the ordinal λ, H is itself an ordinal, thus having order type κ^+.

Now consider the sequence $\{f_\alpha : \alpha \in H\} \subset \{0,1\}^\kappa$. If the graph on H is coloured entirely in green, this is a monotonically increasing sequence (according to the lexicographic order), while if the graph on H is coloured entirely in red, it is monotonically decreasing. In either case, we have constructed a monotonic sequence of order type κ^+ on the set $\{0,1\}^\kappa$, which contradicts Proposition 3.2.

□

However, it turns out that if we set $\kappa > 2^{\aleph_0}$, then the statement $\kappa \to (\aleph_1)^2_2$ will in fact hold true. This is a consequence of a more general theorem due to Erdős and Rado, for which we introduce another new piece of notation:

Definition 3.4 ($\beth_\alpha$)**.** For any ordinal α, define the ordinal $\beth_\alpha$ inductively as follows:

$$\beth_0 = \aleph_0$$
$$\beth_{\alpha+1} = 2^{\beth_\alpha}.$$

For a limit ordinal κ,

$$\beth_\kappa = \sup\{\beth_\alpha \mid \alpha < \kappa\}.$$

3.1. **The Erdős–Rado Theorem.**

Theorem 3.5 (Erdős–Rado)**.** *For any natural number n,*

$$\beth_n^+ \to (\aleph_1)^{n+1}_{\aleph_0}.$$

Proof. (based on [4])
We will prove this theorem for $n = 1$, i.e., we will show that every $\aleph_0$-colouring of the complete graph on $\kappa = (2^{\aleph_0})^+$ vertices contains a monochromatic subgraph of size $\aleph_1$. This is the case of most immediate interest to us in this paper; the full proof of the theorem follows by an inductive argument, which can be found in [4].

Let us use the notation $a =_X b$ (read: *a agrees with b on* X) to mean that for every element $x \in X$, the edge between a and x is of the same colour as the edge between b and x. Observe that $=_X$ is an equivalence relation. Our first step will be to prove the following claim, which may seem contrived, but from which an elegant proof will follow:

Claim 1. For any set $A_\alpha \subset \kappa$ of cardinality $|A_\alpha| = 2^{\aleph_0}$, there exists a superset $A_{\alpha+1} \supset A_\alpha$ also of cardinality $|A_{\alpha+1}| = 2^{\aleph_0}$, such that for every countable set $C \subset A_\alpha$ and for every $u \in \kappa \setminus C$, there exists $v \in A_{\alpha+1} \setminus C$ such that $u =_C v$.

Proof of Claim 1. Given A_α, there exist at most $\aleph_0 \cdot 2^{\aleph_0} = 2^{\aleph_0}$ distinct countable sets $C \in A_\alpha$. Given such a C, there exist at most $\aleph_0 \cdot \aleph_0 = \aleph_0$ ways in which $u \in \kappa \setminus C$ can be coloured with respect to the points of C, i.e., the relation $=_C$ has at most $\aleph_0$ equivalence classes.

We then construct $A_{\alpha+1}$ as follows: for every countable $C \in A_\alpha$, add one element of each nonempty equivalence class of $=_C$ to the set A_α. $A_{\alpha+1}$ evidently satisfies the desired property. Furthermore, in the construction of $A_{\alpha+1}$, we have added at most $\aleph_0 \cdot 2^{\aleph_0} = 2^{\aleph_0}$ points to A_α. The set $A_{\alpha+1}$ therefore itself has cardinality $2^{\aleph_0}$.

Having proven the claim, let us use it to construct a nested ω_1-sequence of subsets of κ, $A_0 \subset A_1 \subset \ldots \subset A_\alpha \subset \ldots$, as follows: Let A_0 be an arbitrary subset of κ with cardinality $2^{\aleph_0}$. For each limit ordinal α, let $A_\alpha = \cup_{\beta<\alpha} A_\beta$. Otherwise, given A_α, choose $A_{\alpha+1}$ following the above construction. Now let $A = \cup_{\alpha<\omega_1} A_\alpha$, and observe that A itself has cardinality $2^{\aleph_0}$. We claim that A satisfies the following property:

Claim 2. For every countable set $C \subset A$ and for every $u \in \kappa \setminus C$, there exists $v \in A \setminus C$ such that $u =_C v$.

Proof of Claim 2. Let $A_0 \subset A_1 \subset \ldots \subset A_\alpha \subset \ldots$ denote the ω_1-sequence of sets of which A is the union, as above. Given a countable $C \subset A$, there exists some $\alpha < \omega_1$ such that $C \subset A_\alpha$. By the previous claim, all subsequent sets in the ω_1-sequence contain the desired v, and therefore so does A itself.

We now use this property to construct a sequence $\langle x_\alpha : \alpha < \omega_1 \rangle$ of elements of κ as follows. Choose an arbitrary $a \in \kappa \setminus A$ and $x_0 \in A$, and given $\{x_\beta : \beta < \alpha\} = C$, choose x_α such that $x_\alpha =_C a$. Note that because we are constructing an ω_1-sequence, the set C is always at most countable, and so the sequence $\langle x_\alpha : \alpha < \omega_1 \rangle$ is indeed well defined.

Now let X be the set of all elements in this sequence, i.e., $X = \{x_\alpha : \alpha < \omega_1\}$. For each element $x_\alpha \in X$, let $F(x_\alpha)$ denote the color with which x_α is connected to a.

Because X is of cardinality $\aleph_1$ while the range of F is of cardinality $\aleph_0$, by the pigeonhole principle, there must exist some $H \subset X$ of cardinality $|H| = \aleph_1$ such that F is homogenous on H, i.e., all the elements of H are connected to a with the same color, which we call F_H. But by the construction of X, if $\alpha < \beta$, then x_α is connected to x_β with the color $F(x_\alpha)$. Therefore, all the elements of H are connected with the same color $F(H)$. H then forms a monochromatic subgraph of size $\aleph_1$.

□

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