DYNAMICS OF MAPS OF THE CIRCLE

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Abstract. This paper applies the basics of dynamical systems to circle homeomorphisms. We lift the circle to the real line and introduce a dynamical invariant, the rotation number, to analyze a map's behavior. We demonstrate many properties of rotation number, such as its continuous dependence on choice of homeomorphism. We finish by proving the Poincaré classification theorem, which states that all circle maps with irrational rotation number are semi-conjugate to a rotation.

Contents

1. Introduction

Dynamics studies the patterns in a map under repeated iteration. In this paper, we will focus on the dynamics of circle homeomorphisms. We begin by studying the simplest circle homeomorphisms, rotations. We will then discuss how we can use rotations to study the dynamics of arbitrary circle homeomorphisms. Our goal is to examine when circle homeomorphisms are dynamically equivalent to rotations. Our main tool in this discussion is an invariant called the rotation number, which is due to Poincaré.

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2. Preliminary Definitions — An Introduction to Dynamics

In this section, we introduce some basic definitions. A (discrete) dynamical system is a pair (X, f), where X is a space and $f : X \to X$ is a map. In this paper, we restrict our attention to the behavior of homeomorphisms of the circle and the real line.

Definition 2.1. A map $f : X \to Y$ is a *homeomorphism* if f is continuous, injective, surjective, and f^{-1} is continuous. If the first r derivatives $f, f', f'', \cdots, f^{(r)}$ exist and are continuous, then f is said to be C^r *continuously differentiable.* A map $f : X \to Y$ is said to be a C^r *diffeomorphism* if f is a C^r homeomorphism and f^{-1} is also C^r.

For a given pair (X, f), as before, dynamics studies the orbits of f.

Definition 2.2. Let $f : X \to X$ be a map. The *forward orbit* of a point $x \in X$, denoted by $\mathcal{O}^+(x)$, is the sequence $\{f^n(x)\}_{n=0}^{n=\infty}$. If f is invertible then the *backwards orbit* of a point $x \in X$, denoted by $\mathcal{O}^-(x)$, is the sequence $\{f^n(x)\}_{n=-\infty}^{n=0}$. For invertible f we define the *full orbit* of a point $x \in X$, denoted by $\mathcal{O}(x)$, as the sequence $\{f^n(x)\}_{n=-\infty}^{n=\infty}$.

Notation 2.3. By $f^n(x)$ we will always mean the nth iterate of a point x. If we want to refer to the nth derivative of a function f at point x, we will write $f^{(n)}(x)$.

The simplest orbits to understand are those of periodic points and fixed points.

Definition 2.4. Let $f : X \to X$ be a map. A point $x \in X$ is a *fixed point* of f if $f(x) = x$. A point $x \in X$ is a *periodic point* of f with period n, if $f^n(x) = x$. The smallest $n \in \mathbb{N}$ such that $f^n(x) = x$ is called the *prime period* of x.

Remark 2.5. If x is a period n point of f, then x is a fixed point for f^n.

Notions of equivalence are useful because it can often be easier to analyze one map or object than another similar one.

Definition 2.6. Let $f : X \to X$ and $g : Y \to Y$ be maps. Then, f is *semi-conjugate* to g if there exists a surjective continuous map $h : X \to Y$ such that $h \circ f = g \circ h$. If h is also injective (so h is a bijection), then f and g are *topologically conjugate.*

The definition of topological conjugacy, unlike that of topological semi-conjugacy, is symmetric in f and g. The definition of a topological conjugacy is parallel to that of a change of basis in Linear Algebra. That is, it can be helpful to think of topological conjugacy as changing the space of a map while maintaining its dynamics. Topological conjugacy is a notion of equivalence because it preserves orbits. On the other hand, for a semi-conjugacy, the dynamics of f are contained in g but not necessarily the other way round. For further reading, Chapter 1.7 in [2] has a good discussion of the material.

Sometimes we want to analyze how much of a space an orbit of a dynamical system covers. To this end, we introduce the following definitions.

Definition 2.7. Let $E \subset X$ be a subset of space X. A point $p \in E$ is a *limit point* of E if every neighborhood of p contains some point $q \neq p \in E$.

Definition 2.8. The *closure* of $D \subset X$, denoted $\bar{D}$, is defined as the union of D and all it's limit points. The set D is said to be *dense* in X if $\bar{D} = X$.

For example, the set of all rational numbers is dense in the real numbers. In dynamics, we care about the density of orbits.

Definition 2.9. A map $f : X \to X$ is said to be *topologically transitive* if, for any two open intervals $U, V \subset X$, there exists $k \in \mathbb{N}$ such that $f^k(U) \cap V \neq \emptyset$.

In this paper, we only consider dynamical systems on the circle or on the real line. For a dynamical system $f : X \to X$ in one dimension, topological transitivity is equivalent to having at least one point $x \in X$, such that $\mathcal{O}_f^+(x)$ is dense in X.

Definition 2.10. A dynamical system $f : X \to X$ is called *minimal* if every orbit of $x \in X$ is dense in X.

3. Maps of the Circle

3.1. **What is a map of the circle?** There are several equivalent ways of viewing the unit circle S^1. One logical way is to think about the unit circle in the complex plane,

$$S^1 = \{x \in \mathbb{C} : |x| = 1\}.$$

Euler's identity, $e^{2\pi ix} = \cos(2\pi x) + i\sin(2\pi x)$, allows us to lift the circle to the real line, which helps simplify constructions for many of the arguments in this paper. If unfamiliar with Euler's identity observe that it follows by considering the Taylor series of sin, cos, and the exponential function. It follows that

$$S^1 = \{x \in \mathbb{C} : |x| = 1\} = \{\cos(2\pi x) + i\sin(2\pi x) : x \in \mathbb{R}\} = \{e^{2\pi ix} : x \in \mathbb{R}\} = \mathbb{R}/\mathbb{Z}.$$

Here, we wrap the whole real line around S^1 as shown in Figure 1. With some thought, we can see that we can cover the circle with just the points in an integer interval. For example, consider the interval $[0, 1]$. It follows that $S^1 = \{e^{2\pi ix} : x \in [0, 1]\}$. This gives rise to what we call additive notation, $\mathbb{R}/\mathbb{Z}$ (read real numbers modulo the integers), is an equivalent definition of the circle. We can actually use

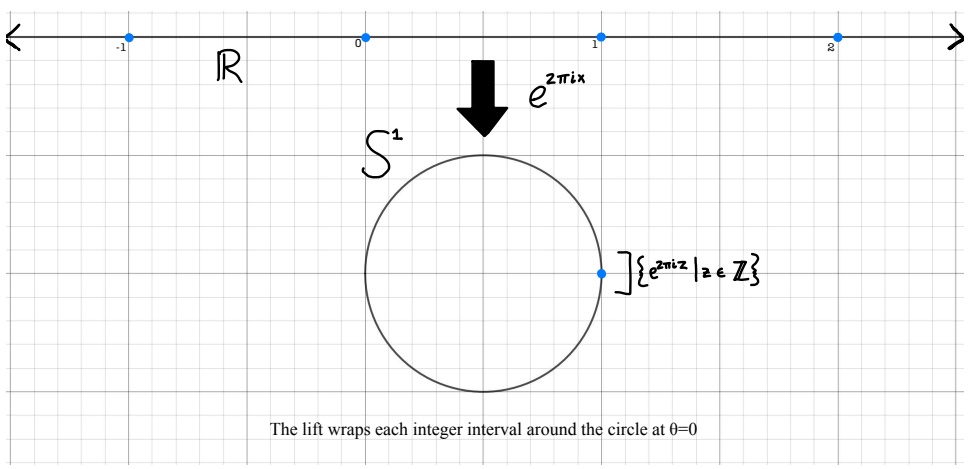


Figure 1. Lifting S^1 to the real line.

Euler's identity to lift a map $f : S^1 \to S^1$ of the circle to a map $F : \mathbb{R} \to \mathbb{R}$ of the real line. Define the exponential function $\pi : \mathbb{R} \to S^1$, which takes us from the real line to the circle, as $\pi(x) = e^{2\pi ix}$. This gives rise to the following definition,

Definition 3.1. Let $f : S^1 \to S^1$ be a homeomorphism of the circle. We call a function $F : \mathbb{R} \to \mathbb{R}$ a *lift* of the homeomorphism f if $f \circ \pi = \pi \circ F$. See Figure 2.

$$\begin{tikzcd} \mathbb{R} \arrow[r, "F"] \arrow[d, "\pi"'] & \mathbb{R} \arrow[d, "\pi"] \\ S^1 \arrow[r, "f"'] & S^1 \end{tikzcd}$$

FIGURE 2. Commutative diagram of the lift of a circle map to the real line.

In this paper, we only consider orientation-preserving homeomorphisms of the circle.

Definition 3.2. Let $f : S^1 \to S^1$ be a homeomorphism of the circle. Then, f is said to be *orientation-preserving* if we can lift f to a monotone increasing homeomorphism $F : \mathbb{R} \to \mathbb{R}$.

Remark 3.3. Suppose $f : S^1 \to S^1$ is an orientation-preserving homeomorphism of the circle, and let $F : \mathbb{R} \to \mathbb{R}$ be a lift of f. For any $\theta \in S^1$, there are countably many $x \in \mathbb{R}$ such that $\pi(x) = \theta$; that is, if $\pi(x) = \theta$, then $\pi(x + 1) = \theta$. Since f is an orientation-preserving homeomorphism, $f(\theta)$ is constant. Therefore, we have that $F(x + 1) = F(x) + 1$ and more generally that $F(x + z) = F(x) + z$ for $z \in \mathbb{Z}$.

The property in Remark 3.3 allows us to bound the difference of lifts of two points.

Proposition 3.4. *Let $f : S^1 \to S^1$ be an orientation-preserving homeomorphism of the circle, and let F be a lift of f. If $|x - y| < 1$, then $|F(x) - F(y)| < 1$. It follows that $|F^n(x) - F^n(y)| < 1$.*

Proof. Since f is orientation-preserving F is monotone increasing. Hence,

$$|F(x) - F(y)| < |F(x + 1) - F(x)| = F(x) + 1 - F(x) = 1.$$

□

Remark 3.5. At this point, it is important to note that for each orientation-preserving circle homeomorphism, f, there are countably many lifts F. The lift only cares about where in an integer interval each point on the circle is placed. Since there are infinitely many of these intervals, we can lift to any one of them. Hence, there are countably infinite lifts differing only by integer values.

3.2. **Rotations of the Circle.** The simplest example of an orientation-preserving circle homeomorphism is rotation.

Definition 3.6. Rotation of the circle by $\alpha \in \mathbb{R}$ is the map $\mathcal{R}_\alpha(e^{2\pi i\theta}) : S^1 \to S^1$, defined by $e^{2\pi i\theta} \mapsto e^{2\pi i(\theta+\alpha)}$. In additive notation, rotation is the map $\mathscr{R}_\alpha(x) : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, defined by $x \mapsto x + \alpha$.

The dynamics of the rotation map are very different when α is rational versus when α is irrational. In fact, the dichotomy between rational and irrational rotation will be a thread throughout this paper.

Suppose that α is rational, that is $\alpha = \frac{p}{q}$ for $p, q \in \mathbb{Z}$ relatively prime. Then, any point $\theta \in S^1$ is fixed by $\mathcal{R}_\alpha^q$, since $\mathcal{R}_\alpha^q(\theta) = \theta + 2\pi\frac{p}{q}q = \theta + 2p\pi = \theta$. Therefore, every point $\theta \in S^1$ is a periodic point of $\mathcal{R}_\alpha$ with period q. The story is entirely different for irrational α.

Proposition 3.7. *If α is irrational, then the map $\mathcal{R}_\alpha(\theta) = \theta + 2\pi\alpha$ is minimal (see Definition 2.10) in S^1.*

Proof. Fix $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and $\theta \in S^1$. First, we will show that all points in the orbit of θ are distinct. Then, we will show that the orbit of θ is dense in S^1. Suppose that all the points on the orbit of θ are not distinct, i.e that

$$\mathcal{R}_\alpha^n(\theta) = \theta + m2\pi\alpha = \theta + n2\pi\alpha = \mathcal{R}_\alpha^n(\theta), \text{ and } m \neq n.$$

Then, we must have that $(n-m)\alpha \in \mathbb{Z}$. However, since α is irrational, any non-zero integer multiple of α is also irrational. So, n must equal m, which is a contradiction. Hence, all points in the orbit of θ are distinct.

Fix $\epsilon > 0$. We need to show that the orbit $\mathcal{O}(\theta)$ is dense in S^1, i.e that for any arc $I \subset S^1$ of length ϵ, we have that $\mathcal{O}(\theta) \cap I \neq \emptyset$. The circle is compact. Therefore, given any infinite sequence of points in $\mathcal{O}(\theta) \subset S^1$, there exists a convergent subsequence. Thus, there exist $n, m \in \mathbb{N}$ such that $n > m$ and $|\mathcal{R}_\alpha^n(\theta) - \mathcal{R}_\alpha^m(\theta)| < \epsilon$. Rotation is an *isometry*, or a map that preserves distance. So, since we are rotating intervals around the circle, the lengths of these intervals must be constant. Thus, we have $|\mathcal{R}_\alpha^k(\theta) - \theta| < \epsilon$.

Let us consider the shortest closed arc between $R_\alpha^k(\theta)$ and θ. Iteration by the rotation map gives us

$$(\theta, \mathcal{R}_\alpha^k(\theta)) \mapsto (\mathcal{R}_\alpha^k(\theta), \mathcal{R}_\alpha^{2k}(\theta)) \mapsto (\mathcal{R}_\alpha^{2k}(\theta), \mathcal{R}_\alpha^{3k}(\theta)) \mapsto \cdots .$$

We have partitioned the circle into arcs of length ϵ each containing a point of the orbit of θ. Since ϵ and θ were picked arbitrarily, $\mathcal{O}(\theta)$ is dense in S^1 and $\mathcal{R}_\alpha(\theta)$ is minimal in S^1. □

3.3. **An example of a lift.** In this section, we introduce a family of circle maps to visualize the lift of a circle homeomorphism.

Definition 3.8. The family of circle maps $\{f_{\omega,\gamma} : S^1 \to S^1 | \omega, \gamma \in \mathbb{R}\}$, which we will call the *standard family*, is defined as

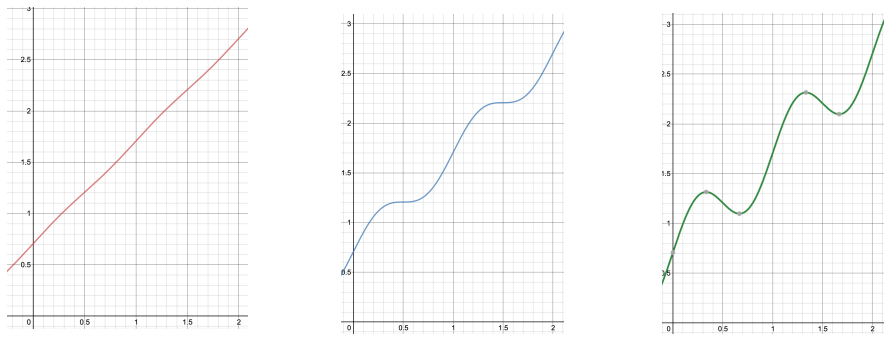
$$f_{\omega,\gamma}(\theta) = \theta + 2\pi\omega + \gamma\sin(\theta). \tag{3.9}$$

The map $f_{\omega,\gamma}$ is not always a homeomorphism. The choice of ω rotates the map and does not affect whether $f_{\omega,\gamma}$ is a homeo/diffeomorphism. When $\gamma = 0$, $f_{\omega,0}$ is just rotation by ω. When $0 \leq \gamma < 1$, $f_{\omega,\gamma}$ is a diffeomorphism. When $\gamma = 1$, $f_{\omega,\gamma}$ is only a homeomorphism. Finally, when $\gamma > 1$, $f_{\omega,\gamma}$ fails to be injective. We will not prove these statements, but the graphs in Figure 3 should make them believable. For a more rigorous treatment, see Chapter 14 in [2].

For fixed $\omega, \gamma \in \mathbb{R}$, the collection of maps $F_{\omega,\gamma,k} : \mathbb{R} \to \mathbb{R}$, defined by

$$F_{\omega,\gamma}(x) = x + \omega + \frac{\gamma}{2\pi}\sin(2\pi x) + k, \text{ where } k \in \mathbb{Z},$$

is the collection of lifts of $f_{\omega,\gamma}$.



(A) $F_{.1}(x) = x + \frac{\sqrt{2}}{2} + .1\sin(x)$ (B) $F_1(x) = x + \frac{\sqrt{2}}{2} + \sin(x)$ (C) $F_2(x) = x + \frac{\sqrt{2}}{2} + 2\sin(x)$

FIGURE 3. Lift $F_{\omega,\gamma}$ of $f_{\omega,\gamma}$ with $\omega = \frac{\sqrt{2}}{2}$ and $\gamma = 0.1, 1,$ and 2 respectively.

Figure 3 shows the lifts of the standard family where $\omega = \frac{\sqrt{2}}{2}$ and γ is equal to $0.1, 1,$ and 2 respectively.

3.4. **Rotation Number.** The central question in this paper is when does an arbitrary circle homeomorphism behave like a rotation? To answer this question, we introduce an invariant called *rotation number*. Given an orientation-preserving homeomorphism of the circle f, rotation number will tell us which rotation, $\mathcal{R}_\alpha$ by $\alpha \in \mathbb{R}$, f most behaves like. In fact, we will be able to understand a lot about a circle homeomorphism's dynamics just by knowing it's rotation number. Our first step will be to introduce what is called a *lift number*. The lift number is a value determined by the choice of lift F of a circle homeomorphism f. We will see that all lift numbers differ only by integer values, which gives us a natural way to define the rotation number which will depend only on choice of circle homeomorphism f.

Theorem 3.10. *Let $f : S^1 \to S^1$ be an orientation-preserving homeomorphism of the circle, and let $F : \mathbb{R} \to \mathbb{R}$ be a lift of f. The limit $lim_{|n|\to\infty} \frac{F^n(x)-x}{n}$ exists for all $x \in \mathbb{R}$, and its value is independent of choice of $x \in \mathbb{R}$. We will define the* lift number *$\tau(F) := lim_{|n|\to\infty} \frac{F^n(x)-x}{n}$. Furthermore, given any two lifts, F_1 and F_2, their lift numbers differ by an integer value, i.e. $\tau(F_1) - \tau(F_2) = z \in \mathbb{Z}$.*

Proof. Assume that the limit $lim_{|n|\to\infty} \frac{F^n(x)-x}{n}$ exists for all $x \in \mathbb{R}$. We will show that the lift number, $\tau(F)$, is independent of choice of $x \in \mathbb{R}$. Fix $x, y \in [0, 1)$. Then,

$$\left| \frac{(F^n(x) - x) - (F^n(y) - y)}{n} \right| \leq \frac{|F^n(x) - F^n(y)| + |x - y|}{n} \leq \frac{2}{n}.$$

Since $\frac{2}{n} \to 0$ as $n \to \infty$, the limit does not depend on choice of $x \in \mathbb{R}$.

We will now show that given any orientation-preserving circle homeomorphism f, and any lift $F : \mathbb{R} \to \mathbb{R}$ of f, the lift number exists. For this proof, we will break up the argument into two cases: when f has a periodic point and when f does not have a periodic point.

Case 1: Suppose f has a periodic point of period $n \in \mathbb{N}$, i.e for some $\theta \in S^1$, there exists $m > 0$ such that $f^m(\theta) = 0$. Fix $x \in \mathbb{R}$ such that $\pi(x) = \theta$. We know that $F^m(x) = x + z$ for some $z \in \mathbb{Z}$. It follows from Remark 3.3 that $F^{jm} = x + jz$ for $j \in \mathbb{Z}$. Immediately, we have that

$$\lim_{j\to\infty} \frac{|F^{jm}(x) - x|}{jm} = \lim_{j\to\infty} \frac{jz}{jm} = \frac{z}{m}.$$

We want to show that

$$\lim_{j\to\infty} \frac{|F^{jm}(x) - x|}{jm} = \lim_{n\to\infty} \frac{F^n(x) - x}{n}.$$

We can write any integer $n \in \mathbb{N}$ as $n = jm + r$, where $r \in \mathbb{N}$ is the remainder, $0 \le r < m$. Since r is bounded above, we know that there must exist some $M > 0$ such that, for all $x \in \mathbb{R}$, $|F^r(x) - x| < M$. It immediately follows that

$$\frac{|F^n(x) - x - (F^{jm}(x) - x)|}{n} = \frac{|F^r(F^{jm}(x)) - F^{jm}(x)|}{n} \le \frac{M}{n}.$$

Since $\frac{M}{n} \to 0$ as $n \to \infty$, we have shown that

$$\tau(F) = \lim_{n\to\infty} \frac{F^n(x) - x}{n} = \lim_{j\to\infty} \frac{|F^{jm}(x) - x|}{jm} = \frac{k}{m}.$$

We have not only shown that $\tau(F)$ exists if f has a periodic point, but also that $\tau(F)$ must be rational.

Case 2: Suppose that f does not have any periodic points. Fix a lift F of f. Since f has no periodic points, we know that, for all $x \in \mathbb{R}$, $F^n(x) - x$ is never an integer for $n \neq 0$. Therefore, there must exist some integer $z \in \mathbb{Z}$ such that $z < F^n(x) - x < z + 1$ for all $x \in \mathbb{R}$. For simplicity, fix $x = 0$. It follows that

$$z < F^n(0) - 0 < z + 1. \tag{3.11}$$

From (3.4) if we take $x = F^n(0)$, then $z < F^{2n}(0) - f^n(0) < z + 1$. Furthermore, for $m \in \mathbb{N}$, we have that $z < F^{mn}(0) - F^{(m-1)n}(0) < z + 1$. Therefore, we can sum over these m inequalities to get $mz < F^{mn}(0) - 0 < m(z + 1)$. Dividing by n, we have that

$$\frac{z}{n} < \frac{F^{mn}(0) - 0}{mn} < \frac{z+1}{n}. \tag{3.12}$$

We can divide by n in (3.11) and combine with (3.12) to get

$$\left| \frac{F^{mn}(0) - 0}{mn} - \frac{F^n(0)}{n} \right| < \frac{1}{n}. \tag{3.13}$$

Repeating the exact same argument with m and n interchanged we get

$$\left| \frac{F^{mn}(0) - 0}{mn} - \frac{F^m(0)}{m} \right| < \frac{1}{m}. \tag{3.14}$$

Combining (3.13) and (3.14), it is immediate that

$$\left| \frac{F^n(0) - 0}{n} - \frac{F^m(0) - 0}{m} \right| < \frac{1}{n} + \frac{1}{m}.$$

So, the sequence $\{\frac{F^n(0)-0}{n}\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ is Cauchy and must converge.

Let F_1 and F_2 be different lifts of f. We want to show that $\tau(F_1) - \tau(F_2) = z \in \mathbb{Z}$. Since there are countably many lifts F of f that differ only by integer values,

we know that there exists $k \in \mathbb{Z}$ such that $F_1(x) = F_2(x) + k$. It follows that $F_1^n(x) = F_2^n(x) + nk$. Hence, $\tau(F_1) = \tau(F_2) + k$. □

We have shown that the lift numbers $\tau(F)$ are well defined and differ only by integer values, which naturally leads to the following definition:

Definition 3.15. Let $f : S^1 \to S^1$ be a homeomorphism and let $F : \mathbb{R} \to \mathbb{R}$ be a lift of f. We define the *rotation number* of f as

$$\tau(f) := \tau(F) \bmod 1 = \lim_{|n|\to\infty} \frac{F^n(x) - x}{n} \bmod 1.$$

One of the useful properties of rotation number is that it is a *dynamical invariant*. A dynamical invariant is a property that is preserved under topological conjugacy.

Notation 3.16. Let $[x] := \max\{k \in \mathbb{Z} | k \leq x\}$.

Theorem 3.17. *If $f : S^1 \to S^1$ and $h : S^1 \to S^1$ are orientation-preserving homeomorphisms of the circle, then $\tau(h^{-1}fh) = \tau(f)$.*

Proof. This proof follows that of Proposition 11.1.3 in [3]. Let F and H be lifts of f and h respectively such that $F(0), H(0) \in [0, 1)$. We note that $(H^{-1}FH)^n(x)$ telescopes and is equal to $(H^{-1}F^nH)(x)$. So, by definition of rotation number, we need to show that

$$\lim_{n\to\infty} \left| \frac{(H^{-1}F^nH)(x) - F^n(x)}{n} \right| = 0.$$

Our first step will be to show that H^{-1} is indeed a lift of h^{-1}. This is true because

$$\pi \circ H^{-1} = h^{-1} \circ h \circ \pi \circ H^{-1} = h \circ H^{-1} \circ \pi \circ H \circ H^{-1} = h^{-1} \circ \pi.$$

Here, the second equality follows because H is a lift of h, so $h \circ \pi = \pi \circ H$. We also need to establish that $H^{-1} \circ F \circ H$ is a lift of $h^{-1} \circ f \circ h$. To do this, we compute

$$\pi \circ H^{-1} \circ F \circ H = h^{-1} \circ \pi \circ F \circ H = h^{-1} \circ f \circ \pi \circ H = h^{-1} \circ f \circ h \circ \pi,$$

where in each of the equalities, we use the fact that H^{-1}, F, and H are lifts of h^{-1}, f, and h respectively.

Since $H(0) \in [0, 1)$, for $x \in [0, 1)$ we have that $-1 < H(x) - x \leq H(x) < H(1) < 2$. It follows that $|H(x) - x| < 2$ for all $x \in \mathbb{R}$. A similar process with H^{-1} gives us the estimate

$$|H^{-1}(x) - x| < 2 \ \forall x \in \mathbb{R}. \tag{3.18}$$

If $|y - x| < 2$, then $|F^n([y]) - F^n([x])| < 3$. This gives us the estimate

$$\begin{aligned} -3 \leq [y] - [x] - 1 &= F^n([y]) - F^n([x] + 1) \\ &\leq F^n(y) - F^n(x) \\ &\leq F^n([y] + 1) - F^n([x]) \\ &= [y] + 1 - [x] \leq 3. \end{aligned} \tag{3.19}$$

Combining (3.18) and (3.19) we get

$$|(H^{-1}F^nH)(x) - F^n(x)| \leq |(H^{-1}F^nH)(x) - F^nH(x)| + |F^nH(x) - F^n(x)| < 2 + 3 = 5,$$

where the first inequality is just the triangle inequality. The second inequality follows by taking $F^nH(x)$ as x in (3.18) and taking $H(x)$ as y in (3.19). We are now done as

$$\lim_{n\to\infty} \left| \frac{(H^{-1}F^nH)(x) - F^n(x)}{n} \right| \leq \lim_{n\to\infty} \frac{5}{n} = 0.$$

□

We now prove a lemma that allows us to work with the rotation number of iterated circle homeomorphisms, f^m.

Lemma 3.20. *Suppose $f : S^1 \to S^1$ is an orientation-preserving homeomorphism of the circle with rotation number $\tau(f)$. Then, for $m \in \mathbb{Z}\backslash\{0\}, \tau(f^m) = m\tau(f)$.*

Proof. Observe that

$$\tau(f^m) = \lim_{n\to\infty} \frac{(F^m)^n(x) - x}{n} = m \lim_{n\to\infty} \frac{F^{mn}(x) - x}{mn} = m\tau(f) \bmod 1.$$

□

Proposition 3.21. *Let $f : S^1 \to S^1$ be an orientation preserving homeomorphism of the circle. Then, f has rational rotation number if and only if it has a periodic point.*

Proof. We have already shown that if f has a periodic point then f has rational rotation number in Theorem (3.10)

We now show the other direction. Suppose $\tau(f) = \frac{p}{q} \in \mathbb{Q}$, where p and q are relatively prime. Lemma 3.20 tells us that $\tau(f^q) = q\tau(f) = p \bmod 1 = 0$. Hence, it suffices to show that if $\tau(f) = 0$, then f has a fixed point.

Suppose that f does not have a fixed point. Choose a lift F of f such that $F(0) \in [0, 1)$. Then, for all $x \in \mathbb{R}, F(x) - x \notin \mathbb{Z}$. If not, $\pi(x)$ would be a fixed point for f. Since $F(0) \in [0, 1)$, then $0 < F(x) - x < 1$. Furthermore, $F(x) - x$ is continuous on the compact interval $[0, 1]$, so it must attain a minimum and a maximum on $[0, 1]$. Hence, there exists some $\delta > 0$ such that $0 < \delta \leq F(x) - x \leq 1 - \delta < 1$, which, by periodicity, holds for any choice of $x \in \mathbb{R}$. In particular, we can take $x = F^i(0)$, where $0 \leq i \leq n - 1$. It follows that

$$n\delta \leq \sum_{i=0}^{n-1} F^{i+1}(0) - F^i(0) = F^n(0) \leq n(1 - \delta).$$

The sum telescopes, so we have that $\delta \leq \frac{F^n(0)}{n} \leq 1 - \delta$. So, as $n \to \infty$, $\tau(f)$ does not approach 0. But, we assumed $\tau(f) = 0$. Hence, we have arrived at a contradiction, which means that f must have a fixed point. □

Proposition 3.21 immediately implies that if $f : S^1 \to S^1$ is an orientation-preserving homeomorphism, then $\tau(f)$ is irrational if and only if f has no periodic points. In addition, if $\tau(f)$ is rational, then in fact all periodic points have the same period.

Proposition 3.22. *Let $f : S^1 \to S^1$ be a homeomorphism with rational rotation number $\tau(f) = \frac{p}{q} \in \mathbb{Q}$, $p, q \in \mathbb{Z}$ relatively prime. Then, all periodic points are of period q.*

Proof. Choose $x \in \mathbb{R}$ such that $\pi(x) \in S^1$ is a periodic point of f. We need to show that there exists some lift F of f, for which $F^q(x) = x + p$. Since $\pi(x)$ is periodic, for any lift F we have that $F^i(x) = x + j$ where $i, j \in \mathbb{Z}$. Take $k \in \mathbb{Z}$. Then, because $t(f) = \frac{p}{q}$ we have

$$k + \frac{p}{q} = \tau(f) = \lim_{n\to\infty} \frac{F^{in}(x) - x}{in} = \lim_{n\to\infty} \frac{jn}{in} = \frac{j}{i},$$

where the second equality follows from the definition of rotation number and the third equality holds by Lemma 3.20.

Fix a lift F of f such that $k = 0$. We have that $\frac{p}{q} = \frac{j}{i}$. It follows that there exists $m \in \mathbb{N}$ such that $j = mp$ and $i = mq$. We want to show that $F^q(x) - p = x$ by showing that $F^q(x) - p \geq x$ and that $F^q(x) - p \leq x$. We will show both inequalities by contradiction. First, assume that $F^q(x) - p > x$. Since f is orientation-preserving, then F is monotone increasing. Thus, it follows that $F^{2q}(x) - 2p = F^q(F^q(x) - p) - p \geq F^q(x) - p > x$. Furthermore, by induction it follows that $F^{mq}(x) - mp(x) > x$ for all $m \in \mathbb{N}$. But, we know that $F^{mq}(x) - mp(x) = F^i(x) - j = x$, so we have a contradiction! The other case, where $F^q(x) - p < x$, follows similarly. Hence, we know that $F^q(x) - p \leq x$ and that $F^q(x) - p \geq x$. Therefore, $F^q(x) - p = x$. □

3.5. **Continuity and ordering of circle homeomorphisms.** The goal of this subsection is to show that the rotation number depends continuously on the choice of the homeomorphism f. To this end, we will define an ordering of orientation-preserving circle homeomorphisms. Then, given a one-parameter family of orientation-preserving homeomorphisms $\{f_\omega\}$, we will see that for $f_{p/q} \subset \{f_\omega\}$ with rational rotation number, there is an interval around $f_{p/q}$ where all other circle homeomorphisms have the same rotation number. On the other hand, there is never such an interval for $f_{ir} \subset \{f_\omega\}$ with irrational rotation number.

To show that the rotation number depends continuously on choice of orientation-preserving homeomorphisms, we must specify a distance for $C^r(X, X)$ maps.

Definition 3.23. Let $f : X \to X$ and $g : X \to X$ be two C^r maps defined on X. Then, we define the C^r*-distance* between f and g as

$$d_r(f, g) := \sup_{x \in X} \{|f(x) - g(x)|, |f'(x) - g'(x)|, \cdots, |f^{(r)}(x) - g^{(r)}(x)|\}.$$

Proposition 3.24. *Let $f : S^1 \to S^1$ be an orientation-preserving homeomorphism of the circle. For every $\epsilon > 0$, there exists $\delta > 0$ such that if $g : S^1 \to S^1$ is an orientation-preserving homeomorphism of the circle and $d_0(f, g) < \delta$, then $|\tau(f) - \tau(g)| < \epsilon$.*

Proof. This proof follows that of Corollary 14.7 in [2]. Fix $\epsilon > 0$, and choose $n \in \mathbb{N}$ such that $\frac{2}{n} < \epsilon$. Let F be a lift of f. Then, $r - 1 < F^n(0) < r + 1$ for some $r \in \mathbb{Z}$. Choose $\delta > 0$, and a lift G of g such that $r - 1 < G^n(0) < r + 1$. From Lemma 3.20 we have that $m(r-1) < F^{mn}(0) < m(r+1)$ and that $m(r-1) < G^{mn}(0) < m(r+1)$. Since the length of the interval $(r - 1, r + 1)$ is 2, we get that for all $m \in \mathbb{N}$,

$$\left| \frac{F^{nm}(0)}{nm} - \frac{G^{nm}(0)}{nm} \right| < \frac{2}{n} < \epsilon.$$

Since, by definition, $\lim_{m\to\infty} \frac{F^{nm}(0)}{nm} = \tau(F)$ and $\lim_{m\to\infty} \frac{G^{nm}(0)}{nm} = \tau(G)$, it follows that $|\tau(f) - \tau(g)| < \epsilon$. □

We will now present an ordering of orientation-preserving homeomorphisms of the circle, adapted from Chapter 14B of [4]. This is not the only such ordering (see Proposition 11.1.8 of [3]).

In Section 2, we presented topological conjugacy between maps as a way to study the orbit structure of a simpler map to understand the orbit structure of a more complicated map. In that vein, we introduce symbolic dynamics here to create

a topological conjugacy between the circle and the sequence space. Representing the orbit of a point via a sequence of 0s and 1s gives us a methodical method for comparing orientation-preserving circle homeomorphisms.

Definition 3.25. We define the *sequence space* $\Sigma_2 := \{s = (s_0 s_1 s_2 \cdots) : s_j = 0 \text{ or } 1\}$ of all infinite sequences of 0s and 1s. We define a distance function for $s = (s_0 s_1 s_2 \cdots), t = (t_0 t_1 t_2 \cdots) \in \Sigma_2$ by

$$d[s,t] = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}.$$

In fact, Σ_2 paired with $d[s,t]$ is a metric space. For an in depth discussion of the construction of the sequence space see Chapter 6 in [2].

Let $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be an orientation-preserving homeomorphism of the circle, and choose $F : \mathbb{R} \to \mathbb{R}$ to be the unique lift of f such that $0 \leq F(0) < 1$. By Remark 3.3, $F(1) = F(0)+1 > 1$. Since F is continuous on $[0,1]$ and by Remark 3.3 we know that $F(1) = F(0) + 1 > 1$. The intermediate value theorem therefore tells us that there must be some point $\xi_{-1} \in [0,1]$ such that $F(\xi_{-1}) = 1$. Then, we define $I_0 := [0, \xi_{-1})$, which by monotonicity of F is the half closed, half open interval consisting of all $x \in [0,1]$ such that $F(x) < 1$. Similarly, let $I_1 = [\xi_{-1}, 1]$, which by monotonicity of F is the closed interval containing all points $x \in [0,1]$ such that $F(x) \geq 1$.

We define the discontinuous lift map $\mathcal{F} : [0,1] \to [0,1)$ as

$$\mathcal{F}(x) = F(x) - s_j(x) \text{ where } \begin{cases} s_j(x) = 0 \text{ if } x \in I_0 \\ s_j(x) = 1 \text{ if } x \in I_1. \end{cases}$$

It follows that $\mathcal{F}$ is just the normal lift map for $x \in I_0$, but for $x \in I_1$, $\mathcal{F}$ shifts $F(x)$ from $[1,2)$ to $[0,1)$.

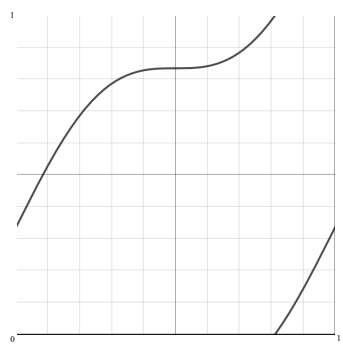


FIGURE 4. $\mathcal{F}$ for $F(x) = x + \frac{1}{3} + \frac{1}{2\pi}\sin(2\pi x)$.

Definition 3.26. Let $\xi_{-1} \mapsto 0 \mapsto \xi_1 \mapsto \xi_2 \mapsto \cdots$ be the orbit of ξ_{-1} under $\mathcal{F}$. We define the *itinerary* of ξ_{-1} to be the sequence $S(F) = s_1 s_2 s_3 \cdots$, where $s_k = 0$ if $\xi_k \in I_0$, and $s_j = 1$ if $\xi_k \in I_1$. Notice that the itinerary of ξ_{-1} begins with $s_1(\xi_1)$, because for any map $s_{-1} = 1$ and $s_0 = 0$, by design.

Proposition 3.27. *If f is an orientation-preserving homeomorphism, and F is the unique lift of f such that $0 \leq F(0) < 1$, then*

$$\tau(F) = \lim_{m\to\infty} \frac{s_1 + \cdots + s_m}{m}.$$

Proof. First, we will prove by induction that $F^m(0) = \sum_{i=1}^{m} s_m + \xi_m$. Our induction hypothesis holds for $m = 1$ as $F(0) = s_1 + \xi_1 = 0 + \xi_1 = F(0)$ (since we chose F with $F(0) \in [0, 1)$). Now we prove the induction step. Suppose that $F^m(0) = s_1 + \cdots + s_m + \xi_m$ for some $m \in \mathbb{N}$. Then,

$$F^{m+1}(0) = F(s_1+\cdots+s_m+\xi_m) = s_1+\cdots+s_m+F(\xi_m) = s_1+\cdots s_m+s_{m+1}+\xi_{m+1}.$$

Now, it follows that $s_1 + \cdots + s_m \leq F^m(0) < s_1 + \cdots + s_m + 1$. Then,

$$\lim_{m\to\infty} \frac{s_1 + \cdots + s_m}{m} \leq \tau(F) \leq \lim_{m\to\infty} \frac{s_1 + \cdots + s_m + 1}{m}. \tag{3.28}$$

Since $\lim_{m\to\infty} \frac{s_1+\cdots+s_m}{m} = \lim_{m\to\infty} \frac{s_1+\cdots+s_m+1}{m}$, the squeeze theorem implies $\tau(f) = \lim_{m\to\infty} \frac{s_1+\cdots+s_m}{m}$. □

We now have the machinery to introduce an ordering of the orientation-preserving homeomorphisms!

Definition 3.29. Let F and G be lifts of orientation preserving homeomorphisms f and g of the circle such that $0 \leq F(0), G(0) < 1$. We say $S(F) < S(G)$ if there exists $m \in \mathbb{N}$ such that $s_k(F) = s_k(G)$ for $k < m$, and $s_m(F) < s_m(G)$.

We now have an ordering of circle homeomorphisms! We will show that this ordering carries over for rotation numbers of circle homeomorphisms.

Proposition 3.30. *Let f and g be orientation preserving homeomorphisms of the circle. Let F and G be lifts of f and g, respectively, such that $0 \leq F(0), G(0) < 1$. If $S(F) < S(G)$, then $0 \leq \tau(F) \leq \tau(G) < 1$. It follows that $\tau(f) \leq \tau(g)$. The inequalities become strict when $\tau(F)$ or $\tau(G)$ is irrational.*

Proof. Assume that $S(F) < S(G)$. It follows, by definition, that $s_1(F) + \cdots + s_m(F) + 1 = s_1(G) + \cdots + s_m(G)$, since $s_m(F) + 1 = 0 + 1 = s_m(G)$. Then, directly from (3.28), we get,

$$\tau(F) \leq \frac{s_1(G) + \cdots + s_m(G)}{m} \leq \tau(G).$$

Suppose without loss of generality that $\tau(F)$ is irrational. Then the left inequality is strict because $\frac{s_1(G)+\cdots+s_m(G)}{m} \in \mathbb{Q}$.

□

Proposition 3.31. *Suppose $f : S^1 \to S^1$ is an orientation preserving homeomorphism with rational rotation number $\tau(f) = \frac{p}{q} \in \mathbb{Q}$. Then there exists a neighborhood I_f around f such that all orientation-preserving circle homeomorphisms $f_i \in I_f$ have rotation number $\tau(f_i) = \frac{p}{q}$.*

Proof. See Proposition 11.1.10 in [3]. □

We now give an example to illustrate the results of this section.

Example 3.32. Let's return to the standard family $f_{\omega,\epsilon}(\theta) = \theta + 2\pi\omega + \epsilon\sin(\theta)$, where $\omega, \epsilon \in \mathbb{R}$. Consider the lift $F_{\omega,\epsilon}(x) = x + \omega + \frac{\epsilon}{2\pi}\sin(2\pi x)$. Fix $\epsilon \in (0, 1]$ so that f is a homeomorphism. We can order the lift, as before, by choice of ω; notice that if $\omega_1 > \omega_2$, then $F_{\omega_1}(x) > F_{\omega_2}(x)$ for all $x \in \mathbb{R}$. Therefore it follows that $F^n_{\omega_1} > F^n_{\omega_2}$, so $\tau(f_{\omega_1}) \geq \tau(f_{\omega_2})$. We have already shown that at irrational values of rotation number this inequality must be strict. Now, suppose that $\tau(f_{\omega_0}) = \frac{p}{q} \in \mathbb{Q}$. For this family, we can show directly that there exists a neighborhood η about ω_0 such that $\tau(f_\omega) = \frac{p}{q}$ for $\omega \in \eta$.

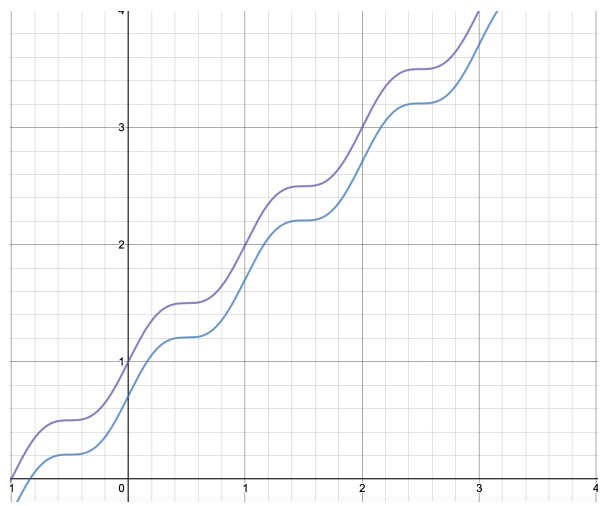


FIGURE 5. Lift of $f_{\omega,1}(\theta) = \theta + 2\pi\omega + \sin(\theta)$, where $\omega_1 = \frac{\sqrt{2}}{2}$ in blue and $\omega_2 = 1$ in purple.

We know that there exists $\theta \in S^1$ such that $f^q(\theta) = \theta$. So, choose $x_0 \in [0, 1)$ such that $\pi(x_0) = \theta$. Now, we break the proof up into cases.

First, suppose that $(F^q_{\omega_0})'(x_0) \neq 1$. Then, the graph of F^q intersects the line $y = x+p$ at (x_0, x_0+p). We can apply the implicit function theorem to the function $G(x, \omega) = (x + p) - (F^q_\omega(x_0))$, since $G(x_0, \omega_0) = 0$ and $\frac{\partial G}{\partial \omega} \neq 0$. This immediately tells us that there exists a neighborhood η of ω_0 such that $F^q_\omega(x_0) = x_0 + p$ for all $\omega \in \eta$. So, the maps f_ω with $\omega \in \eta$ have rotation number $\tau(f_\omega)) = \frac{p}{q}$.

For our second case, suppose that $(F^q_{\omega_0})'(x_0) = 1$. Notice that $F^q_{\omega_0}$ is analytic (can be represented by a convergent power series). Thus, there exists $j \in \mathbb{N}$ such that $(F^q_{\omega_0})^{(n)} \neq 0$; otherwise, $F^q_{\omega_0}$ would be identically equal to $x + p$. If j is odd, then $F^q_{\omega_0}(x_0)$ cannot be a maximum or minimum since $(F^q_{\omega_0})^{(j)}(x_0) \neq 0$. Therefore, sufficiently nearby perturbations must also cross the line $y = x+p$. If j is even, then $F^q_{\omega_0}$ is either concave up or concave down at x_0 (not an inflection point). Hence, the one sided neighborhoods, where $\omega < \omega_0$ and $\omega > \omega_0$ respectively, must cross $y = x + p$. So, they must also have rotation number $\tau(f_\omega)) = \frac{p}{q}$. For more details, see Example 14.10 in [2].

So, we have shown that for this family of functions, rotation number depends monotonically and continuously on ω. At rational values of $\tau(f)$, nearby circle homeomorphisms have the same rotation number. Whereas at irrational values of $\tau(f)$, there are no constant intervals of $\tau(f)$. Hence, the graph of $\tau(f_\omega)$ is a

Cantor function (see Proposition 11.1.11 in [3]). Figure 6 is an example of a Cantor function.

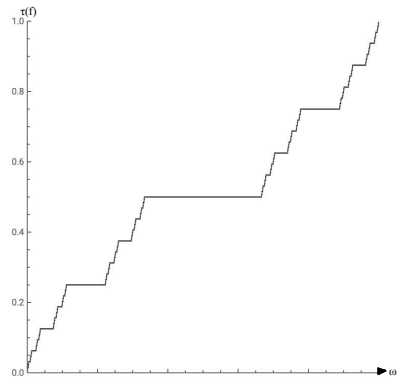


FIGURE 6. Devil's staircase of rotation number $\tau(f_{\omega,\epsilon})$ for $f_{\omega,\epsilon}(\theta) = \theta + 2\pi\omega + \epsilon\sin(\theta)$.

4. Homeomorphisms of the Circle — The Poincaré Classification

In this section we answer the big question: when does a circle homeomorphism behave like a rotation? First, we will look at orientation-preserving circle homeomorphisms $f : S^1 \to S^1$ with rational rotation numbers $\tau(f) = \frac{p}{q} \in \mathbb{Q}$. We will show that the orbits of periodic points of f have the same order (see Proposition 4.2) in the unit interval as the orbit of 0 under $\mathscr{R}_{\frac{p}{q}}$. We also see that circle homeomorphisms with rational rotation number are very rarely even semi-conjugate to rational rotation $\mathcal{R}_{\frac{p}{q}}$. Since all orbits of $\mathcal{R}_{\frac{p}{q}}$ are periodic and have the same period, q, any sort of conjugacy will result in a map with only periodic points. However, most orientation-preserving circle homeomorphisms that are not rotation do not have this sort of orbit structure. Due to the difference in behavior of irrational rotation of the circle (described in Proposition 3.7), orientation-preserving circle homeomorphisms with irrational rotation number **can** be conjugate to an irrational rotation, which is the main result of this paper.

Theorem 4.1 (Poincaré Classification Theorem)**.** *Let $f : S^1 \to S^1$ be an orientation-preserving homeomorphism with irrational rotation number $\tau(f) \in \mathbb{R}\backslash\mathbb{Q}$.*

(1) If f is topologically transitive in S^1 then f is conjugate to rotation by $\tau(f)$, $\mathcal{R}_{\tau(f)}$.
(2) If f is not topologically transitive in S^1 then f is semi-conjugate to rotation by $\tau(f)$, $\mathcal{R}_{\tau(f)}$.

4.1. Rational Rotation Number.

We begin this section by discussing the order of periodic orbits of orientation-preserving circle homeomorphisms with rational rotation number.

Proposition 4.2. *Let $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be an orientation-preserving circle homeomorphism with rational rotation number $\tau(f) = \frac{p}{q} \in \mathbb{Q}$, with p, q relatively prime. We know that there exists $\bar{x} \in S^1$ such that $f^q(\bar{x}) = \bar{x}$. Let $m, n \in \mathbb{N} \cup 0$ be such that $0 \leq m, n \leq q-1$. The orbit of $\bar{x}$ under f has the same* order *in the unit interval as*

the orbit of 0 *under* $\mathscr{R}_{\frac{p}{q}}$. *By order, we mean that if* $m\frac{p}{q} \bmod 1 \leq n\frac{p}{q} \bmod 1$, *then* $f^m(\bar{x}) \leq f^n(\bar{x})$.

Proof. For a complete proof refer to Proposition 11.2.1 in [3]. □

Definition 4.3. Let (X, d) be a metric space, and let $f : X \to X$ be a homeomorphism. A point $x \in X$ is said to be *homoclinic* to a point $y \in X$ if

$$\lim_{|n|\to\infty} d(f^n(x), f^n(y)) = 0.$$

A point $x \in X$ is said to be *heteroclinic* to points y_1 and $y_2 \in X$ if

$$\lim_{n\to\infty} d(f^n(x), f^n(y_1)) = 0 \text{ and } \lim_{n\to\infty} d(f^n(x), f^n(y_2)) = 0.$$

See Figure 7.

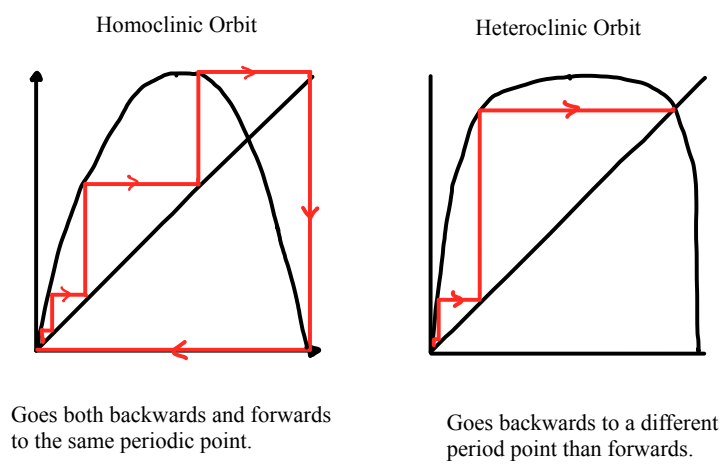


FIGURE 7. Example of Homoclinic and Heteroclinic Orbits.

Definition 4.4. Let $f : X \to X$ be a map. Suppose that $p \in X$ is a periodic point of f with period n. A point $x \in X$ is *forward asymptotic* to p if $\lim_{m\to\infty} f^{mn}(x) = p$. Similarly, if f is invertible, a point $y \in X$ is *backwards asymptotic* to p if $\lim_{m\to-\infty} f^{mn}(y) = p$.

Definition 4.5. Let $f : X \to X$ be a map. A point $y \in X$ is said to be

(1) an ω-*limit point* for $x \in X$ if there exists a sequence $\{n_k\}_{k=0}^{\infty}$ going to $+\infty$ such that $f^{n_k}(x) \to y$ as $k \to \infty$.
(2) an α-*limit point* for $x \in X$ if there exists a sequence $\{n_k\}_{k=0}^{\infty}$ going to $-\infty$ such that $f^{n_k}(x) \to y$ as $k \to -\infty$.

Lemma 4.6. *Let* $J = [a, b] \subset \mathbb{R}$ *be a closed interval. Suppose that* $f : J \to J$ *is an orientation-preserving homeomorphism. Then, all* $x \in J$ *either are fixed points or are positively and negatively asymptotic to adjacent fixed points. That is, for* $x_0 \in J$ *not fixed,* x_0 *is negatively asymptotic to the maximum of* $Fix(f) \cap [a, x_0]$, *and positively asymptotic to the minimum of* $Fix(f) \cap [x_0, b]$ *(note that we can take a maximum and a minimum as the set of fixed points for* f *cannot be dense in* J*).*

Proof. Since f is orientation-preserving, we know that it is monotone increasing. Fix $x \in J$. Without loss of generality, suppose $f(x) > x$ (the case where $f(x) < x$ is similar). Then, $f^2(x) = f(f(x)) > f(x)$, and inductively, $f^n(x) = f(f^{n-1}(x)) > f^{n-1}(x)$. So, $\{f^n(x)\}_{n\in\mathbb{N}}$ is monotone increasing and bounded above by b. Therefore, the sequence $\{f^n(x)\}$ converges to some $y = \sup_{n\in\mathbb{N}}\{f^n(x)\} \in J$. Furthermore, by continuity of f, we get that

$$f(y) = f(\lim_{n\to\infty} f^n(x)) = \lim_{n\to\infty} f^{n+1}(x) = y.$$

So, $x \in J$ is forward asymptotic to fixed point $y \in J$, and the $\omega-$limit set of $\{f^n(x)\}$ is y. Since f is a bijection, we can apply the same argument to f^{-1} to show that x is backwards asymptotic to some fixed $z \in J$. □

When we apply this lemma to the circle, it shows that for an orientation-preserving circle homeomorphism with rational rotation number, non-periodic orbits are asymptotic to periodic orbits. The following proposition adapted from Proposition 11.2.2 in [3] fully explains the details.

Proposition 4.7. *Let $f : S^1 \to S^1$ be an orientation-preserving homeomorphism with rational rotation number $\frac{p}{q} \in \mathbb{Q}$. Then, there are two possible outcomes for the non-periodic orbits of f.*

(1) Suppose that f has exactly one periodic orbit. Then every point not in the periodic orbit is heteroclinic under f^q to two adjacent points on the periodic orbit. These points are distinct if $\tau(f) \neq 0$, i.e periodic points do not have period one.

(2) Suppose f has more than one periodic orbit. Then, each non-periodic point is heteroclinic under f^q to two points on separate periodic orbits.

Proof. Let $\theta \in S^1$ be fixed by f^q, i.e. $f^q(\theta) = \theta$. Let F be a lift of f. Then, fix some $x \in \mathbb{R}$ such that $\pi(x) = \theta$. We know that $F^q(x) - p = x$ and $F^q(x+1) - p = x+1$. If we restrict F^q to the closed interval $[x, x+1]$, then (1) follows immediately from Lemma 4.6.

Now, suppose that f has more than one periodic orbit. Lemma 4.6 tells us that any non-periodic $y \in S^1$ is asymptotic to adjacent periodic points in S^1. Hence, all we need to show is that these two periodic orbits are always distinct. Suppose not, i.e that there exists some interval $I = [a, b] \subset \mathbb{R}$ such that a and b are adjacent zeros of $F^q(x) - x - p$, and a, b are on the same periodic orbit. Therefore, we have that $\pi(a) = x \in S^1$ and $\pi(b) = y \in S^1$, so $f^k(x) = y$ for $0 < l \leq q-1$. By assumption, there are no periodic points in the interval $\pi(a, b) \subset S^1$. This means that $\bigcup_{n=0}^{q-1} f^{nk}\pi(a, b)$ contains no periodic points. We can see that $\bigcup_{n=0}^{q-1} f^{nk}\pi(a, b)$ covers $S^1\backslash\{f^n\pi(a)\}$, which is the whole circle minus the periodic orbit of a, b. But, this means there is only one periodic orbit, which is a contradiction! Hence, we cannot have two adjacent zeros of $F^q(x)-x-p$ from the same orbit. So, Lemma 4.6 proves (2). □

4.2. **Irrational Rotation Number.** We will now build up machinery to prove the Poincaré Classification Theorem.

Definition 4.8. Let (X, d) be a metric space. A set $E \subset X$ is a *Cantor set* if it is totally disconnected, perfect subset of X. In other words, E contains no intervals, is closed, and every point of E is a limit point of E.

Note that our presentation of the Poincaré classification follows that of [3] from Proposition 11.2.4 onwards.

Lemma 4.9. *Let $f : S^1 \to S^1$ be an orientation preserving homeomorphism with irrational rotation number $\tau \in \mathbb{R}\backslash\mathbb{Q}$. Let m and n be integers such that $m < n$. For any point $\theta \in S^1$, define an interval $I = [f^m(\theta), f^n(\theta)]$. Note that I is non-trivial since f cannot have periodic points and it does not matter which of the two possible arcs for I is chosen. Then, for any $x \in S^1$, the positive and negative orbits of x both intersect I, i.e $\{f^n(x)\}_{n=0}^{\infty} \cap I \neq \emptyset$ and $\{f^n(x)\}_{n=0}^{-\infty} \cap I \neq \emptyset$.*

Proof. Fix $\theta \in S^1$. Without loss of generality, consider the forward orbit $\{f^n(x)\}_{n=0}^{\infty}$. Our strategy will be to show that the backwards iterates of I cover S^1, so all points eventually end up in I under forward iteration. Define the interval $I_k := f^{-k(n-m)}(I)$. Notice that for all $k \in \mathbb{N}$, I_k and I_{k-1} share an endpoint.

Suppose that S^1 is not covered by backwards iterates of I, i.e that that $S^1 \nsubseteq \cup_{k\in\mathbb{N}} I_k$. Then, the sequence of endpoints converge to some point $\xi \in S^1$. So,

$$\begin{aligned}
\xi = \lim_{k\to\infty} f^{-k(m-m)} f^m(\theta) &= \lim_{k\to\infty} f^{-k+1(n-m)} f^m(\theta) \\
&= \lim_{k\to\infty} f^{-k+1(n-m)} f^m(\theta) \\
&= \lim_{k\to\infty} f^{n-m} f^{(-k)(n-m)} f^m(\theta) \\
&= f^{(n-m)} \lim_{k\to\infty} f^{-k(m-m)} f^m(\theta) \\
&= f^{(n-m)}(\xi),
\end{aligned}$$

where the first equality holds because the end points of I_k and I_{k-1} are the same. The next two equalities hold by expanding parenthesis. The final equality holds by definition of ξ. This is a contradiction as ξ would be a periodic point of f, but f has irrational rotation number so it cannot have any periodic points. Hence, the backwards iterates of I cover the whole circle, and we are done. □

Proposition 4.10. *Let $f : S^1 \to S^1$ be an orientation-preserving homeomorphism with irrational rotation number $\tau(f) \in \mathbb{R}\backslash\mathbb{Q}$. Then the ω-limit set $\omega(x)$ is independent of x for any $x \in S^1$. We will denote $\omega := \omega(x)$.*

Proof. We need to show that for all $x, y \in S^1$, $\omega(x) = \omega(y)$. Fix $x, y \in S^1$ and $z \in \omega(x) \subset S^1$. Then, there exists a sequence $\{l_n\} \subset \mathbb{N}$ such that $f^{l_n}(x) \to z$ as $l_n \to \infty$. Define an interval $I = [f^m(x), f^n(x)]$. By Lemma 4.9, there exists $k_m \in \mathbb{N}$ such $f^{k_m}(y) \in I_m = [f^{l_m}, f^{l_{m+1}}]$. Then, we can take the limit $\lim_{m\to\infty} f^{k_m}(y) = \lim_{m\to\infty}[f^{l_m}, f^{l_{m+1}}] = z$. Hence, if $z \in \omega(x)$, then $z \in \omega(y)$. So, $\omega(y) \subset \omega(x)$ and similarly, $\omega(x) \subset \omega(y)$, and we have that $\omega(x) = \omega(y)$. □

Proposition 4.11. *Let $f : S^1 \to S^1$ be an orientation-preserving homeomorphism with irrational rotation number $\tau(f) \in \mathbb{R}\backslash\mathbb{Q}$. The set ω from Proposition 4.10 is either the whole of S^1 or a Cantor set in S^1.*

Proof. The key to this proof is the observation that ω is the only minimal, closed, nonempty, f-invariant proper subset of S^1. We will now show this observation. Since the orbit of any point $x \in S^1$ is well-defined, ω must be nonempty. Because $\omega \subset S^1$ is the ω-limit set of any point $x \in S^1$, every limit point of $x \in S^1$ must be contained in ω, so ω is closed.

Suppose that ω is not the only f-invariant set of this type, i.e. suppose that $A \subset S^1$ is non-empty, closed, and f-invariant. But, then for $x \in A$, $\{f^n(x)\}_{n\in\mathbb{Z}} \subset A$. So, $\omega(x) \subset A$. This means that A must be the union of all its ω-limit sets. Hence, $A = \omega$. Therefore, the only closed invariant subsets of ω are the empty-set and ω itself.

Let $\partial\omega$ denote the boundary of ω in S^1. We know from analysis that $\partial\omega$ must be a closed invariant subset of ω. So, we have two possible cases, either $\partial\omega = \emptyset$ or $\partial\omega = \omega$. If $\partial\omega = \emptyset$, then $\omega = S^1$.

If, on the other hand, $\partial\omega = \omega$, then we want to show that ω is a Cantor set. To do this, we need to show that ω is nowhere dense and perfect. Since $\partial\omega = \omega$, ω is nowhere dense. So, all that remains is to show that ω is perfect. We have already shown that ω is closed. Fix $x \in \omega$. We need to show that x is a limit point of ω. From Proposition 4.10 we know that $\omega = \omega(x)$. So, there exists a sequence $\{l_n\} \subset \mathbb{Z}$, such that $f^{l_n}(x) \to x$ as $n \to \infty$. Since ω is f-invariant and $x \in \omega$, $f^{l_n} \in \omega$ for all $n \in \mathbb{N}$. Hence, x is a limit point of ω. □

Lemma 4.12. *Let $f : S^1 \to S^1$ be an orientation-preserving homeomorphism with irrational rotation number $\tau(f) \in \mathbb{R}\backslash\mathbb{Q}$ (we will denote $\tau(f)$ as τ). Let $F : \mathbb{R} \to \mathbb{R}$ be a lift of f. Let n_1, n_2, m_1, m_2 be (not necessarily distinct) integers. Finally, let x be a point in $\mathbb{R}$. Then, $n_1\tau+m_1 < n_2\tau+m_2$ if and only if $F^{n_1}(x)+m_1 < F^{n_2}(x)+m_2$.*

Proof. This proof follows that of Proposition 11.2.4 in [3]. Fix $n_1, n_2, m_1, m_2 \in \mathbb{Z}$. We will show the if direction first, i.e if $F^{n_1}(x) + m_1 < F^{n_2}(x) + m_2$, then $n_1\tau + m_1 < n_2\tau + m_2$. Our first step is to show that the inequality $F^{n_1}(x) + m_1 < F^{n_2}(x) + m_2$ is independent of choice of $x \in \mathbb{R}$. To this end, define $p : \mathbb{R} \to \mathbb{R}$ as $p(a) = F^{n_1}(a)+m_1-(F^{n_2}(a)+m_2)$. We claim that p never changes sign. Given the properties of the lift, we know that p is continuous. Hence, if it did change sign then there would exist some $z \in \mathbb{R}$ such that $p(z) = 0$, i.e $F^{n_1}(z) - F^{n_2}(z) = m_2 - m_1$. Since m_2 and m_1 are integers, $m_2 - m_1 \in \mathbb{Z}$, which implies that $\pi(z)$ is a periodic point for f. This is a contradiction, since f has irrational rotation number and thus cannot have periodic points. Therefore, $F^{n_1}(x)+m_1 < F^{n_2}(x)+m_2$ is independent of choice of $x \in \mathbb{R}$.

Now, fix $x = 0$ and define $y = F^{n_2}(0)$. Since F is increasing, we have that $F^{n_1-n_2}(y) - y < m_1 - m_2$. We already showed that this inequality is independent of the choice of $y \in \mathbb{R}$, so we can set $y = 0$. It then directly follows that $F^{n_1-n_2}(0) < m_2-m_1$. Replacing 0 with $F^{n_1-n_2}(0)$, we get that $F^{n_1-n_2}(F^{n_1-n_2}(0))-F^{n_1-n_2}(0) < m_2 - m_1$. So, $F^{2(n_1-n_2)}(0) < (m_2 - m_1) + F^{n_1-n_2}(0) < 2(m_2 - m_1)$. Inductively, it follows that $F^{n(n_1-n_2)} < n(m_2 - m_1)$. Hence,

$$\tau(f) = \lim_{n\to\infty} \frac{F^{n(n_1-n_2)}(0) - 0}{n(n_1 - n_2)} < \lim_{n\to\infty} \frac{n(m_1 - m_2)}{n(n_1 - n_2)} = \frac{m_2 - m_1}{n_1 - n_2},$$

where the inequality is strict since $\tau(f)$ is irrational. It directly follows that $n_1\tau + m_1 < n_2\tau + m_2$.

To show the only if direction, we first apply the exact same argument to $F^{n_1}(0)+ m_1 > F^{n_2}(0) + m_2$, to imply that $n_1\tau + m_1 > n_2\tau + m_2$. We can never have equality in $F^{n_1}(0) + m_1 > F^{n_2}(0) + m_2$, because f has no periodic points. We can also never have equality in $n_1\tau + m_1 > n_2\tau + m_2$, because τ is irrational. Suppose that the only if direction did not follow, i.e that $n_1\tau + m_1 < n_2\tau + m_2$ implies $F^{n_1}(0)+m_1 > F^{n_2}(0)+m_2$. Immediately, we have that $F^{n_1}(0)+m_1 > F^{n_2}(0)+m_2$

implies $n_1\tau + m_1 > n_2\tau + m_2$, which is a clear contradiction. Therefore, the only if direction must follow. □

Finally, we have reached the point where we can prove the Poincaré classification theorem (Theorem 4.1).

Proof of the Poincaré classification theorem. We are going to construct a continuous monotone semi-conjugacy between any circle homeomorphism with irrational rotation number and its corresponding rotation.

Pick a lift $F : \mathbb{R} \to \mathbb{R}$ of f. We will denote $\tau(f)$ by τ. Fix $\theta \in S^1$. Choose $x \in \mathbb{R}$ such that $\pi(x) = \theta$. Define the set $B = \{F^n(x) + m | m, n \in \mathbb{Z}\}$, which is the complete lift of the orbit $\{f^n(\pi(x))\}_{n=-\infty}^{n=\infty}$. Define the map $R_\tau : \mathbb{R} \to \mathbb{R}$ by $R_\tau(x) = x+\tau$. The notation R_τ is a bit of an abuse of notation, which is deliberate as we want to think of the map as additive rotation on the real line. Our strategy is to construct a monotone continuous map $H : B \to \mathbb{R}$ that is a semi-conjugacy between $F|_B$ and R_τ on $\mathbb{R}$. We will then extend this map to the whole real line to create a semi-conjugacy between F and R_τ. Finally, we will show that H descends to a semi-conjugacy between $f : S^1 \to S^1$ and $\mathcal{R}_\tau$.

Let us define $H : B \to \mathbb{R}$ by $F^n(x) + m \mapsto n\tau + m$. Immediately, Lemma 4.12 tells us that for $n_1, n_2, m_1, m_2 \in \mathbb{Z}$, if $F^{n_1}(x)+m_1 < F^{n_2}(x)+m_2$, then $n_1\tau+m_1 < n_2\tau+m_2$. So, $H(F^{n_1}(x)+m_1) = n_1\tau+m_1 < n_2\tau+m_2 = H(F^{n_2}(x)+m_2)$. Hence, H is monotone.

It follows immediately from Example 3.7 that $H(B)$ is dense in $\mathbb{R}$. So, on $B \subset \mathbb{R}$ we have that $H \circ F = \mathcal{R}_\tau \circ H$, since for any $(F^n(x) + m) \in B$

$$\begin{aligned} H \circ F(F^n(x) + m) &= H \circ (F^{n+1}(x) + m) \\ &= (n+1)\tau + m \\ &= \mathcal{R}_\tau \circ (n\tau + m) \\ &= \mathcal{R}_\tau \circ H(F^n(x) + m). \end{aligned}$$

Now, in order to extend H to $\mathbb{R}$, we will first extend H continuously to the closure of B, denoted $\bar{B}$. Fix $y \in \bar{B}\backslash B$. By definition, there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subset B$ such that $y = \lim_{n\to\infty} x_n$. We want to be able to define $H(y) = \lim_{n\to\infty} H(x_n)$. To do this, we need to show that $\lim_{n\to\infty} H(x_n)$ exists and doesn't depend on the choice of our sequence x_n. Since H is monotone, we know that the liminf and limsup, estimating y from below and above, must exist and be independent of choice of sequence. Now, suppose that these limits did not agree. Then, $\mathbb{R}\backslash H(B)$ would have to contain an interval, as H is continuous on B. But, we have already shown that $H(B)$ is dense in $\mathbb{R}$, so this cannot be the case. We have thus shown that the left and right limits agree, and that we can extend $H : \bar{B} \to \mathbb{R}$ as a continuous monotone map.

Finally, we want to extend H to the whole real line. Notice that $\bar{B}$ projects, under π, to the closure of the orbit of $\pi(x)$, which is exactly $\omega(\pi(x)) = \omega$. So, $\bar{B}$ is the lift of ω. From Proposition 4.11, we know that ω is either the whole of S^1 or a Cantor set in S^1. We will break our argument up into two cases. In the case that f is transitive, $\bar{B} = \mathbb{R}$, so we have already extended H to $\mathbb{R}$. Furthermore, we showed that H must be injective, and thus a conjugacy.

Otherwise, $\bar{B}$ is a Cantor set. We know that $H : \bar{B} \to \mathbb{R}$ is continuous, monotone and surjective, since H is monotone and continuous on $\bar{B}$, and $H(B)$ is dense in $\mathbb{R}$. Hence, to extend H to $\mathbb{R}$ we need to define H on intervals $\mathbb{R}\backslash B$. We can simply set

H as constant and equal to the endpoints on these intervals. This is possible since H is continuous on Cantor set $\bar{B}$, which is just a set of points. Hence, if on the intervals between these points, H is constant, then H must still be continuous on the extension. Note that since H is constant on intervals, it fails to be injective and is only a semi-conjugacy. We have now constructed a semi-conjugacy $H : \mathbb{R} \to \mathbb{R}$ such that $H \circ F = \mathcal{R}_\tau \circ H$ (see Figure 8).

$$\begin{tikzcd} \mathbb{R} \arrow[r, "F"] \arrow[d, "H"'] & \mathbb{R} \arrow[d, "H"] \\ \mathbb{R} \arrow[r, "R_\tau"'] & \mathbb{R} \end{tikzcd}$$

FIGURE 8. Commutative diagram of semi-conjugacy between lift F and $\mathcal{R}_\tau = x + \tau$.

We now want to show that $H : \mathbb{R} \to \mathbb{R}$ descends to a semi-conjugacy $h : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$. For $z \in B$ we have

$$H(z+1) = H(F^n(z+1) + m) = H(F^n(z) + m + 1) = n\tau + m + 1 = H(z) + 1.$$

Therefore, our conjugacy is periodic at integer intervals, and we have shown that $f : S^1 \to S^1$ is semi-conjugate to $\mathcal{R}_\tau(\theta) = \theta + 2\pi\tau$ by the monotone continuous map $h : S^1 \to S^1$ (see Figure 9).

$$\begin{tikzcd} \mathbb{R}/\mathbb{Z} \arrow[r, "f"] \arrow[d, "h"'] & \mathbb{R}/\mathbb{Z} \arrow[d, "h"] \\ \mathbb{R}/\mathbb{Z} \arrow[r, "R_{\tau(f)}"'] & \mathbb{R}/\mathbb{Z} \end{tikzcd}$$

FIGURE 9. Commutative Diagram of the Semi-Conjugacy between f and $\mathcal{R}_{\tau(f)}$.

$\square$

Table 1 is based on the table on page 399 of [3] and summarizes the results we proved in this section. Let $f : S^1 \to S^1$ be an orientation-preserving homeomorphism of the circle. We have proved the following

$\tau(f) = \frac{p}{q} \in \mathbb{Q}$	$\tau(f) \in \mathbb{R}\backslash\mathbb{Q}$
Points in periodic orbit.	**Case one: f is transitive.**
All points in periodic orbit have prime period q. Orbit is ordered in the same way as orbit $\mathcal{R}_{\frac{p}{q}}$.	The map f is topologically transitive, so orbit is dense in S^1 and ordered in the same way as $\mathcal{R}_{\tau(f)}$.
Points outside periodic orbit.	**Case two: f is not transitive.**
Case 1: f has one periodic orbit, so points are heteroclinic under f^q to two points on that one periodic orbit.	The map f is not topologically transitive, so the orbit is dense in a Cantor set.
Case 2: f has more than one periodic orbit, so points are heteroclinic under f^q to two points on separate orbits.	Points outside Cantor set are homoclinic to the Cantor set.

Table 1. Poincaré Classification Table.

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