FOURIER SERIES AND ITS APPLICATIONS TO GEOMETRY AND EQUIDISTRIBUTION THEOREM

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Abstract. This expository paper starts with the definitions of Fourier coefficients and Fourier series on the torus $\mathbf{T}^n$. It then explores the convergence problem of Fourier series on $\mathbf{T}^n$, when, with the help of concepts including the *kernels* and the *approximate identities*, some convergence conditions for the Fourier series of L^1, L^2 functions on $\mathbf{T}^n$ are derived. Afterwards, a generalization of the convergence problem to functions on $\mathbf{R}^n$ is given by the *Poisson Summation Formula*. The second half of this paper applies the Fourier series to prove the *isoperimetric inequality* and *Weyl's theorem on equidistribution*, which are fundamental in Geometry and Number Theory respectively.

Contents

1. Fourier Series of Functions on the Torus $\mathbf{T}^n$

We start with the Fourier series of functions on the torus $\mathbf{T}^n$. These functions are 1-periodic on the unit interval $[-1/2, 1/2]^n$, and so are their Fourier series, which we define in Definition 1.6.

Definition 1.1. With

$$x \equiv y \text{ if } x - y \in \mathbf{Z}^n,$$

a n-torus $\mathbf{T}^n$ is the set $\mathbf{R}^n/\mathbf{Z}^n$ of all equivalence classes.

Example 1.2. $(a, b) \equiv (c, d)$ on $\mathbf{T}^2 \iff (a, b) - (c, d) = (a - c, b - d) \in \mathbf{Z}^2$. For example, $-(1/2, 1/3) \equiv (1/2, 2/3)$ on $\mathbf{T}^2$.

Remarks 1.3. (i) For any integrable function f on $\mathbf{T}^n$,

$$\int_{\mathbf{T}^n} f(x)dx = \int_{[-1/2,1/2]^n} f(x)dx.$$

(ii) By periodicity, which implies the cancellation of boundary terms,

$$\int_{\mathbf{T}^n} \partial_j f(x)g(x)dx = -\int_{\mathbf{T}^n} \partial_j g(x)f(x)dx,$$

where f, g are continuously differentiable functions on $\mathbf{T}^n$.

Notations 1.4. We use the following notations:

(i) For multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in (\mathbf{Z}^+ \cup \{0\})^n$,
$$\partial^\alpha f = \partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n} f.$$
(ii) For $x = (x_1, x_2, \ldots, x_n) \in \mathbf{T}^n$, $|x| = (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2}$ denotes the distance of x from the origin. Thus, for a given $n \in \mathbf{N}$, $|x| \in [0, \sqrt{n}/2]$ for any $x \in \mathbf{T}^n$.

(iii) For $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in \mathbf{R}^n$,
$$x \cdot y = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n.$$

Now, we define *Fourier coefficients* and *Fourier series* for functions in $L^1(\mathbf{T}^n)$:

Definition 1.5. For $f \in L^1(\mathbf{T}^n)$ and $m \in \mathbf{Z}^n$, the mth *Fourier coefficient* of f is
$$\widehat{f}(m) = \int_{\mathbf{T}^n} f(x) e^{-2\pi i m \cdot x} dx.$$
For a finite Borel measure μ on $\mathbf{T}^n$ and $m \in \mathbf{Z}^n$, the mth *Fourier coefficient* of μ is
$$\widehat{\mu}(m) = \int_{\mathbf{T}^n} e^{-2\pi i m \cdot x} d\mu.$$

Definition 1.6. The *Fourier series* of $f \in L^1(\mathbf{T}^n)$ at $x \in \mathbf{T}^n$ is
$$\sum_{m \in \mathbf{Z}^n} \widehat{f}(m) e^{2\pi i m \cdot x}$$
if the sum converges.

Remarks 1.7. (i) $\widehat{f}(\xi)$ is not defined for any $\xi \in \mathbf{R}^n \setminus \mathbf{Z}^n$, given the map $x \mapsto e^{-2\pi i \xi \cdot x}$ is not 1-periodic and thus not well-defined on $\mathbf{T}^n$.

(ii) For any $f, g \in L^1(\mathbf{T}^n)$ and $m, m_1, m_2 \in \mathbf{Z}^n$,
  (a) $\widehat{f+g}(m) = \widehat{f}(m) + \widehat{g}(m)$,
  (b) $\widehat{cf}(m) = c\widehat{f}(m)$ for all $c \in \mathbf{C}$,
  (c) $\widehat{f \otimes g}(m_1, m_2) = \widehat{f}(m_1)\widehat{g}(m_2)$, where $\otimes$ denotes tensor multiplication.

2. Reproduction of Functions on $\mathbf{T}^n$ from Fourier Coefficients

Next, we would like to study the conditions for the Fourier series of a function $f \in L^1(\mathbf{T}^n)$ to converge and to further give back f. Our method expresses the partial sums of the Fourier series as the *convolution* $f * k_R$, where $\{k_R\}_{R \geq 0}$ is a collection of *trigonometric polynomials* named the *kernels*. We will find that certain "good kernels", called the *approximate identities*, imply the convergence of $f * k_R$ to f in L^p norm. The uniqueness of $\{\widehat{f}(m)\}_{m \in \mathbf{Z}^n}$ for each $f \in L^1(\mathbf{T}^n)$, the almost everywhere convergence of the Fourier series to $f \in L^1(\mathbf{T}^n)$ with absolutely convergent Fourier series, and the almost everywhere convergence of the Fourier series to $f \in L^2(\mathbf{T}^n)$ then follow.

For the rest of this section, let G be a locally compact group, such as $\mathbf{T}^n$, and let λ be any left invariant Haar measure.

We first define *convolution* and *trigonometric polynomials*.

Definition 2.1. For $f, g \in L^1(G)$, the *convolution* $f * g$ is defined by

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)dy$$

where $x, y \in G$.

Remarks 2.2. (i) It is trivial to check that the convolution of functions on $L^1(G)$ satisfies associativity, commutativity and distributivity.

(ii) In the case where $G = \mathbf{T}^n$ with the usual additive structure $y^{-1} = -y$,

$$(f * g)(x) = \int_{\mathbf{T}^n} f(y)g(x-y)dy.$$

Definition 2.3. A *trigonometric polynomial* on $\mathbf{T}^n$ is a function $P(x)$ of the form

$$P(x) = \sum_{m \in \mathbf{Z}^n} a_m e^{2\pi i m \cdot x},$$

where $\{a_m\}_{m \in \mathbf{Z}^n}$ is a finitely supported sequence in $\mathbf{Z}^n$.
The *degree* of P is the largest number $|m_1| + \ldots + |m_n|$ such that $a_m \neq 0$, where $m = (m_1, \ldots, m_n)$.

Remarks 2.4. For all $m \in \mathbf{Z}^n$, we have

$$\widehat{P}(m) = \int_{\mathbf{T}^n} \sum_{k \in \mathbf{Z}^n} a_k e^{2\pi i k \cdot x} e^{-2\pi i m \cdot x} dx = a_m.$$

Now, we define the *Dirichlet kernel* as a trigonometric polynomial on $\mathbf{T}^n$.

Definition 2.5. With $0 \leq R < \infty$, the *square Dirichlet kernel* on $\mathbf{T}^n$ is

$$D_R^n(x) = \sum_{\substack{m \in \mathbf{Z}^n \\ |m_j| \leq R}} e^{2\pi i m \cdot x}.$$

Remarks 2.6. For $N \in \mathbf{N}$ and $x \in \mathbf{T}^1$, we have

$$D_N^1(x) = \sum_{m=-N}^{N} e^{2\pi i m \cdot x} = \frac{\sin\left((2N+1)\pi x\right)}{\sin(\pi x)}, \tag{2.7}$$

since

$$\sum_{m=-N}^{N} e^{2\pi i m x} = e^{-2\pi i N x} \sum_{m=0}^{2N} e^{2\pi i m x} = e^{-2\pi i N x} \frac{\left(e^{2\pi i x}\right)^{2N+1} - 1}{e^{2\pi i x} - 1}$$

which further gives

$$\sum_{m=-N}^{N} e^{2\pi i m x} = \frac{e^{\pi i (2N+1)x} - e^{-\pi i (2N+1)x}}{e^{\pi i x} - e^{-\pi i x}} = \frac{\sin((2N+1)\pi x)}{\sin(\pi x)}.$$

We then show that the partial sums of Fourier series can be expressed by the convolution $f * D_R^n$.

Proposition 2.8. *With $R \geq 0$, we may define*

$$S_R(f)(x) = (f * D_R^n)(x) = \sum_{\substack{m \in \mathbf{Z}^n \\ |m_j| \leq R}} \widehat{f}(m) e^{2\pi i m \cdot x}, \tag{2.9}$$

which is called the square partial sums of the Fourier series of f. S_R is an operator on f that maps $L^p(\mathbf{T}^n)$ onto $L^p(\mathbf{T}^n)$.

Proof. For (2.9),

$$\begin{aligned}(f * D_R^n)(x) &= \int_{\mathbf{T}^n} f(y) D_R^n(x-y)dy \\ &= \int_{\mathbf{T}_n} f(y)\left(\sum_{\substack{m\in\mathbf{Z}^n\\ |m_j|\le R}} e^{2\pi i m\cdot(x-y)}\right)dy \\ &= \sum_{\substack{m\in\mathbf{Z}^n\\ |m_j|\le R}} \left(\int_{\mathbf{T}_n} f(y)e^{-2\pi i m\cdot y}dy\right)\cdot e^{2\pi i m\cdot x} \\ &= \sum_{\substack{m\in\mathbf{Z}^n\\ |m_j|\le R}} \widehat{f}(m)e^{2\pi i m\cdot x}.\end{aligned}$$

□

In what sense, and under what conditions, does the partial sum of the Fourier series converge back to give f? To explore this fundamental question in Fourier analysis, we need the concept of *approximate identity.*

Definition 2.10. An *approximate identity* (as $R \to \infty$) is a family of $L^1(G)$ functions $\{k_R\}_{R\ge 0}$ satisfying the following properties:

(i) There exists some constant $c > 0$ such that $\|k_R\|_{L^1(G)} \le c$ for all $R \ge 0$.
(ii) $\int_G k_R(x)d\lambda(x) = 1$ for all $R \ge 0$.
(iii) For any neighbourhood V of the identity element e of the group G,

$$\int_{V^C} |k_R(x)|d\lambda(x) \to 0 \text{ as } R \to \infty,$$

where V^C denotes the complement of V.

Theorem 2.11. *For an approximate identity $\{k_R\}_{R\ge 0}$ on a locally compact group G with left Haar measure λ, if $f \in L^p(G)$ for $1 \le p < \infty$, then $\|k_R * f - f\|_{L^p(G)} \to 0$ as $R \to \infty$.*

Proof. Let $1 \le p < \infty$. Consider any $f \in L^p(G)$. We may approximate f by some $g \in \mathscr{C}_0^\infty(G)$ supported in a compact set K, given $\mathscr{C}_0^\infty(G)$ is dense in $L^p(G)$. Since $|g(h^{-1}x) - g(x)|^p \le (2\|g\|_{L^\infty})^p \chi_{W^{-1}K}$ where $h \in W$ with W being a compact neighbourhood of e, by Dominated Convergence Theorem,

$$\begin{aligned}&\lim_{h\to e}\int_G |g(h^{-1}x) - g(x)|^p d\lambda(x) = 0 \\ \Longrightarrow &\lim_{h\to e}\int_G |f(h^{-1}x) - f(x)|^p d\lambda(x) = 0.\end{aligned} \tag{2.12}$$

Fix any $\epsilon > 0$. Then (2.12) implies that there exists some neighbourhood V of e such that for all $h \in V$,

$$\left\|f(h^{-1}x) - f(x)\right\|^p_{L^p(G,d\lambda(x))} = \int_G |f(h^{-1}x) - f(x)|^p d\lambda(x) < \left(\frac{\epsilon}{2c}\right)^p, \tag{2.13}$$

where c is the constant in Definition 2.10 (i).
Meanwhile, by Definition 2.10 (iii), there exists some $R_0 > 0$ such that for all

$R \geq R_0$,

$$\int_{V^C} |k_R(x)| d\lambda(x) < \frac{\epsilon}{4(\|f\|_{L^p} + 1)}. \tag{2.14}$$

Then, by Definition 2.10 (ii),

$$\begin{aligned}(k_R * f)(x) - f(x) &= \int_G f(y^{-1}x) k_R(y) d\lambda(y) - f(x) \int_G k_R(y) d\lambda(y) \\ &= \int_G \left(f(y^{-1}x) - f(x)\right) k_R(y) d\lambda(y) \\ &= \int_V \left(f(y^{-1}x) - f(x)\right) k_R(y) d\lambda(y) \\ &+ \int_{V^C} \left(f(y^{-1}x) - f(x)\right) k_R(y) d\lambda(y),\end{aligned}$$

which combines with

$$\begin{aligned}&\left\| \int_V \left(f(y^{-1}x) - f(x)\right) k_R(y) d\lambda(y) \right\|_{L^p(G, d\lambda(x))} \\ \leq &\int_V \left\| f(y^{-1}x) - f(x) \right\|_{L^p(G, d\lambda(x))} |k_R(y)| d\lambda(y) \\ \leq &\int_V \frac{\epsilon}{2c} |k_R(y)| d\lambda(y) \text{ (by (2.13))} \\ \leq &\frac{\epsilon}{2c} \cdot c = \frac{\epsilon}{2} \text{ (by Definition 2.10 (i))}\end{aligned}$$

and

$$\begin{aligned}&\left\| \int_{V^C} \left(f(y^{-1}x) - f(x)\right) k_R(y) d\lambda(y) \right\|_{L^p(G, d\lambda(x))} \\ \leq &\int_{V^C} 2 \|f\|_{L^p(G)} |k_R(y)| d\lambda(y) \\ \leq &2 \|f\|_{L^p(G)} \cdot \frac{\epsilon}{4(\|f\|_{L^p(G)} + 1)} = \frac{\|f\|_{L^p(G)}}{\|f\|_{L^p(G)} + 1} \cdot \frac{\epsilon}{2} < \frac{\epsilon}{2} \text{ (by (2.14))}\end{aligned}$$

to give

$$\|(k_R * f)(x) - f(x)\|_{L^p(G, d\lambda(x))} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

proving the theorem. □

Unfortunately, we cannot conclude the convergence of Fourier series to f for all $f \in L^1(\mathbf{T}^n)$ from Theorem 2.11, as $\{D_R^n\}_{R \geq 0}$ is not an approximate identity on $\mathbf{T}^n$. This is because $\|D_R^n\|_{L^1} \approx (\log R)^n \to \infty$ as $R \to \infty$, which implies that $\{D_R^n\}_{R \geq 0}$ fails Definition 2.10 (i).

However, we may consider another kernel defined by the arithmetic means of Dirichlet kernels, which turns out to be an approximate identity.

Definition 2.15. Consider the arithmetic means of Dirichlet kernels in dimension 1,

$$F_N(x) = \frac{1}{N+1} \sum_{k=0}^{N} D_k(x) \tag{2.16}$$

with $0 \leq N < \infty$. The *Fejér kernel* on $\mathbf{T}^n$ is

$$F_N^n(x_1, \ldots, x_n) = \prod_{j=1}^{n} F_N(x_j). \tag{2.17}$$

In particular, the *Fejér kernel* on $\mathbf{T}^1$ is just F_N.

Remarks 2.18. (i) Given

$$\frac{1}{N+1}\sum_{k=0}^{N} D_k(x) = \frac{1}{N+1}\sum_{k=0}^{N}\sum_{|j|\leq k} e^{2\pi i j x} = \sum_{|j|\leq N} \frac{N-|j|+1}{N+1} e^{2\pi i j x},$$

we have

$$F_N(x) = \sum_{j=-N}^{N}\left(1 - \frac{|j|}{N+1}\right) e^{2\pi i j x}, \tag{2.19}$$

and further, by orthonormality of $e^{2\pi i m_k x_k}$,

$$F_N^n(x_1, \ldots, x_n) = \sum_{\substack{m\in\mathbf{Z}^n \\ |m_j|\leq N}} \left(\prod_{k=1}^{n}\left(1 - \frac{|m_k|}{N+1}\right)\right) e^{2\pi i m\cdot x}. \tag{2.20}$$

(ii) By (2.7) and (2.16),

$$\begin{aligned}
F_N(x) &= \frac{1}{(N+1)\sin(\pi x)}\sum_{k=0}^{N}\sin((2k+1)\pi x) \\
&= \frac{1}{2(N+1)\sin^2(\pi x)}\sum_{k=0}^{N} 2\sin((2k+1)\pi x)\sin(\pi x) \\
&= \frac{1}{2(N+1)\sin^2(\pi x)}\sum_{k=0}^{N}\left[\cos(2k\pi x) - \cos\left((2k+2)\pi x\right)\right] \\
&= \frac{1-\cos(2(N+1)\pi x)}{2(N+1)\sin^2(\pi x)} = \frac{2\sin^2((N+1)\pi x)}{2(N+1)\sin^2(\pi x)} \\
&= \frac{1}{N+1}\left(\frac{\sin((N+1)\pi x)}{\sin(\pi x)}\right)^2.
\end{aligned} \tag{2.21}$$

By (2.17),

$$F_N^n(x_1, \ldots, x_n) = \frac{1}{(N+1)^n}\prod_{j=1}^{n}\left(\frac{\sin((N+1)\pi x_j)}{\sin(\pi x_j)}\right)^2. \tag{2.22}$$

Definition 2.23. The *square Cesàro means* (or *square Fejér means*) of f is

$$(F_N^n * f)(x) = \sum_{\substack{m\in\mathbf{Z}^n \\ |m_j|\leq N}} \left(1 - \frac{|m_1|}{N+1}\right)\cdots\left(1 - \frac{|m_n|}{N+1}\right)\widehat{f}(m)e^{2\pi i m\cdot x}.$$

Proposition 2.24. *$\{F_N^n\}_{N=0}^{\infty}$ is an approximate identity on $\mathbf{T}^n$.*

Proof. We first prove the proposition for $\{F_N\}_{N=0}^{\infty}$. We know from (2.21) that $F_N \geq 0$ for all $N \geq 0$, and thus $\|F_N\|_{L^1(\mathbf{T}^1)} = \int_{\mathbf{T}^1} F_N(x)dx = (1-0)^n \cdot e^0 = 1$,

which satisfies (i) and (ii) in Definition 2.10.
Moreover, given $1 \leq \frac{|t|}{|\sin t|} \leq \frac{\pi}{2}$ for $|t| \leq \frac{\pi}{2}$, if $|x| \leq \frac{1}{2}$, then

$$F_N(x) \leq \frac{1}{N+1} \min\left(\frac{(N+1)|\pi x|}{|\sin(\pi x)|}, \frac{1}{|\sin(\pi x)|}\right)^2 \leq \frac{1}{N+1}\frac{\pi^2}{4} \min\left(N+1, \frac{1}{|\pi x|}\right)^2.$$

Hence, for any neighbourhood around the origin $[-\delta, \delta]$, we have
(2.25)

$$\int_{\mathbf{T}^1 \setminus [-\delta,\delta]} F_N(x)dx \leq \frac{1}{N+1}\frac{\pi^2}{4} \int_{\delta \leq |x| \leq \frac{1}{2}} \frac{dx}{|\pi\delta|^2} \leq \frac{1}{4\delta^2(N+1)} \to 0 \text{ as } N \to \infty,$$

which proves (iii).

By (2.17) from Definition 2.15, $\{F_N^n\}_{N=0}^\infty$ clearly satisfies Definition 2.10 (i)(ii). Meanwhile, it follows from (2.25) that

$$\int_{\substack{x \in \mathbf{T}^n \\ |x| \geq \delta}} F_N^n(x)dx \leq \sum_{j=1}^n \int_{|x_j| \geq \frac{\delta}{\sqrt{n}}} F_N(x_j)dx_j \prod_{k \neq j} \int_{\mathbf{T}^1} F_N(x_k)dx_k \leq \frac{n}{4(\delta/\sqrt{n})^2(N+1)} \to 0$$

as $N \to \infty$, which proves (iii) for $\{F_N^n\}_{N=0}^\infty$. □

The conclusions above allow us to derive several statements about the convergence of the Fourier series of functions in $L^1(\mathbf{T}^n)$ and $L^2(\mathbf{T}^n)$.

Lemma 2.26. *If $f, g \in L^1(\mathbf{T}^n)$ satisfy $\widehat{f}(m) = \widehat{g}(m)$ for all $m \in \mathbf{Z}^n$, then $f = g$ almost everywhere.*

Proof. Consider $h = f - g \in L^1(\mathbf{T}^n)$. Remarks 1.7 (ii) implies $\widehat{h}(m) = \widehat{f}(m) - \widehat{g}(m) = 0$ for all $m \in \mathbf{Z}^n$. Then by Definition 2.23, $F_N^n * h = 0$ for all $N \in \mathbf{Z}^n$. And given Proposition 2.24, we have

$$\|h\|_{L^1} = \|F_N^n * h - h\|_{L^1} \to 0 \text{ as } N \to \infty.$$

Thus, $\|h\|_{L^1} = 0$, which means $h = f - g = 0$ almost everywhere. □

Proposition 2.27 (Fourier inversion)**.** *Suppose that $f \in L^1(\mathbf{T}^n)$ and that*

$$\sum_{m \in \mathbf{Z}^n} |\widehat{f}(m)| < \infty. \tag{2.28}$$

Then

$$f(x) = \sum_{m \in \mathbf{Z}^n} \widehat{f}(m) e^{2\pi i m \cdot x} \text{ almost everywhere,}$$

an everywhere continuous function.

Proof. (2.28) implies $\sum_{m \in \mathbf{Z}^n} |\widehat{f}(m) e^{2\pi i m \cdot x}| < \infty$; hence, $g(x) = \sum_{m \in \mathbf{Z}^n} \widehat{f}(m) e^{2\pi i m \cdot x}$ is well-defined. And given $\widehat{f}(m) = \widehat{g}(m)$ for all $m \in \mathbf{Z}^n$, Lemma 2.26 gives $f = g$ almost everywhere. □

Next, we prove a few important results for functions $f, g \in L^2(\mathbf{T}^n)$.

Theorem 2.29 (Plancherel's identity)**.** *For any $f \in L^2(\mathbf{T}^n)$,*

$$\|f\|_{L^2}^2 = \sum_{m \in \mathbf{Z}^n} |\widehat{f}(m)|^2.$$

Theorem 2.30. *For any $f \in L^2(\mathbf{T}^n)$,*

$$f(t) = \lim_{M\to\infty} \sum_{|m|\leq M} \widehat{f}(m)e^{2\pi i m\cdot t} \text{ almost everywhere.}$$

Proofs for Theorem 2.29 and Theorem 2.30. The two theorems above are consequences of $\{\phi_k\}_{k\in\mathbf{Z}^n}$, with $\phi_k(x) = e^{2\pi i k\cdot x}$ for each $k \in \mathbf{Z}^n$, being a complete orthonormal system in the separable Hilbert space $\mathbf{T}^n$. Indeed, with inner product defined as

$$\langle f|g\rangle = \int_{L^2(\mathbf{T}^n)} f(t)\overline{g(t)}dt,$$

$\{\phi_k\}_{k\in\mathbf{Z}^n}$ forms an orthonormal system since

$$\langle \phi_m|\phi_k\rangle = \int_{L^2(\mathbf{T}^n)} e^{2\pi i(m-k)\cdot t}dt = \begin{cases} 1 & \text{if } m = k \\ 0 & \text{if } m \neq k. \end{cases}$$

Furthermore, since $\langle f|\phi_m\rangle = \widehat{f}(m)$ for all $f \in L^2(\mathbf{T}^n)$ by Definition 1.5, it follows from Lemma 2.26 that if $\langle f|\phi_m\rangle = 0$ for all $m \in \mathbf{Z}^n$, then $f = 0$ almost everywhere. Hence, $\{\phi_k\}_{k\in\mathbf{Z}^n}$ forms a complete orthonormal system in $L^2(\mathbf{T}^n)$, implying Theorem 2.29 and Theorem 2.30. □

Theorem 2.31 (Parseval's relation)**.** *For $f, g \in L^2(\mathbf{T}^n)$,*

$$\int_{\mathbf{T}^n} f(t)\overline{g(t)}dt = \sum_{m\in\mathbf{Z}^n} \widehat{f}(m)\overline{\widehat{g}(m)}.$$

Proof. Applying Theorem 2.29 to $f + g \in L^2(\mathbf{T}^n)$ and expanding both sides of the equation, we have

$$\|f\|_{L^2}^2 + \|g\|_{L^2}^2 + 2Re\langle f|g\rangle = \sum_{m\in\mathbf{Z}^n} |\widehat{f}(m)|^2 + \sum_{m\in\mathbf{Z}^n} |\widehat{g}(m)|^2 + 2Re \sum_{m\in\mathbf{Z}^n} \widehat{f}(m)\overline{\widehat{g}(m)}.$$

Since $\|f\|_{L^2}^2 = \sum_{m\in\mathbf{Z}^n} |\widehat{f}(m)|^2$ and $\|g\|_{L^2}^2 = \sum_{m\in\mathbf{Z}^n} |\widehat{g}(m)|^2$ by Theorem 2.29,

$$Re\langle f|g\rangle = Re \sum_{m\in\mathbf{Z}^n} \widehat{f}(m)\overline{\widehat{g}(m)}.$$

Meanwhile, applying Theorem 2.29 to $f + ig \in L^2(\mathbf{T}^n)$ and expanding both sides, we have, using the fact that $Re(-iw) = Im(w)$, that

$$\|f\|_{L^2}^2 + \|g\|_{L^2}^2 + 2Im\langle f|g\rangle = \sum_{m\in\mathbf{Z}^n} |\widehat{f}(m)|^2 + \sum_{m\in\mathbf{Z}^n} |\widehat{g}(m)|^2 + 2Im \sum_{m\in\mathbf{Z}^n} \widehat{f}(m)\overline{\widehat{g}(m)},$$

and hence

$$Im\langle f|g\rangle = Im \sum_{m\in\mathbf{Z}^n} \widehat{f}(m)\overline{\widehat{g}(m)}.$$

Since both the real and imaginary parts of $\langle f|g\rangle$ and $\sum_{m\in\mathbf{Z}^n} \widehat{f}(m)\overline{\widehat{g}(m)}$ are equal for all $f, g \in L^2(\mathbf{T}^n)$, the statement in Theorem 2.31 holds true. □

3. Fourier Series on $\mathbf{R}^n$

In this section, we want to extend our discussions of Fourier series on $\mathbf{T}^n$ to $\mathbf{R}^n$. Since $\sum_{m\in\mathbf{Z}^n} \widehat{f}(m)e^{2\pi i m\cdot x}$, if the series converges, is 1-periodic, it would never converge back to a non-periodic f on $\mathbf{R}^n$. However, as we show below, if a continuous function f on $\mathbf{R}^n$ satisfies certain conditions, then its Fourier series reproduces its periodization.

Theorem 3.1 (Poisson summation formula)**.** *Let f be a continuous function on $\mathbf{R}^n$ such that for some $C, \delta > 0$,*

$$|f(x)| \le C(1+|x|)^{-(n+\delta)} \tag{3.2}$$

for all $x \in \mathbf{R}^n$ and

$$\sum_{m\in\mathbf{Z}^n} |\widehat{f}(m)| < \infty. \tag{3.3}$$

Then, for any $x \in \mathbf{R}^n$,

$$\sum_{m\in\mathbf{Z}^n} \widehat{f}(m)e^{2\pi i m\cdot x} = \sum_{k\in\mathbf{Z}^n} f(x+k). \tag{3.4}$$

Proof. We define the periodization of f by a function on $\mathbf{T}^n$, that is

$$F(x) = \sum_{k\in\mathbf{Z}^n} f(x+k).$$

Then $\|F\|_{L^1([0,1]^n)} = \|f\|_{L^1(\mathbf{R}^n)} < \infty$, implying $F \in L^1(\mathbf{T}^n)$. Moreover, given $0 \le |x| \le \sqrt{n}/2$ for all $x \in \mathbf{T}^n$, we have $|x+k| \ge |k| - |x| \ge |k| - \sqrt{n}/2 \ge |k| - \sqrt{n}$ for all $x \in \mathbf{T}^n$ and $k \in \mathbf{Z}^n$, which gives

$$\sum_{k\in\mathbf{Z}^n} \frac{1}{(1+|x+k|)^{n+\delta}} \le \sum_{k\in\mathbf{Z}^n} \frac{(1+\sqrt{n})^{n+\delta}}{(1+\sqrt{n}+|x+k|)^{n+\delta}} \le \sum_{k\in\mathbf{Z}^n} \frac{C_{n,\delta}}{(1+|k|)^{n+\delta}} < \infty$$

for some constant $C_{n,\delta}$ depending on n and δ. And since (3.2) holds for all $x \in \mathbf{R}^n$, the Weierstrass M-test implies the uniform convergence of $\sum_{k\in\mathbf{Z}^n} f(x+k)e^{-2\pi i m\cdot x}$. Thus we may compute, for each $m \in \mathbf{Z}^n$, that

$$\begin{aligned}
\widehat{F}(m) &= \int_{\mathbf{T}^n} \sum_{k\in\mathbf{Z}^n} f(x+k)e^{-2\pi i m\cdot x}dx \\
&= \sum_{k\in\mathbf{Z}^n} \int_{\mathbf{T}^n} f(x+k)e^{-2\pi i m\cdot x}dx \\
&= \sum_{k\in\mathbf{Z}^n} \int_{\mathbf{T}^n - k} f(x)e^{-2\pi i m\cdot x}dx \\
&= \int_{\mathbf{R}^n} f(x)e^{-2\pi i m\cdot x}dx \\
&= \widehat{f}(m).
\end{aligned}$$

(3.3) then implies that

$$\sum_{m\in\mathbf{Z}^n} |\widehat{F}(m)| < \infty.$$

And since F is continuous, (3.4) holds for all $x \in \mathbf{T}^n$ by Proposition 2.27. The statement, which applies to all $x \in \mathbf{R}^n$, then follows by periodicity. $\square$

4. Applications of Fourier Series to Isoperimetric Inequality

Fourier series can be applied to prove the isoperimetric inequality, which informs us of the relation between the length L of a closed, positively oriented, nonself intersecting, regular, $\mathscr{C}^1$ planar curve C^1 and the area A of the region R it encloses. The theorem is stated as follows:

Theorem 4.1 (Isoperimetric Inequality)**.** *For a closed, positively oriented, nonself intersecting, regular, $\mathscr{C}^1$ planar curve C^1 which encloses a region R of area A,*

$$A \leq \frac{L^2}{4\pi}. \tag{4.2}$$

Moreover, equality holds if and only if C^1 is a circle.

We first prove that C^1 can be parameterized such that it has constant speed.

Lemma 4.3. *Suppose C^1 can be produced by parametric equations $x = x(t)$ and $y = y(t)$, with $0 \leq t \leq 1$. Then there exists a reparameterization of the curve with constant speed, i.e. a reparameterization $(x(t), y(t))$ which satisfies*

$$\sqrt{|x'(t)|^2 + |y'(t)|^2} = L. \tag{4.4}$$

Proof. Consider the "normalized arc length function", that is

$$s(t) = \frac{1}{L}\int_0^t \sqrt{|x'(u)|^2 + |y'(u)|^2}du.$$

Given $|(x'(t), y'(t))| \neq 0$, the Inverse Function Theorem implies that $s : [0,1] \to [0,1]$ is a bijective map with some inverse function $\gamma = s^{-1}$.
Consider $\gamma : [0,1] \to [0,1]$. Since

$$|x'(\gamma(t))|^2|\gamma'(t)|^2 + |y'(\gamma(t))|^2|\gamma'(t)|^2 = \frac{|x'(s^{-1}(t))|^2 + |y'(s^{-1}(t))|^2}{|s'(s^{-1}(t))|^2} = \frac{1}{1/L^2} = L^2,$$

the curve parameterized by $(x(\gamma(t)), y(\gamma(t)))$ satisfies (4.4), i.e. it has constant speed. □

Hence, when proving Theorem 4.1, we may denote C^1 by $f(t) = x(t) + iy(t)$, with $|f'(t)| = L$ for all $t \in [0,1]$.

Proof for Theorem 4.1. We first show that the inequality holds. By Green's Theorem,

$$A = \frac{1}{2}\int_0^1 (x(t)y'(t) - x'(t)y(t))\, dt = \frac{1}{2} Im \int_0^1 f'(t)\overline{f(t)}dt.$$

Furthermore, given $\int_0^1 \overline{\widehat{f}(0)}dt = 0$ and the assumption that $|f'(t)| = L$, we have

$$A = \frac{1}{2}\int_0^1 (x(t)y'(t) - x'(t)y(t))\, dt = \frac{1}{2} Im \int_0^1 f'(t)\overline{(f(t) - \widehat{f}(0))}dt \leq \frac{1}{2}L \left\| f - \widehat{f}(0) \right\|_{L^2}.$$

We also have

$$\left\| f - \widehat{f}(0) \right\|_{L^2} = \left[\sum_{m \in \mathbf{Z}\setminus\{0\}} |\widehat{f}(m)|^2 \right]^{\frac{1}{2}} \leq \left[\sum_{m \in \mathbf{Z}} |m\widehat{f}(m)|^2 \right]^{\frac{1}{2}} = \frac{1}{2\pi} \|f'\|_{L^2}, \tag{4.5}$$

which implies

$$A \leq \frac{1}{2}L \cdot \frac{1}{2\pi} \|f'\|_{L^2} = \frac{L^2}{4\pi}.$$

Having proven (4.2), we now want to show that $A = L^2/4\pi$ if and only if $f(t) = Ce^{2\pi it} + \widehat{f}(0)$ for some $C \in \mathbf{C}$ with $|C| = L/2\pi$. That is, the equality holds if and only if C^1 is a circle with radius $L/2\pi$ and centre $\widehat{f}(0)$.
($\Longrightarrow$) Suppose $A = L^2/4\pi$. Then (4.5) holds with equality, implying $\widehat{f}(m) = 0$ for all $|m| \geq 2$. Therefore, there must exist some $c_1, c_2 \in \mathbf{C}$ with $\widehat{f}(1) = c_1$, $\widehat{f}(-1) = c_2$ such that

$$f(t) = \sum_{m\in\mathbf{Z}} \widehat{f}(m)e^{2\pi imt} = \widehat{f}(1)e^{2\pi it} + \widehat{f}(-1)e^{-2\pi it} + \widehat{f}(0) = c_1e^{2\pi it} + c_2e^{-2i\pi t} + \widehat{f}(0).$$

Since $\|f'\|_{L^2} = L$,

$$4\pi^2(|c_1|^2 + |c_2|^2) = L^2; \tag{4.6}$$

also, since $|f'(t)| = L$ for all $t \in [0,1]$,

$$\left(\frac{L}{2\pi}\right)^2 = |c_1|^2 + |c_2|^2 - 2Re[c_1\overline{c_2}e^{2\pi i2t}] \text{ for } t \in [0,1]. \tag{4.7}$$

It follows from (4.6) and (4.7) that

$$Re[c_1\overline{c_2}e^{2\pi i2t}] = 0 \text{ for } t \in [0,1]. \tag{4.8}$$

At $t = 0$, (4.8) gives $Re[c_1\overline{c_2}] = 0$, while at $t = 1/8$, we have $Im[c_1\overline{c_2}] = 0$, thus either $c_1 = 0$ or $c_2 = 0$. Which means,

$$f(t) = Ce^{2\pi it} + \widehat{f}(0)$$

for some $C \in \mathbf{C}$ with $|C| = L/2\pi$ by (4.6).
($\Longleftarrow$) Suppose that C^1 is a circle with radius $L/2\pi$ and centre $\widehat{f}(0)$. Then C^1 can be denoted by $f(t) = Ce^{2\pi it} + \widehat{f}(0)$ with $|C| = L/2\pi$. We have the length of the curve C^1 being $L = 2\pi \cdot \frac{L}{2\pi}$ and the area it encloses being $A = (\frac{L}{2\pi})^2 \cdot \pi = \frac{L^2}{4\pi}$, proving the statement. □

5. Applications of Fourier Series to Weyl's Theorem on Equidistribution

Fourier series also plays an essential role in the statement and proof of Weyl's theorem on equidistributed sequences.

Definition 5.1. Let a cube in $\mathbf{T}^n \equiv [0,1]^n$ be defined by any $[a_1, b_1] \times \ldots \times [a_n, b_n] \subset \mathbf{T}^n$ with $|b_1 - a_1| = |b_2 - a_2| = \ldots = |b_n - a_n|$. A sequence $\{u_k\}_{k=1}^{\infty}$ on $\mathbf{T}^n$ is *equidistributed* if for every cube Q in $\mathbf{T}^n$,

$$\lim_{N\to\infty} \frac{Z(N;Q)}{N} = |Q|, \tag{5.2}$$

where

$$Z(N;Q) = \#\{k \in \mathbf{Z} : 1 \leq k \leq N, u_k \in Q\} \tag{5.3}$$

with $\#\{S\}$ denoting the cardinality of the set S and $|Q|$ denoting the measure of the cube Q.

Theorem 5.4 (Weyl's theorem on equidistribution)**.** *The following statements are equivalent:*

(a) $\{u_k\}_{k=1}^{\infty}$ is equidistributed.

(b) For every smooth function f on $\mathbf{T}^n$,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} f(u_k) = \int_{\mathbf{T}^n} f(x)dx. \tag{5.5}$$

(c) For every $m \in \mathbf{Z}^n \setminus \{0\}$,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} e^{2\pi i m\cdot u_k} = 0. \tag{5.6}$$

Proof. We first prove $(a) \implies (b)$. Suppose that (5.2) holds for any cube $Q \subset \mathbf{T}^n$. Fix $\epsilon > 0$, and consider any smooth function f on $\mathbf{T}^n$. Given f is uniformly continuous, there exists some step function $g = \sum_{j=1}^{m} c_j \chi_{Q_j}$, with $c_j \in \mathbf{C}$ and Q_j being a cube in $\mathbf{T}^n$ for each $j \in [m]$, such that

$$\left|\int_{\mathbf{T}^n} f(x)dx - \int_{\mathbf{T}^n} g(x)dx\right| \le \|f-g\|_{L^\infty} < \frac{\epsilon}{3}.$$

Given that for each $N \in \mathbf{Z}$ and cube $Q \subset \mathbf{T}^n$, $Z(N;Q) = \sum_{k=1}^{N} \chi_Q(u_k)$ by inspection, and since $\int_{\mathbf{T}^n} \chi_Q(x)dx = |Q|$, we know by (a) that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} \chi_Q(u_k) = \int_{\mathbf{T}^n} \chi_Q(x)dx,$$

hence for g, which is a finite linear combination of characteristic functions for cubes, there exists some $N_0 \in \mathbf{N}$ such that for all $N \ge N_0$,

$$\left|\frac{1}{N}\sum_{k=1}^{N} g(u_k) - \int_{\mathbf{T}^n} g(x)dx\right| < \frac{\epsilon}{3}.$$

And given

$$\left|\frac{1}{N}\sum_{k=1}^{N} f(u_k) - \frac{1}{N}\sum_{k=1}^{N} g(u_k)\right| < \frac{\epsilon}{3}$$

for all $N \ge N_0$, we have

$$\begin{aligned}
&\left|\frac{1}{N}\sum_{k=1}^{N} f(u_k) - \int_{\mathbf{T}^n} f(x)dx\right| \\
\le &\left|\frac{1}{N}\sum_{k=1}^{N} f(u_k) - \frac{1}{N}\sum_{k=1}^{N} g(u_k)\right| + \left|\frac{1}{N}\sum_{k=1}^{N} g(u_k) - \int_{\mathbf{T}^n} g(x)dx\right| + \left|\int_{\mathbf{T}^n} g(x)dx - \int_{\mathbf{T}^n} f(x)dx\right| \\
< &\frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon
\end{aligned}$$

for all $N \ge N_0$, which yields (b).

Then, we prove $(b) \implies (a)$. Fix any $\epsilon > 0$, and consider any cube $Q \subset \mathbf{T}^n$. There exist smooth functions g and h that approximate χ_Q as follows: $g(x) = 1$

on $(1-\epsilon)Q$ and vanishes off Q, while $h(x)=1$ on Q and vanishes off $h(x)=0$ on $(1+\epsilon)Q$. Then

$$\frac{1}{N}\sum_{k=1}^{N} g(u_k) \leq \frac{1}{N}\sum_{k=1}^{N} \chi_Q(u_k) \leq \frac{1}{N}\sum_{k=1}^{N} h(u_k)$$

and

$$|Q| - c_n\epsilon \leq \int_{\mathbf{T}^n} g(x)dx \leq |Q| \leq \int_{\mathbf{T}^n} h(x)dx \leq |Q| + c_n\epsilon$$

for some $c_n > 0$. Supposing (b) is true for g and h, we have, by the Sandwich Theorem, that

$$|Q| - c_n\epsilon \leq \liminf_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} \chi_Q(u_k) \leq \limsup_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} \chi_Q(u_k) \leq |Q| + c_n\epsilon.$$

Taking $\epsilon \to 0$ gives (a).

Given $f(x) = e^{2\pi i m\cdot x}$ is a smooth function on $\mathbf{T}^n$, (b) directly implies (c).

Finally, we show $(c) \implies (b)$.
Consider any smooth function f on $\mathbf{T}^n$, for which we have

$$\frac{1}{N}\sum_{k=1}^{N} f(u_k) = \frac{1}{N}\sum_{k=1}^{N}\sum_{m\in\mathbf{Z}^n} \widehat{f}(m)e^{2\pi i m\cdot u_k} = \widehat{f}(0) + \sum_{m\in\mathbf{Z}^n\setminus\{0\}} \widehat{f}(m)\left(\frac{1}{N}\sum_{k=1}^{N} e^{2\pi i m\cdot u_k}\right).$$

Taking the limit as $N\to\infty$, it follows from (c) that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N} f(u_k) = \widehat{f}(0) = \int_{\mathbf{T}^n} f(x)dx.$$

□

Remarks 5.7. Define U_N as the measure that places unit point-masses at points of $\{u_k\}_{k=1}^{N}$. Then for each $m \in \mathbf{Z}^n \setminus \{0\}$,

$$\widehat{U}_N(m) = \int_{\mathbf{T}^n} e^{-2\pi i m\cdot\alpha} dU_N = \sum_{k=1}^{N} e^{-2\pi i m\cdot u_k}.$$

We may then reformulate (c) as

$$\widehat{U}_N(m) = o(N) \text{ as } N\to\infty \text{ for each } m\in\mathbf{Z}^n\setminus\{0\}.$$

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