A SIMPLE PROOF OF THE ATIYAH-SEGAL COMPLETION THEOREM

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Abstract. In this paper, we present a simple proof of the Atiyah-Segal completion theorem. We follow the proof in [1] with progroup language and prove the original version, that the completion of $K_G^*(X)$ at the augment ideal I is isomorphic to $K_G^*(EG \times X) \cong K^*(EG \times_G X)$.

Contents

1. Introduction

Paper [1] gives a geodesic route from Bott periodicity to a generalized version of the Atiyah-Segal completion theorem that generalized to completion at a family of subgroups. Basic progroup language is used in the proof of that paper. In this paper there will be a more accessible proof explaining firstly the simple progroup language involved and then the topological proof.

To make the proof more explicit, we will focus only on the complex case since the proof for the real case is identical to the complex case. Also, we will focus only on the original form of the Atiyah-Segal Completion Theorem, and ignore the generalization to a family of subgroups that is mentioned in [1].

2. equivariant K-theory

In this paper, we use the definition of equivariant K theory in [5].

Let G be a compact Lie group and we understand a G-space as a G-CW-complex. Define a G-vector bundle to be a G-map $p : E \to X$ which is a complex vecter bundle. For any $g \in G$ and $x \in X$, $g : E_x \to E_{gx}$ is a homomorphism of vector spaces. Let $K_G(X)$ be the Grothendieck group of the monoid of G-vector bundles over some G-space X.

Call two G-vector bundles E, E' *stably equivalent* if there exist trivial G-vector bundles M and M' such that $E \oplus M \cong E' \oplus M'$. The stable equivalence classes

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form an abelian group $\tilde{K}_G(X)$. For example, $R(G)$, which is the representation ring of G, is also equal to $K_G(\mathrm{pt})$, and $\tilde{K}_G(pt) = 0$ since we only have trivial G-vector bundles over a point.

Define $\tilde{K}_G^{-k}(X) = \tilde{K}_G(S^k X)$, $\tilde{K}_G^{-k}(X, A) = \tilde{K}_G(S^k(X \cup_A CA))$, $K_G^{-k}(X) = \tilde{K}_G^{-k}(X_+)$ and $\tilde{K}_G^{-k}(X, A) = \tilde{K}_G^{-k}(X_+, A_+)$ where X_+ is the one-point compactification of X and $S^k X$ is the k-th suspension which is equal to the smash product $S^k \wedge X$. For example, $K_G^n(pt) = 0$ if n is odd and $K_G^n(pt) = R(G)$ if n is even by equivariant Bott periodicity explained in section 4.

[5] shows that $K_G^*(X)$ is a generalized cohomology theory and $\tilde{K}_G^*(X)$ is a reduced generalized cohomology theory. It satisfies the following axioms.
(i). (suspension) $\tilde{K}_G^{*+1}(SX)$ is isomorphic to $\tilde{K}_G^*(X)$;
(ii). (homotopy invariance) If $f_1 \simeq f_2 : X \to Y$, then the induced homomorphism $f_1^* = f_2^*$;
(iii). (exactness) If A is a closed G-subspace of X, $\tilde{K}_G^*(X \sqcup_A CA) \to \tilde{K}_G^*(X) \to \tilde{K}_G^*(A)$ is exact.
And (i'). (homotopy invariance) if $f : (X_1, A_1) \to (X_2, A_2)$ is a homotopy equivalence, $f^* : K_G^*(X_2, A_2) \to K_G^*(X_1, A_1)$ is an isomorphism;
(ii'). (exactness) if A is a closed G-subspace of X, $\cdots \to K_G^*(X, A) \to K_G^*(X) \to K_G^*(A) \to K_G^{*+1}(X, A) \to \cdots$ is exact;
(iii'). (excision) if A is a closed G-subspace of X and U is a G-subspace of X such that $\bar{U} \subset \mathrm{Int}(A)$, then $i^* : K_G^*(X, A) \to K_G^*(X - U, A - U)$ is an isomorphism;

Let EG be the universal space of G, $E^n G$ be the n-skeleton of EG. Then EG is the geometric realization of the simplicial space $E_n(G) = G^{n+1}$ with certain faces and degeneracies. So EG is a G-CW complex. Explicit construction and properties can be find in 16.5 of [3]. Define I_G to be the augmentation ideal of $R(G)$, that is, the kernel of the restriction $R(G) \to R(\{e\})$. We will simply denote I_G as I if causing no confusion.

Then the Atiyah-Segal completion theorem can be expressed as following. The meaning of notations in the following two theorems can be found in the next section.

Theorem 2.1. *(the Atiyah-Segal Completion Theorem) Let X be a G-space. Then the projection $EG \times X \to X$ induces an isomorphism of progroup $K_G^*(X)_{\widehat{I}} \to K_G^*(\mathrm{EG} \times X)$.*

To prove this theorem, we will prove the following theorem stated in [1] firstly in section 4.

Theorem 2.2. *If X_1 and X_2 are G-spaces, and a G-map $f : X_1 \to X_2$ is a homotopy equivalence (but not essentially a G-homotopy equivalence), then f induces an isomorphism $K_G^*(X_2)_{\widehat{I}} \to K_G^*(X_1)_{\widehat{I}}$.*

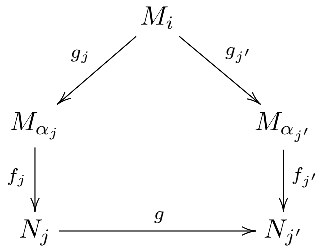
3. Progroups

Definition 3.1. a *progroup* is a inverse system of abelian groups, indexed on a filtered directed poset.

Definition 3.2. If $\{M_\alpha\}, \{N_\beta\}$ are two progroups, then define $\mathrm{Hom}(\{M_\alpha\}, \{N_\beta\}) = \varprojlim_\beta \varinjlim_\alpha \mathrm{Hom}(M_\alpha, N_\beta)$.

To be explicit, an arrow in $\mathrm{Hom}(\{M_\alpha\}, \{N_\beta\})$ can be represented by a set $\{f_j : M_{\alpha_j} \to N_j\}$ of group homomorphisms, one for each j, such that for each arrow

$g : N_j \to N_{j'}$ of $\{N_\beta\}$, there is some i, an arrow $g_j : M_i \to M_{\alpha_j}$ and an arrow $g_{j'} : M_i \to M_{\alpha_{j'}}$ such that $g \circ f_j \circ g_j = f_{j'} \circ g_{j'}$.



Note that f_j is a representative of an equivalence class in $\varinjlim_\alpha \mathrm{Hom}(M_\alpha, N_j)$. Each such f_j is called a *representative of* f. On the other hand, two sets $\{f_j : M_{\alpha_j} \to N_j\}$ and $\{f'_j : M_{\alpha'_j} \to N_j\}$ are representatives of the same arrow if for every j, there exists some i, an arrow $g_j : M_i \to M_{\alpha_j}$ and an arrow $g'_j : M_i \to M'_{\alpha_j}$ such that $f_j \circ g_j = f'_j \circ g'_j$.

Then it gives a unique composition of two morphisms, and the collection of identity morphisms in $\mathrm{Hom}(M_\alpha, M_\alpha)$ gives the identity morphism of $\{M_\alpha\}$. So progroups form a category.

For example, each profinite group can be identified with a progroup of finite groups. But generally, a progroup contains more information than the limit of the groups. Also, $\{0\}$ gives a zero progroup.

Proposition 3.3. *A progroup $\{M_i\}$ is isomorphic to zero if and only if for any i, there exists a zero homomorphism $f_i : M_j \to M_i$. Such progroup is called pro-zero.*

Proof. A progroup $\{M_i\}$ isomorphic to zero means that id is equal to 0. So by the explanation above, it is equal to the right hand side. □

Appendix of [7] shows that progroups form an Abelian category. But we would give some explicit description of exact sequences of progroups. The (co)kernel of a morphism f between progroups is the progroup $\{(\mathrm{co})\ker(f_j) \mid f_j$ is a representative of $f\}$ with arrows induced naturally. And the (co)image of a morphism f between progroups is the progroup $\{(\mathrm{co})\mathrm{im}(f_j) \mid f_j$ is a representative of $f\}$ with arrows induced naturally. So we have

Proposition 3.4. *Given a sequence of two morphisms of progroups $\{L_i\} \xrightarrow{f} \{M_j\} \xrightarrow{g} \{N_k\}$ such that the composition is zero. The sequence is pro-exact if for every representative $f_j : L_i \to M_j$ of f, there is some $h_j^m : M_m \to M_j$ and some representative $g_k : M_m \to N_k$ of g such that $h_j^m(\ker(g_k)) \subset \mathrm{im}(f_j)$.*

To give the statement of the Atiyah-Segal completion theorem, we need the notion of the I*-adic completion*. Given a ring R, an ideal I and a progroup $\{M_i\}$ with index A which consists of R-modules.

Definition 3.5. Define the I*-adic completion* of $\{M_i\}$ to be progroup $\{M_i/I^k M_i \mid i \in A, k \in \mathbb{Z}_+\}$ with index $A \times \mathbb{Z}_+$, and denote it as $\{M_i\}_{\hat{I}}$.

Then it is an exact functor as follows.

Lemma 3.6. *Suppose R is a Noetherian ring, I is an ideal of R and $f : A \to B$ is a morphism between finitely generated R-modules. Then there exists $c \in \mathbb{N}$ such that for any $n \in \mathbb{N}_+$, $\ker(A \to B/I^{n+c}B) \subset \ker(f) + I^n A$.*

Proof. We can decompose $A \to B/I^{n+c}B$ into $A \to \mathrm{im}(f) \to B/I^{n+c}B$ and let C be $\mathrm{im}(f)$. Then $0 \to C \to B$ is exact. By Artin-Rees lemma, we can choose $c > 0$ such that for any $n > 0$, $\ker(C \to B/I^{n+c}B) = C \cap I^{n+c}B \subset I^n C$.

Since tensoring $R/I^n R$ is right exact and $C = \mathrm{coker}(\ker(f) \to A)$, $\ker(f)/I^n \ker(f) \to A/I^n A \to C/I^n C \to 0$ is exact. So $\ker(A \to C/I^n C) \subset \ker(f) + I^n A$. Combining the two results we come to the conclusion that $\ker(A \to B/I^{n+c}B) \subset \ker(f) + I^n A$. □

Theorem 3.7. *If R is a Noetherian ring, the I-adic completion is an exact functor in the subcategory of progroup consisting of finitely generated R-modules.*

Proof. For any morphism $f : M \to N$ between R-modules, $f(I^k M) \subset I^k N$. Let $\tilde{f}^k$ be the corresponding morphism $M/I^k M \to N/I^k N$. It commutes with composition. So formal completion is a functor.

Then it suffices to prove that for pro-exact sequence $\{A_i\} \xrightarrow{f} \{B_j\} \xrightarrow{g} \{C_k\}$, the sequence $\{A_i\}_{\widehat{I}} \to \{B_j\}_{\widehat{I}} \to \{C_k\}_{\widehat{I}}$ is pro-exact.

For any representative $f_j : A_i \to B_j$ and any $m \in \mathbb{Z}_+$, suppose there is $r_j^{j'} : B_{j'} \to B_j$ and $g_k : B_{j'} \to C_k$ such that $r_j^{j'}(\ker(g_k)) \subset \mathrm{im} f_j$. By the lemma above, we have some $c > 0$. Suppose $\tilde{g}_k$ is the map $B_{j'}/I^{m+c}B_{j'} \to C_k/I^{m+c}C_k$ induced from g_k. Then we have that $\ker(\tilde{g}_k) \subset \ker(g_k) + I^m$. Let $\tilde{f}_j$ be $A_i/I^m A_i \to B_j/I^m B_j$ induced from f_j and $\tilde{r}$ be $B_{j'}/I^{m+c} \to B_j/I^m$ induced from $r_j^{j'}$. Then $\mathrm{im}(\tilde{f}_j) = \mathrm{im}(f_j) + I^m$. So by Lemma 3.6, $\tilde{r}(\ker(\tilde{g}_k)) \subset \tilde{r}(\ker(g_k) + I^m B_{j'}) = r(\ker(g_k)) + I^m B_j \subset \mathrm{im}(\tilde{f}_j)$. So the sequence $\{A_i\}_{\widehat{I}} \to \{B_j\}_{\widehat{I}} \to \{C_k\}_{\widehat{I}}$ is pro-exact. □

If X is a finite G-CW-complex, then $X \to \{\mathrm{pt}\}$ gives $R(G) = K_G^0(pt) \to K_G^0(X)$. So $K_G^*(X)$ has the structure of a $R(G)$-module.

Definition 3.8. For any G-CW-complex X(not necessarily finite), $K_G^n(X)$ is a progroup $K_G^n(X_\alpha)$ where X_α runs over the finite G-subcomplexes of X.

Then by the theorem above, $K_G^*(X)_{\widehat{I}}$ is a generalized cohomology theory.

4. Proof of Theorem 2.2

Let M be a term in the cofiber sequence $X_1 \to X_2 \to M$. Since X_1 is homotopy equivalent to X_2, M is contractible. So by the cofiber exact sequence it suffices to prove the following theorem.

Theorem 4.1. *If X is contractible, then $\tilde{K}_G^*(X)_{\widehat{I}}$ is pro-zero.*

Suppose U is the collection of G representations V such that $V^G = \{0\}$ and $V^H \neq \{0\}$ for some proper subgroup $H < G$. For any proper subgroup $H < G$, notice that the induced representation $\mathrm{Ind}_H^G \mathbb{1}_H$ is nontrivial, so we can choose a nontrivial irreducible sub-G-representation $V \subset \mathrm{Ind}_H^G \mathbb{1}_H$. Then by Frobenius reciprocity, $\mathrm{Hom}_H(\mathbb{1}_H, \mathrm{Res}_H^G V) = \mathrm{Hom}_G(\mathrm{Ind}_H^G \mathbb{1}_H, V)$. Thus $V^G = \{0\}$ and $V^H = V \neq \{0\}$. So we have $V^{\oplus k} \in U$ for any positive integer k. Let I be a finite set of representations in U, and let Y^I be the one point compactification of the direct sum of elements in I. Then $(Y^I)^G = S^0$. The inclusion $I \subset J$ gives an inclusion $Y^I \to Y^J$. Define Y to be the colimit of all Y^I's. We will first show that $\{\tilde{K}_H^*(Y)\}_{\widehat{I_H}}$ is pro-zero for all $H < G$ where Y is constructed above.

Lemma 4.2. *$\{\tilde{K}^*_H(Y)\}_{\widehat{I_H}}$ is pro-zero for all $H < G$.*

Proof. If H is a proper subgroup of G, then for Y^I, choose V' such that H acts trivially on V'. Let J be $I + \{V'\}$, V be the direct sum of elements in I, and W be $V \oplus V'$. Then $Y^I = S^V$ and $Y^J = S^W$. So $((v, v'), t) \mapsto (v, v'/t)$ for $0 < t \leq 1$, $((v, v'), 0) \mapsto \infty$, and $(\infty, t) \mapsto \infty$ give a homotopy between the null map to infinity and the inclusion map from S^V to S^W. So such inclusion map is null-homotopic. Since I is arbitrary, Y is H-contractible. Then the lemma is obvious.

If $H = G$, it suffices to show that for any Y^I and m, there exists $Y^I \to Y^J$ such that the map $\tilde{K}^*_H(Y^J)/I^m\tilde{K}^*_H(Y^J) \to \tilde{K}^*_H(Y^I)/I^m\tilde{K}_H$ is 0.

We need the Bott periodicity for equivariant K theory that is described at Proposition 3.2 of [5] and Theorem 4.3 of [8]. It says the following. For any complex G-module V, there is an element $\lambda_V \in K^0_G(V) = \tilde{K}^0_G(S^V)$, $\lambda_V = \sum_{i=0}^{\infty}(-1)^i\Lambda^i V$ where Λ^i is the i-th wedge power. Then multiply by λ_V induces an isomorphism $K^*_G(X) \to K^*_G(V \times X)$. If $W = V \oplus V'$, then we have $\lambda_W = \lambda_{V'}\lambda_V$.

By Bott periodicity, taking $X = \mathrm{pt}$, then $\tilde{K}^*_G(S^V)$ is the free $\tilde{K}^*_G(S^0)$-module generated by the Bott class $\lambda_V \in \tilde{K}^0_G(S^V)$. Suppose $W = V \oplus V'$. The inclusion $i : S^V \to S^W$ is equal to $\mathrm{id}_V \wedge i'$ where i' is the inclusion $S^0 \to S^{V'}$. Then $i^*(\lambda_W) = i'^*(\lambda_{V'})\lambda_V$. Notice that we have a commutative diagram

$$\begin{array}{ccc} \tilde{K}^n_G(S^V) & \xrightarrow{i'^*_G} & \tilde{K}^n_G(S^0) \\ \downarrow r & & \downarrow r \\ \tilde{K}^n(S^V) & \xrightarrow{i'^*_e} & \tilde{K}^n(S^0) \end{array}$$

where each column is the restriction. Since i' is null homotopic (not equivariantly), $r(i'^*_G(\lambda_{V'})) = i'^*_e(r(\lambda_{V'})) = 0$. So $i'^*(\lambda_{V'}) \in I$. Choose $Y^J = Y^I \oplus V \oplus \cdots \oplus V$(the number of V's is m) then we are done. $\square$

We have a cofiber sequence $S^0 \to Y^I \to Y^I/S^0$ and take smash product with some X to get a cofiber sequence $X \to X \wedge Y^I \to X \wedge (Y^I/S^0)$.

Theorem 4.3. *For any finite G-CW-complex X, the progroup $\{\tilde{K}^*_G(Y \wedge X)_{\hat{I}} \mid k \in \mathbb{N}\}$ is zero.*

Proof. Use induction on dimension d of X. The case when $d = 0$ follows from Lemma 4.2. It suffices to show that the G-space X' obtained by attaching an d cell to X still satisfies the property. There is a pro-exact sequence

$$\cdots \to \tilde{K}^*_G((G/H)_+ \wedge S^d \wedge Y)_{\hat{I}} \to \tilde{K}^*_G(X')_{\hat{I}} \to \tilde{K}^*_G(X)_{\hat{I}} \to \cdots$$

. By the suspension axiom, it suffices to show that $\tilde{K}^*_G((G/H)_+ \wedge Y)_{\hat{I}}$ is pro-zero. Example (iii) of section 2 in [5] shows that $\tilde{K}^*_G((G/H)_+ \wedge Y) = \tilde{K}^*_H(Y)$. Corollary 3.9 of [4] shows that I_G (acts through restriction) and I_H topology are the same. Then it follows from pro-exactness and lemma 4.2. $\square$

Consider the lexicographical ordering given by dimension and the number of connected components of subgroups of the compact Lie group G. We have that any descending chain of subgroups of G is finally constant. So we can use transfinite induction on the poset of inclusion of subgroups of G. Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. We use induction on subgroup $H < G$.

Suppose H is the trivial subgroup $\{e\}$. Since X is contractible, we consider the homotopy $h : X \times [0,1] \to E_G$ such that $h(X \times \{0\}) = \mathrm{id}_X$ and $h(X \times \{1\}) = \mathrm{pt}$. Since every finite G-subcomplex X^n has a open neighborhood that contracts to itself and it is compact, $h(X^n \times [0,1]) \subset X^m$ for some m. So $X^n \hookrightarrow X^m$ is null homotopic. As a result, the progroup $\tilde{K}^*(X)$ is zero.

Suppose H is not $\{e\}$. Let Y be the same construction below Theorem 4.1 with G changed to H, and without loss of generosity, suppose $H = G$. Cofiber sequence $X = X \wedge S^0 \to X \wedge Y \to X \wedge (Y/S^0)$ gives a long pro-exact sequence

$$\cdots \to \tilde{K}_G^k(X \wedge Y)_{\widehat{I}} \to \tilde{K}_G^k(X)_{\widehat{I}} \to \tilde{K}_G^{k+1}(X \wedge (Y/S^0))_{\widehat{I}} \to \cdots$$

By Theorem 4.3, $\tilde{K}_G^k(X \wedge Y)_{\widehat{I}}$ is pro-zero. So it suffices to show that $\tilde{K}_G^k(X \wedge (Y/S^0))$ is pro-zero.

In fact we can prove that $(\tilde{K}_G^k(X \wedge Z))_{\widehat{I}}$ is pro-zero for every k and finite G-space Z such that $Z^G = pt$. We use the same induction as Theorem 4.3. We induct on the dimension d of Z. Notice that $Z^0 = pt$, the $d = 0$ case is trivial. Consider attaching a d-cell $(G/H)_+ \wedge S^d$ to Z and getting Z'. Since $Z^G = pt$, $H \neq G$. There is a pro-exact sequence

$$\cdots \to \tilde{K}_G^k((G/H)_+ \wedge S^d \wedge X)_{\widehat{I}} \to \tilde{K}_G^k(Z' \wedge X)_{\widehat{I}} \to \tilde{K}_G^k(Z \wedge X)_{\widehat{I}} \to \cdots$$

. By the suspension axiom, it suffices to show that $\tilde{K}_G^k((G/H)_+ \wedge X)_{\widehat{I}}$ is pro-zero for each k. Since $\tilde{K}_G^*((G/H)_+ \wedge X)_{\widehat{I_G}} = \tilde{K}_H^*(X)_{\widehat{I_H}}$ and H is a proper subgroup of G, it follows from the induction hypothesis. ☐

5. Applications

To deduce Theorem 2.1 from Theorem 2.2, it suffices to show that $K_G^*(EG \times X)$ is I-adically complete.

Lemma 5.1. *If G acts freely on some G-space X, then $K_G^*(X)$ is discrete in the I-adic topology, so it is complete.*

This lemma can be found at Proposition 4.3 in [6]. The proof mainly involves the fact that $R(G)$ is noetherian and considering $K_G^*(X)$ as $K^*(X/G)$. Then it uses some calculation in commutative algebra. Here we omit the proof.

From the construction of E^nG we know that G acts freely on EG. So G acts freely on $EG \times X$. Then Theorem 2.1 holds.

If we take $X = \mathrm{pt}$, then we have that $K(BG) = K_G(EG) \cong R(G)_{\widehat{I}}$.

If X is compact, then $K_G(EG \times X) \cong K_G(X)_{\widehat{I}}$. So $K_G(EG \times X)$ satisfies the Mittag-Leffler condition. Then we can identify $K_G^*(EG \times X)$ with $\lim_n K_G^*(E^nG \times X)$.

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References

[1] J.F.Adams, J.-P.Haeberly, S.Jackowski and J.P.May. A Generalization of the Atiyah-Segal Completion Theorem. Topology Vol. 27, No. 1, pp. 1-6, 1988
[2] J.F.Adams, J.-P.Haeberly, S.Jackowski and J.P.May. A Generalization of the Segal's Conjecture. Topology Vol. 27, No. 1, pp. 7-21, 1988
[3] J. P. May. A Concise Course in Algebraic Topology. University of Chicago Press. 1999.
[4] G. B. Segal. The Representation Ring of a Compact Lie Group. Inst. Haures Etudes Sci. Publ. Math. 34 (1968), 113-128.
[5] G. B. Segal. Equivariant K-Theory. Publications mathématiques de l'I.H.É.S., tome 34 (1968), p. 129-151
[6] M. F. Atiyah, G. B. Segal. Equivariant K-Theory and Completion. J. Differential Geometry. 3(1969), p. 1-18
[7] M. Artin, B. Mazur. Etale Homotopy. Springer. 1969.
[8] M. F. Atiyah. Bott Periodicity and the Index of Elliptic Operators. Quart. J. Math. Oxford Ser. 19(1983), p. 113-140.