THE NUMBER OF COUNTABLE MODELS OF A COMPLETE COUNTABLE THEORY

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Abstract. In this expository paper, we will explore the number of non-isomorphic countable models a complete theory has. We will prove Vaught's Never-Two Theorem that no complete theory has two countable models, and in the process, we will learn about atomic and countably saturated models. From there, we will give examples of complete theories with n non-isomorphic models for any $n \neq 2$, and we will give examples of complete theories with $\aleph_0$ many non-isomorphic models and of $2^{\aleph_0}$ many non-isomorphic models. We will also briefly mention some of the deeper results in the study of the spectrum of theories along with Vaught's Conjecture as areas of further mathematical study.

Contents

1. Introduction

This paper will investigate different models of a theory in first-order logic. For a given theory T, we denote the number of models that T of cardinality κ up to isomorphism by $I(T, \kappa)$, and we call this function the *spectrum* of T. By the Löwenheim-Skolem-Tarski Theorem, if T has an infinite model, then

$$I(T, \kappa) \geq 1$$

for every infinite κ greater than or equal to the size of the language. We are often interested in when

$$I(T, \kappa) = 1$$

and when this happens, we say that the theory T is *κ-categorical*.

Many important breakthroughs in model theory relate to the study of the spectrum of theories T. For example, Morley proved that if a countable theory T is κ-categorical for some uncountable κ, then T is κ-categorical for every uncountable κ (see [4]). Baldwin and Lachlan proved that when a countable theory T is $\aleph_1$-categorical (and thus, due to Morley, κ-categorical for all uncountable κ), then either T has $\aleph_0$ many countable models or T is $\aleph_0$-categorical (see [1]).

In our paper, we will be focused primarily in the values $I(T, \aleph_0)$ can take on for complete countable theories T. Restricting our study only to countable models makes this study easier. However, there are still interesting questions which arise. For example, Vaught conjectured that independent of the continuum hypothesis if $I(T, \aleph_0) > \aleph_0$, then $I(T, \aleph_0) = 2^{\aleph_0}$. While Vaught's Conjecture becomes trivial if we assume the continuum hypothesis, the conjecture still remains unsolved as of August 2020. However, significant progress on this question was proved by Morley who showed that if $I(T, \aleph_0) > \aleph_1$, then $I(T, \aleph_0) = 2^{\aleph_0}$ (see [5]).

2. Preliminary Model Theory

There are no prerequisites for this paper besides some set theory. However, some introductory knowledge in model theory will help. In this section we will give a quick overview of some of these introductory concepts. There will also be no proofs in this section, and for a fuller picture of the basics of elementary model theory most introductory model theory textbooks should do (for example [2]).

A *first order language* $\mathcal{L}$ is a set of constant symbols relation symbols and function symbols where each relation symbol is an n-placed relation for some $n \geq 1$ and each function symbol is an m-placed function for $m \geq 1$. We say that such a relation has an arity of n and such a function has an arity of m. Whenever the symbols used in our language are commonly used in mathematics we often assume the arity of them without explicitly stating it. For example, the language $\mathcal{L} = \{0, 1, <, +, *\}$ is the language with constant symbols 0 and 1, a binary relation symbol $<$, and binary function symbols $+$ and $*$.

A *model* $\mathfrak{A}$ of a language $\mathcal{L}$ consists of a set A which we call its *universe* along with interpretations of the constant relation and function symbols in the language. In particular, the interpretation of a constant symbol is an element in the universe A, the interpretation of an m-ary function symbol is a function $f : A^m \to A$, and the interpretation of an n-ary relation symbol is a subset of A^n. For example, a model of the language $\mathcal{L} = \{0, 1, <, +, *\}$ given above is an ordered field where $0, 1, <, +,$ and $*$ all have their usual interpretations in such a field. When it is unambiguous, we will usually use a set A as the universe of $\mathfrak{A}$ without explicitly mentioning it, and we will similarly use B as the universe for $\mathfrak{B}$ and so on.

The *size of a language* $\mathcal{L}$ is denoted by $||\mathcal{L}||$ where $||\mathcal{L}|| = \aleph_0 \cup |\mathcal{L}|$. Likewise, the *size of a model* $\mathfrak{A}$ is $|A|$.

Our languages are equipped with infinitely many variables $v_0, v_1, ...$ along with the following symbols: parenthesis (and), the connectives $\wedge$, $\vee$, $\rightarrow$, $\leftarrow$, $\leftrightarrow$ and $\neg$, the quantifiers $\forall$ and $\exists$, and a binary identity symbol $\equiv$. A *term* of a language is any string from $\mathcal{L}$ along with the list of symbols above that can be constructed by the following inductive process:

- Any variable v_i is a term.
- Every constant is a term.
- If f is an m-ary function and $t_1, ..., t_m$ are terms, then $f(t_1, ..., t_m)$ is a term.

We can similarly define an *atomic formula* by the following process:

- If t_1 and t_2 are terms, then $t_1 \equiv t_2$ is an atomic formula.
- If $t_1, ..., t_n$ are terms and P is an n-ary relation, then $P(t_1, ..., t_n)$ is an atomic formula.

Formulas are built up from atomic formulas through the following inductive process:

- Every atomic formula is a formula.
- If φ and ψ are formulas then $\neg\varphi$, $(\varphi \wedge \psi)$, $\varphi \vee \psi$ $\varphi \to \psi$, $\varphi \leftarrow \psi$, and $\varphi \leftrightarrow \psi$ are all formulas.
- If v is a variable, then $(\forall v)\varphi$ and $(\exists v)\varphi$ are formulas.

When a variable in a formula is not bound by quantifiers, we say that the variable is *free.* A *sentence* is any formula with no free variables. We denote by $\varphi(v_0...v_n)$ any formula φ whose free variables are all within $\{v_0, ..., v_n\}$.

We will omit the details regarding whether a formula is satisfied in a model $\mathfrak{A}$ of a language $\mathcal{L}$ (see [2]). However, whether a formula is satisfied in a model should match the reader's intuition of what it means to be satisfied. When $\varphi(x_1...x_n)$ is satisfied in $\mathfrak{A}$ by some n-tuple, $a_1, ..., a_n$ we write

$$\mathfrak{A} \models \varphi[a_1...a_n].$$

When φ is a sentence, it contains no free variables so we can just write

$$\mathfrak{A} \models \varphi$$

which we can read as φ is true in $\mathfrak{A}$, holds in $\mathfrak{A}$, or $\mathfrak{A}$ models φ.

If $\mathcal{L}' = \mathcal{L} \cup X$, then any model $\mathfrak{A}$ of $\mathcal{L}$ can be expanded to a model of $\mathcal{L}'$ by adding an interpretation for all elements of X. Similarly, given a model $\mathfrak{A}'$ of $\mathcal{L}'$, we can take the reduct of the model by just ignoring all the interpretations of the symbols in X. If we have some set Y of constant symbols not in $\mathcal{L}$ we can add these constants to $\mathcal{L}$ to obtain a new language which we denote $\mathcal{L}_Y$. We will often denote the expanded model by $\mathfrak{A}_Y$ to be the model where we add a constant element a for each $a \in Y$ where we interpret each element of Y by the corresponding element in $\mathfrak{A}_Y$.

We say that two models $\mathfrak{A}$ and $\mathfrak{B}$ are *isomorphic*, denoted $\mathfrak{A} \cong \mathfrak{B}$, if there is a bijective function $f : A \to B$ such that:

- for each n-placed relation R of $\mathfrak{A}$ there is a corresponding relation R' of $\mathfrak{B}$ such that
$$R(a_1...a_n) \text{ iff } R'(f(a_1)...f(a_n))$$
for all $a_1, ..., a_n \in A$.
- For each m-placed function G of $\mathfrak{A}$ there is a function G' of $\mathfrak{B}$ such that
$$f(G(a_1...a_m)) = G'(f(a_1)...f(a_m)).$$
for all $a_1, ..., a_m \in A$
- For each constant $a \in \mathfrak{A}$ with its corresponding constant $b \in \mathfrak{B}$, we have
$$f(a) = b.$$

In such a case we call the function f an *isomorphism.*

Two models $\mathfrak{A}$ and $\mathfrak{B}$ of a language $\mathcal{L}$ are *elementarily equivalent* if

$$\mathfrak{A} \models \varphi \text{ iff } \mathfrak{B} \models \varphi$$

for every sentence φ of $\mathcal{L}$, and we denote this by $\mathfrak{A} \equiv \mathfrak{B}$. It is easy to check that

$$\text{if } \mathfrak{A} \cong \mathfrak{B} \text{ then } \mathfrak{A} \equiv \mathfrak{B}.$$

However, the converse is not always true.

$\mathfrak{A}$ is a *submodel* of $\mathfrak{B}$ if $A \subset B$ contains all the constant symbols in B, and each function and relation in $\mathfrak{A}$ is just the restriction of the corresponding relations and functions in $\mathfrak{B}$. In this case, we say that B is an *extension* of $\mathfrak{A}$, and we also can denote this by $\mathfrak{A} \subset \mathfrak{B}$. If $X \subset A$, then the intersection of all submodels of $\mathfrak{A}$ containing X is called the *submodel generated by* X.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be models for a language $\mathcal{L}$. An injective map $f : A \to B$ is an *elementary embedding* if for every formula $\varphi(v_1...v_n)$ and for any $a_1, ..., a_n \in A$,

$$\mathfrak{A} \models \varphi[a_1...a_n] \text{ iff } \mathfrak{B} \models \varphi[(f(a_1)...f(a_n)].$$

We say that $\mathfrak{A}$ is an *elementary submodel* of $\mathfrak{B}$ and $\mathfrak{B}$ is an *elementary extension* of $\mathfrak{A}$. We often denote such a map f by $f : A \prec B$

A *theory* T is a set of sentences in a language $\mathcal{L}$. We write $\mathfrak{A} \models T$ if every sentence in T is satisfied in $\mathfrak{A}$ in which case we say that T has a model $\mathfrak{A}$. If every sentence of $\mathcal{L}$ can be deduced from T, then T is said to be *inconsistent.* Otherwise, T is said to be *consistent.* Given a model $\mathfrak{A}$ we define $Th(\mathfrak{A})$ to be the set of all sentences which are true in $\mathfrak{A}$.

We say that a theory T in a language $\mathcal{L}$ is *complete* if for every sentence σ exactly one of $T \models \sigma$ or $T \models \neg\sigma$ holds. It should be clear that for any complete theory T any two models of T are elemantarily equivalent.

Finally, we will present three important theorems. We will omit the proofs of these theorems but the reader can see [2].

Theorem 2.1. (Completeness Theorem) *A theory T is consistent iff T has a model.*

Theorem 2.2. (Compactness Theorem) *A theory T has a model iff every finite subset of T has a model.*

Theorem 2.3. (Löwenheim-Skolem-Tarski Theorem) *If a theory T has at least one infinite model, then T has a model of every infinite cardinality $\kappa \geq ||\mathcal{L}||$.*

3. Omitting Types

Given a formula σ with free variables in $x_1, ..., x_n$, we say that a model $\mathfrak{A}$ *realizes* σ if there exists some sequence $a_1, ..., a_n$ of elements in A that satisfies σ in $\mathfrak{A}$ and we denote this by

$$\mathfrak{A} \models \sigma[a_1...a_n].$$

Then, given a set of formulas Σ with free variables in $x_1, ..., x_n$, which we will denote $\Sigma = \Sigma(x_1...x_n)$, we say that

$$\mathfrak{A} \models \Sigma[a_1...a_n]$$

if every $\sigma \in \Sigma$ is satisfied by $a_1, ..., a_n$ in $\mathfrak{A}$. When this occurs we say that the sequence $a_1, ..., a_n$ *satisfies* or *realizes* Σ in $\mathfrak{A}$. If a sequence $a_1, ..., a_n$ of elements of A exists such that

$$\mathfrak{A} \models \Sigma[a_1...a_n]$$

then we say $\mathfrak{A}$ *realizes* Σ or that Σ is *satisfiable* in $\mathfrak{A}$. When this is not the case, we say that $\mathfrak{A}$ *omits* Σ.

We will often want to know when a set of formulas Σ is omitted by some model. We will see that in order to answer this question, it will be helpful to introduce the notion of a theory locally omitting Σ.

Definition 3.1. If $\Sigma = \Sigma(x_1...x_n)$ is a set of formulas in $\mathcal{L}$ then a theory T *locally realizes* Σ if there is some formula $\varphi(x_1...x_n)$ such that φ is consistent with T and

$$T \models \varphi \to \sigma$$

for every $\sigma \in \Sigma$. Equivalently, any n-tuple in a model of T which satisfies φ also realizes Σ. When T does not locally realize Σ, we say that T *locally omits* Σ.

Proposition 3.2. *Let T be a complete theory in $\mathcal{L}$ and $\Sigma(x_1...x_n)$ be a set of formulas in $\mathcal{L}$. If T has a model which omits Σ then T locally omits Σ.*

Proof. We will prove the contrapositive, that if T locally realizes Σ, then every model of T realizes Σ. If T locally realizes Σ, then there exists a formula $\varphi(x_1...x_n)$ consistent with T such that

$$T \models \varphi \to \sigma$$

for every $\sigma \in \Sigma$. Let $\mathfrak{A}$ be a model of T. Since T is complete,

$$T \models (\exists x_1...x_n)\varphi$$

so there is some $a_1, ..., a_n \in A$ satisfying φ in $\mathfrak{A}$. Therefore, $a_1, ..., a_n$ satisfies each $\sigma \in \Sigma$ so $a_1, ..., a_n$ realizes Σ. □

Theorem 3.3. (Omitting Types Theorem) *Let T be a consistent theory in a countable language $\mathcal{L}$ and $\Sigma(x_1...x_n)$ be a set of formulas. If T locally omits Σ then there is a countable model of T which omits Σ.*

Proof. To make notation easier, we will consider $\Sigma(x)$ to be a set of formulas in the one free variable x. All of the arguments in this proof involving formulas in one free variable apply almost identically to formulas in several variables by replacing the occurrences of the one variable x with the n variables $x_1, ..., x_n$. Let $C = \{c_0, c_1, ...\}$ be a countable set of constant symbols not in $\mathcal{L}$. Then let $\mathcal{L}' = \mathcal{L} \cup C$. Since $\mathcal{L}'$ is countable, we can index each sentence of $\mathcal{L}'$ with a natural number, so arrange the sentences of $\mathcal{L}'$ in a sequence $\varphi_0, \varphi_1, ...$ We will construct a sequence of consistent theories

$$T = T_0 \subset T_1 \subset ...$$

each of which are a finite extension of T having the following properties:

(1) $\neg\sigma(c_m) \in T_{m+1}$ for some $\sigma \in \Sigma$.
(2) Either $\varphi_m \in T_{m+1}$ or $\neg\varphi_m \in T_{m+1}$.
(3) If $\varphi_m = (\exists x)\psi(x)$ and $\varphi_m \in T_{m+1}$ then $\psi(c_p) \in T_{m+1}$ for some constant c_p.

If some theory T_m has been constructed and is a finite extension of T, then we can let $T_m = T \cup \{\theta_1, ..., \theta_r\}$ with $r > 0$, and we can let $\theta = \theta_1 \wedge ... \wedge \theta_r$. Then if $c_0, ..., c_n$ are all of the constants from C appearing in θ, we replace each c_i by the variable x_i and prefix $\exists x_i, i \not\equiv m$ to obtain the sentence $\theta(x_m)$. Then since θ is true in T_m, clearly $\theta(x_m)$ is consistent in T_m. Therefore, there exists some $\sigma(x) \in \Sigma(x)$ such that $\theta(x_m) \wedge \neg\sigma(x_m)$ is consistent with T (because if this weren't the case then $\theta(x_m) \to \sigma(x_m)$ must hold in T which contradicts the hypothesis that T locally omits Σ). Then add the sentence $\neg\sigma(c_m)$ into T_{m+1} satisfying (1).

If φ_m is consistent with $T_m \cup \{\neg\sigma(c_m)\}$ add φ_m to T_m and otherwise add $\neg\varphi_m$ satisfying (2).

If $\varphi_m = (\exists x)\psi(x)$is consistent with $T_m \cup \{\neg\sigma(c_m)\}$ put $\psi(c_p)$ in T_{m+1} where c_p is the first constant not yet occurring in T_m or φ_m, satisfying (3).

Since we added finitely many sentences each of which were consistent with T_m, T_{m+1} is consistent in $\mathcal{L}'$ and is a finite extension of T_m, and thus, it is a finite extension of T.

Let $T_\omega = \bigcup_{n<\omega} T_n$. Then by the compactness theorem, T_ω is consistent in $\mathcal{L}'$ and (2) ensures that T_ω is maximal consistent in $\mathcal{L}'$ so it has a model $\mathfrak{B}' = (\mathfrak{B}, a_0, a_1, ...)$ where each a_i is an interpretation of the constant c_i. We can choose $\mathfrak{A}' = (\mathfrak{A}, a_0, a_1, ...)$ to be the submodel generated by $a_0, a_1, ...$ Then from (3) we see that this model's universe is $A = \{a_0, a_1, ...\}$. Using (3) to induct on the complexity of sentences in $\mathcal{L}'$ we can see that

$$\mathfrak{A}' \models \varphi \text{ iff } T_\omega \models \varphi$$

so $\mathfrak{A}'$ is a model of T_ω and $\mathfrak{A}$ is a model of T. Condition (1) shows that $\mathfrak{A}$ omits Σ and since the universe of $\mathfrak{A}$ is $\{a_0, a_1, ...\}$, $\mathfrak{A}$ is countable. □

Corollary 3.4. (Extended Omitting Types Theorem) *Let T be a consistent theory in a countable language $\mathcal{L}$, and $\Sigma_r(x_1, ..., x_{n_r})$ be a set of formulas in n_r variables with $r < \omega$. If T locally omits each Σ_r, then T has a countable model which omits each Σ_r.*

Proof. The proof is similar to the proof of the omitting types theorem. The difference is that here we arrange every n_r-tuple of constants in a list $s^r_r, s^r_{r+1}, ...$ We construct the theories T_m such that, instead of condition (1) of the previous theorem holding, we ensure that for each r there is a $\sigma \in \Sigma_r$ such that $\neg\sigma(s^r_m) \in T_{m+1}$. □

We will see the applications of the omitting types theorem and the extended omitting types theorem in the next section as they will be used on a few occasions to prove results in that section.

4. Atomic Models and ω-Saturated Models

The remainder of this paper will study countable models of complete theories. When a complete theory has an atomic and countably saturated model, we will see that these models become two important types of countable models and will be a major focus of study throughout this paper. The study of these models will prove to be important in the study of the number of non-isomorphic countable models of a complete theory exists, which will be the question this paper ultimately aims to provide some insight into. We will see that intuitively, atomic models can be thought of as small countable models, while countably saturated models can by thought of as large countable models. Throughout this section, when we are studying atomic and countably saturated models of a language $\mathcal{L}$, we will assume that $\mathcal{L}$ is a countable language.

Definition 4.1. A formula $\varphi(x_1...x_n)$ is *complete* in a complete theory T if for every formula $\psi(x_1, ..., x_n)$, exactly one of $T \models \varphi \to \psi$ or $T \models \varphi \to \neg\psi$ holds. A formula $\theta(x_1...x_n)$ is *completable* if there exists a complete formula $\varphi(x_1, ..., x_n)$ such that $T \models \varphi \to \theta$. When a formula θ is not completable, we say that θ is *incompletable.*

Definition 4.2. A theory T is *atomic* if every consistent (with T) formula of $\mathcal{L}$ is completable in T. A model $\mathfrak{A}$ is an *atomic model* if every n-tuple $a_1, ..., a_n \in A$ satisfies a complete formula in $Th(\mathfrak{A})$.

Theorem 4.3. (Existence Theorem for Atomic Models) *Let T be complete. Then T has an atomic model iff T is atomic.*

Proof. Let $\mathfrak{A}$ be an atomic model for T and let $\varphi(x_1...x_n)$ be consistent with T. Since T is complete,

$$T \models (\exists x_1, ..., x_n)\varphi(x_1...x_n).$$

Choose $a_1, ..., a_n$ satisfying φ and since $\mathfrak{A}$ is atomic, choose a complete formula $\psi(x_1...x_n)$ satisfied by $a_1, ..., a_n$. It is clear that

$$T \models \psi \to \neg\varphi$$

cannot happen so since T is complete,

$$T \models \psi \to \varphi.$$

Therefore, φ is completable and T is atomic.

Now let T be atomic, and for $n < \omega$, let $\Gamma_n(x_1...x_n)$ be the set of negations of complete formulas in n variables in T. Any formula $\varphi(x_1, ..., x_n)$ which is consistent with T is completable, so there is some complete formula $\psi(x_1, ..., x_n)$ such that

$$T \models \psi \to \varphi.$$

Thus, $\varphi \wedge \psi$ is consistent with T and $\psi = \neg\gamma$ for some $\gamma \in \Gamma_n$. Therefore, $\varphi \wedge \neg\gamma$ is consistent with T, so T locally omits each Γ_n. Since there are countably many Γ_n, by the extended omitting types theorem, T has a countable model $\mathfrak{A}$ which omits each Γ_n. Thus, each $a_1, ..., a_n$ is not consistent with some $\gamma \in \Gamma_n$, so $a_1, ..., a_n$ satisfies a complete formula $\neg\gamma$. Therefore, $\mathfrak{A}$ is atomic. □

Theorem 4.4. (Uniqueness Theorem for Atomic Models) *If* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are countable atomic models with* $\mathfrak{A} \equiv \mathfrak{B}$ *then* $\mathfrak{A} \cong \mathfrak{B}$

Proof. It is not too difficult to show that for any finite models $\mathfrak{A}$ and $\mathfrak{B}$, $\mathfrak{A} \equiv \mathfrak{B}$ implies $\mathfrak{A} \cong \mathfrak{B}$. We will therefore assume A and B are infinite. We will well-order A and B in order type ω and use a back and forth process as follows:

Let a_0 be the first element of A and because $\mathfrak{A}$ is atomic choose $\varphi_0(x_0)$ be a complete formula satisfied by a_0. Then

$$\mathfrak{A} \models (\exists x_0)\varphi_0(x_0) \text{ implies } \mathfrak{B} \models (\exists x_0)\varphi_0(x_0),$$

so we can choose $b_0 \in B$ satisfying $\varphi_0(x_0)$. Choose b_1 to be the first element of $B \setminus \{b_0\}$ and let $\varphi_1(x_0, x_1)$ be a complete formula satisfied by b_0, b_1 in $\mathfrak{B}$. Then since φ_0 is complete, we know that $\mathfrak{B}$ and therefore also $\mathfrak{A}$ satisfy

$$\forall x_0(\varphi_0(x_0) \to (\exists x_1)\varphi_1(x_0, x_1))$$

so we can choose $a_1 \in A$ such that a_0, a_1 satisfy $\varphi_1(x_0, x_1)$. We then choose a_2 to be the first element of $A \setminus \{a_0, a_1\}$. Continuing this process we obtain $A = \{a_0, a_1, ...\}$ and $B = \{b_0, b_1, ...\}$ and for any n, $a_0, ..., a_n$ and $b_0, ..., b_n$ satisfy the same complete formula. From this, it follows without too much difficulty that $\mathfrak{A} \cong \mathfrak{B}$ by the isomorphism sending $a_n \to b_n$. □

This is our first time seeing such a *back and forth* argument. Such a method of proof is worth remembering as it comes up a lot, and we will see more examples of this method of proof later in this paper.

We now introduce the concept of a prime model in order to illustrate why I mentioned earlier that atomic models are intuitively the small models of a theory T.

Definition 4.5. $\mathfrak{A}$ is a *prime model* if $\mathfrak{A}$ can be elementarily embedded in every model of $Th(\mathfrak{A})$. $\mathfrak{A}$ is *countably prime* if it can be elementarily embedded in every countable model of $Th(\mathfrak{A})$

Theorem 4.6. *The following are equivalent:*

(1) $\mathfrak{A}$ *is a countable atomic model.*
(2) $\mathfrak{A}$ *is prime.*
(3) $\mathfrak{A}$ *is countably prime.*

Proof. (1) $\Longrightarrow$ (2) Let $\mathfrak{A}$ be a countably atomic model with universe $A = \{a_0, a_1, ...\}$, and let $T = Th(\mathfrak{A})$. Let $\mathfrak{B}$ be a model of T. Choose a complete formula $\varphi_0(x_0)$ satisfied by a_0. Then

$$T \models (\exists x_0)\varphi_0(x_0),$$

so choose $b_0 \in B$ satisfying $\varphi_0(x_0)$. Now choose a complete formula $\varphi_1(x_0x_1)$ satisfied by a_0, a_1. Similar to the proof of Theorem 4.4,

$$T \models \varphi_0(x_0) \to (\exists x_1)\varphi_1(x_0x_1),$$

so we can choose $b_1 \in B$ such that b_0, b_1 satisfies φ_1. Continuing this process we get a sequence $\{b_0, b_1, ...\}$, and it is not too difficult to check that the function sending $a_n \to b_n$ is an elementary embedding.

(2) $\Longrightarrow$ (3) This follows immediately from the definition

(3) $\Longrightarrow$ (1) If $\mathfrak{A}$ is countably prime, and $a_1, ..., a_n \in A$, then let $\Gamma(x_1...x_n)$ be the set of all formulas satisfied by $a_1, ..., a_n$. Let $\mathfrak{B}$ be a countable model of T. Then there exists an elementary embedding of $f : \mathfrak{A} \prec \mathfrak{B}$. Then $f(a_1), ..., f(a_n)$ satisfies $\Gamma(x_1, ..., x_n)$ in $\mathfrak{B}$, so Γ is realized in any countable model of T. By the omitting types theorem, T locally realizes Γ. Therefore, there exists a formula φ consistent with T such that for every $\gamma \in \Gamma$,

$$T \models \varphi \to \gamma.$$

Any formula $\psi(x_1, ..., x_n)$ is either in Γ or its negation is in Γ so φ is complete, and thus, $\mathfrak{A}$ is atomic. ☐

We can now begin the study of saturated models which will closely parallel our study of atomic models. In order to begin this study, it is necessary to first define a type.

Definition 4.7. A *type* $\Gamma(x_1...x_n)$ is a maximal consistent set of formulas of $\mathcal{L}$ in the variables $x_1, ..., x_n$.

Example 4.8. Given a model $\mathfrak{A}$ and an n-tuple $a_1, ..., a_n$, the set of all formulas $\gamma(x_1, ..., x_n)$ satisfied by $a_1, ..., a_n$ is the unique type realized by $a_1, .., a_n$ called *the type of* $a_1, ..., a_n$ *in* $\mathfrak{A}$.

Definition 4.9. A model $\mathfrak{A}$ is ω-*saturated* if, for every finite $Y \subset A$, every type in $\mathcal{L}_Y$ consistent with $Th(\mathfrak{A}_Y)$ is realized in $\mathfrak{A}_Y$. If $\mathfrak{A}$ is also countable, then $\mathfrak{A}$ is *countably saturated.*

Intuitively, an ω-saturated model is a model that realizes as many types as possible.

Like in the case of atomic models, we want to show the existence of ω-saturated models, show their uniqueness, and show that they are large.

Theorem 4.10. (Existence Theorem for Countably Saturated Model) *If T is a complete theory, then T has a countable saturated model iff for each $n < \omega$, T has only countably many types in n variables.*

Proof. If T has a countably saturated model $\mathfrak{A}$, then every type of T is realized in $\mathfrak{A}$. Each n-tuple realizes a unique type in n variables, so T has only countably many types.

If for each $n < \omega$, T has only countably many types, then add a set $C = \{c_1, c_2, ...\}$ to $\mathcal{L}$ to form the new language $\mathcal{L}'$. For every finite subset Y of C, the types $\Gamma(x)$ of T in $\mathcal{L}_Y$ are in one-to-one correspondence with the types $\Sigma(x_1...x_n x)$ of T in $\mathcal{L}$, so T has only countably many types in $\mathcal{L}_Y$. There are countably many finite $Y \subset C$ so we can let $\Gamma_1(x), \Gamma_2(x), ...$ be a list of all types of T in all $\mathcal{L}_Y$ for all finite $Y \subset C$. $\mathcal{L}'$ is countable so let $\varphi_1, \varphi_2, ...$ be a list of all sentences in $\mathcal{L}'$. We will form an increasing sequence of theories

$$T = T_0 \subset T_1 \subset ...$$

such that for each $m < \omega$:

(1) T_m is consistent and contains only finitely many constants from C.
(2) Either $\varphi_m \in T_{m+1}$ or $\neg\varphi_m \in T_{m+1}$.
(3) If $\varphi_m = (\exists x)\psi(x)$ is in T_{m+1} then $\psi(c) \in T_{m+1}$ for some $c \in C$.
(4) If $\Gamma_m(x)$ is consistent with T_{m+1} then $\Gamma_m(c) \subset T_{m+1}$ for some $c \in C$.

The proof that we can construct each T_m is similar to the proof seen in the omitting types theorem. Then we can define $T_\omega = \bigcup_{m<\omega} T_m$. T_ω is maximal consistent in $\mathcal{L}'$. Then by (3), we can construct a model $\mathfrak{A}' = (\mathfrak{A}, a_1, a_2, ...)$ of T_ω such that $A = \{a_1, a_2, ...\}$, so $\mathfrak{A}$ is a model of T.

If $Y \subset A$ is finite and $\Sigma(x)$ is consistent with $Th(\mathfrak{A}_Y)$, then extend $\Sigma(x)$ to a type $\Gamma(x)$ in $Th(\mathfrak{A}_Y)$. Then $\Gamma(x) = \Gamma_m(x)$ for some m and Γ_m is consistent with T_ω so Γ_m is consistent with T_{m+1}. By (4) $\Gamma_m(c_i) \subset T_{m+1}$ for some $c_i \in C$ so a_i realizes $\Gamma(x)$, and therefore, $\mathfrak{A}$ is ω-saturated. □

Corollary 4.11. *If T is complete with only countably many non-isomorphic countable models, then T has a countably saturated model.*

Proof. A countable model of T realizes countably many types and T has countably many models, so T has countably many types. Thus T has a countably saturated model. □

Theorem 4.12. (Uniqueness Theorem for Countably Saturated Models) *If $\mathfrak{A}$ and $\mathfrak{B}$ are countably saturated and $\mathfrak{A} \equiv \mathfrak{B}$, then $\mathfrak{A} \cong \mathfrak{B}$.*

Proof. The proof is similar to the proof of the uniqueness of atomic models. Like in the proof of the uniqueness of atomic models, given two countably saturated models $\mathfrak{A}$ and $\mathfrak{B}$, we can use a back and forth argument to construct two sequences $A = \{a_1, a_2, ...\}$ and $B = \{b_1, b_2, ...\}$ such that a_{n+1} realizes the same type in $(\mathfrak{A}, a_1, ..., a_n)$ as b_{n+1} realizes in $(\mathfrak{B}, b_1, ..., b_n)$ where $\{a_1, a_2, ...\}$ and $\{b_1, b_2, ...\}$ are the universe of $\mathfrak{A}$ and $\mathfrak{B}$ respectively. Then the map sending $a_n \to b_n$ is the desired isomorphism. $\square$

Definition 4.13. A *countably universal model* $\mathfrak{A}$ is a countable model where if $\mathfrak{B} \equiv \mathfrak{A}$ and $\mathfrak{B}$ is countable, then $\mathfrak{B}$ can be elementary embedded into $\mathfrak{A}$.

Theorem 4.14. *Every countably saturated model is countably universal.*

Proof. Like the previous theorem, this theorem's proof is similar to the analogous proof for atomic models (Theorem 4.6). If $\mathfrak{A}$ is a countably saturated model and $\mathfrak{B}$ is a countable model with $\mathfrak{A} \equiv \mathfrak{B}$, then similar to Theorem 4.6, if we let $B = \{b_0, b_1, ...\}$ we can obtain a sequence $a_0, a_1, ...$ such that $(\mathfrak{A}, a_0, a_1, ...) \equiv (\mathfrak{B}, b_0, b_1, ...)$ with the map $b_n \to a_n$ an elementary embedding. $\square$

While not needed for this paper, it is worth noting that unlike Theorem 4.6, in this case, there do exist examples of countably universal models which are not countably saturated.

Theorem 4.15. *If T is complete and T has a countably saturated model, then T has a countable atomic model.*

Proof. Assume that T has no countable atomic model. Then T is not atomic, so there exists a formula $\varphi(x_1...x_n)$ which is consistent but not completable in T. Then, φ is not a complete formula for if it was then $T \models \varphi \to \varphi$ would show that φ is completable. Thus, there exists a consistent formula φ_0 such that it is not the case that exactly one of $T \models \varphi \to \varphi_0$ and $T \models \varphi \to \neg\varphi_0$ holds. Then by consistency of φ, they cannot both hold, so neither of them hold. If we let $\varphi_1 = \neg\varphi_0$, by completeness of T, we get $T \models \neg(\varphi \to \varphi_0)$ and $T \models \neg(\varphi \to \varphi_1)$. It follows that $T \models \varphi \wedge \neg\varphi_0$ so $T \models \varphi_0 \to \varphi$ and similarly $T \models \varphi_1 \to \varphi$. Additionally, we have $T \models \neg(\varphi_0 \wedge \varphi_1)$.

φ_0 and φ_1 are incompletable because if they were, that would make φ completable. So for the formula φ_0 identically to what we did with φ, we can obtain consistent formulas φ_{00} and φ_{01} such that

$$T \models \varphi_{00} \to \varphi_0,\ T \models \varphi_{01} \to \varphi_0, \text{ and } T \models \neg(\varphi_{00} \wedge \varphi_{01})$$

and likewise, we can obtain consistent φ_{10} and φ_{11} such that

$$T \models \varphi_{10} \to \varphi_1,\ T \models \varphi_{11} \to \varphi_1 \text{ and } T \models \neg(\varphi_{10} \wedge \varphi_{11}).$$

Any infinite string $s_0, s_1, ...$ of 0s and 1s gives a collection of formulas

$$\Sigma_s = \{\varphi, \varphi_{s_0}, \varphi_{s_0 s_1}, ...\}$$

which is consistent with T. Since each set of formulas Σ_s is incompatible with each other, extending each Σ_s to a type gives uncountably many pairwise incompatible types consistent with T which, from Theorem 4.10, implies that T has no countably saturated model. $\square$

We can now begin the study of the spectrum of a theory T. First, recall that given a theory T and a cardinal κ, we can define $I(T, \kappa)$ to be the number of non-isomorphic models of cardinality κ of a theory T, where we call the function $I(T, \kappa)$ the *spectrum* of T. Also recall that when

$$I(T, \kappa) = 1,$$

T is said to be $\kappa - categorical$.

Theorem 4.16. (Characterization of $\aleph_0$-categorical Theories) *The following are equivalent:*

(1) *T is $\aleph_0$-Categorical.*
(2) *T has a model which is both atomic and ω-saturated.*
(3) *For each $n < \omega$, each type $\Gamma(x_1..., x_n)$ contains a complete formula.*
(4) *For each $n < \omega$, T has only finitely many types in $x_1, ..., x_n$.*
(5) *For each $n < \omega$, there are only finitely many formulas $\varphi(x_1, ..., x_n)$ up to equivalence with respect to T (where two formulas φ and ψ are equivalent in T if $T \models \varphi \leftrightarrow \psi$).*

Proof. (1) $\Longrightarrow$ (2) By Corollary 4.11, T has a countably saturated model, so by Theorem 4.15, T has an atomic model, and since T has only one model, they must be the same model.

(2) $\Longrightarrow$ (3) Let $\mathfrak{A}$ be ω-saturated and atomic. Since $\mathfrak{A}$ is ω-saturated, any type $\Gamma(x_1, ..., x_n)$ is realized by some $a_1, ..., a_n \in A$. $\mathfrak{A}$ is atomic so $a_1, ..., a_n$ satisfies a complete formula $\gamma(x_1...x_n)$. $\neg\gamma \notin \Gamma$ so $\gamma \in \Gamma$.

(3) $\Longrightarrow$ (4) If $\Sigma(x_1...x_n)$ is the negation of all complete formulas $\varphi(x_1...x_n) \in T$, then Σ cannot be extended to a type in T since each type contains a complete formula. Thus, Σ is inconsistent with T, so by the compactness theorem, some finite $\{\neg\varphi_1, ...\neg\varphi_m\} \in \Sigma$ is inconsistent with T. Therefore,

$$T \models \varphi_1 \vee ... \vee \varphi_m.$$

For $i \leq m$, the set $\Gamma_i(x_1...x_n)$ of consequences of $T \cup \{\varphi_i\}$ is a type of T, and every n-tuple satisfies some φ_i, so it realizes some Γ_i. By completeness of each φ_i, we get that $\Gamma_1, ..., \Gamma_i$ are the only types in $x_1, ..., x_n$.

(4) $\Longrightarrow$ (5) Given a formula $\varphi(x_1...x_n)$, let $\varphi*$ be the set of all types $\Gamma(x_1...x_n)$ containing φ. There are finitely many types $\Gamma_1, ..., \Gamma_m$ in $x_1, ..., x_n$, so there are only 2^m sets of types, and thus, there are only finitely many possible sets $\varphi*$.

$$\varphi* = \psi* \text{ implies } T \models \varphi \leftrightarrow \psi,$$

so we get that there are only finitely many formulas $\varphi(x_1...x_n)$ up to equivalence.

(5) $\Longrightarrow$ (1) It suffices to show all models of T are atomic by the uniqueness theorem for atomic models. Let $\mathfrak{A}$ a model of T and $a_1, ..., a_n \in A$. Let $\varphi_1, ...\varphi_k$ be the set of all formulas up to equivalence satisfied by $a_1, ..., a_n$. $\varphi_1 \wedge ... \wedge \varphi_k$ is complete and satisfied by $a_1, ..., a_n$ so $\mathfrak{A}$ is atomic. □

The final theorem of this section will demonstrate our primary reason for our interest in the study of atomic and countably saturated models in the context of the question this paper aims to answer. Its result will rely heavily on our previous results of atomic and countably saturated models, and it will finally provide the insight we have been waiting for regarding the number of non-isomorphic countable models a complete theory T has.

Theorem 4.17. (Vaught's Never-Two Theorem) *If T is complete, then*

$$I(T, \aleph_0) \neq 2.$$

Proof. If $I(T, \aleph_0) = 2$, then by Corollary 4.11, T has a countably saturated model $\mathfrak{B}$, so by Theorem 4.15, T also has an atomic model $\mathfrak{A}$. Our goal will be to construct a third model that is not atomic and not countably saturated, and thus, it is not isomorphic to either of $\mathfrak{A}$ or $\mathfrak{B}$.

If $\mathfrak{A} \cong \mathfrak{B}$, then T is $\aleph_0$-categorical. So let $\mathfrak{B} \ncong \mathfrak{A}$. Then, $\mathfrak{B}$ is not atomic, and thus, there exists $b_1, ..., b_n \in B$ not satisfying any complete formulas. Since $\mathfrak{B}$ is countably saturated, $(\mathfrak{B}, b_1, ..., b_n)$ is countably saturated, so by Theorem 4.15, the theory $T' = Th(\mathfrak{B}, b_1, ..., b_n)$ has an atomic model $(\mathfrak{C}, c_1, ..., c_n)$ whose reduct $\mathfrak{C}$ is a model of T.

$\mathfrak{C}$ is not atomic because $c_1, ..., c_n$ does not satisfy any complete formula. Since T is not $\aleph_0$-categorical it has infinitely many nonequivalent formulas. Thus, T' has infinitely many nonequivalent formulas so T' is not $\aleph_0$-categorical. Therefore, $(\mathfrak{C}, c_1, ..., c_n)$ cannot be both atomic and ω-saturated. Since $(\mathfrak{C}, c_1, ..., c_n)$ is not ω-saturated, it follows that $\mathfrak{C}$ also cannot be ω-saturated. □

5. Spectrum of Complete Theories in $\aleph_0$

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We have proven the result that for any complete countable theory T, $I(T, \aleph_0) \neq 2$. The remainder of this paper will be investigating what values $I(T, \aleph_0)$ can take on. The first thing to note is that by a simple combinatorial argument, it can be shown that for every countable theory T and for every cardinality κ,

$$I(T, \kappa) \leq 2^{\kappa}.$$

This gives an upper bound on the possible values of $I(T, \aleph_0)$

In order to proceed with our study of the values of $I(T, \aleph_0)$ for complete theories T, we will make use the following test in order to determine when a theory is complete.

Theorem 5.1. (Vaught's Test) *If T is a consistent theory with no finite models and every model of T of size κ is elementarily equivalent for some $\kappa \geq ||\mathcal{L}||$, then T is complete.*

Proof. If T is not complete then there is a sentence φ such that

$$T \not\models \varphi \text{ and } T \not\models \neg\varphi.$$

Then $T_0 = T \cup \varphi$ and $T_1 = T \cup \varphi$ are both consistent, and thus, have a model. Both of their models are models of T, and thus, must be infinite. By the Löwenheim-Skolem-Tarski Theorem, T_0 and T_1 both have models of size κ, and these two models are not elementarily equivalent. □

We are now equipped to present several examples of complete theories using Vaught's Test to prove completeness, and for each of these theories T, we will investigate $I(T, \aleph_0)$

Example 5.2. The theory of dense linear orders without endpoints is complete and $\aleph_0$-categorical.

Proof. Let T be the theory of dense linear orders without endpoints. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two models of T. Well-order $A = \{a_0, a_1, ...\}$ and $B = \{b_0, b_1, ...\}$. We will define $f : A \to B$ such that f is bijective and $a_i < a_j$ implies $f(a_i) < f(a_j)$. Begin by sending $a_0 \to b_0$. Assume we have partially defined f as a partial embedding mapping some subset of A to a subset of B. If a_n is in the image of this function then we are done. Otherwise, we can find some $b \in B$ such that for every a in the image of this partially defined f, if $a_n < a$ then $b < f(a)$, and if $a < a_n$ then $f(a) < b$. When this is done, define $f(a_n) = b$. If b_n is not in the partial image of f we can find some $a \in A$ not in the partial domain of f similarly to how we found $b \in B$ in the first half of this back and forth construction and send $a \to b_n$. Then continuing this construction, we get a function $f : A \to B$ which gives the desired isomorphism. Thus T is $\aleph_0$-categorical, and by Vaught's Test, since T has no finite model, T is also complete. □

Example 5.3. Let

$$\mathcal{L} = \{<, c_0, c_1, ...\},$$

and let T be the theory of dense linear orders without endpoints along with sentences that tell us $c_0 < c_1 < ....$ Then T is complete and $I(T, \aleph_0) = 3$.

The three models T are one where the the sequence $\{c_i\}$ is unbounded, one where the sequence is bounded but does not converge to anything, and one where the sequence does converge.

To prove completeness of T consider

$$\mathcal{L}_n = \{<, c_0, ..., c_n\}$$

to be a sublangauge of $\mathcal{L}$. Then similar to the theory of dense linear orders, the theory T_n of dense linear order without endpoints along with the sentences $c_0 < ... < c_n$ is complete. Every sentence $\varphi \in T$ contains only finitely many constants $c_0, ..., c_n$ so we can consider it as a sentence in $\mathcal{L}_n$. Thus, by the completeness of T_n either $T_n \models \varphi$ or $T_n \models \neg\varphi$, so either $T \models \varphi$ or $T \models \neg\varphi$. Therefore, T is complete.

Example 5.4. We can modify the previous example by letting

$$\mathcal{L}_n = \{<, P_1, ..., P_{n-2}, c_0, c_1, ...\}$$

be an expansion of the language in the previous example for $n > 3$ where $P_0, ..., P_{n-3}$ are all unary relations. Our theory T is an expanded version of the previous theory, containing all of the sentences from Example 5.3. Additionally, for each i, we add the sentence

$$\forall x \forall y (x < y \to \exists z (x < z < y \wedge P_i(z))),$$

which states that each P_i is dense. Finally we add sentences that each x is in exactly one of the subsets formed by the P_is, which tells us that the sets $P_1, ..., P_{n-2}$ partition A. The proof of completeness is similar to the completeness of the theory in Example 5.3. If we consider a subset of $\mathcal{L}_n$ containing only the constants $c_0, ..., c_m$, then this language is complete by a similar proof to the proof of completeness for the theory of dense linear orders. Every sentence $\varphi \in T$ contains only finitely many constants $c_0, ..., c_m$ so similar to in Example 5.3, either $T \models \varphi$ or $T \models \neg\varphi$.

This theory has n models. If the sequence $\{c_i\}$ does not converge that gives the same two models as in the previous example. When $\{c_i\}$ converges, we get $n - 2$ models depending on which P_i the sequence converges to.

We have already seen that for any complete theory,

$$I(T, \aleph_0) \neq 2.$$

The previous three examples demonstrate that for every $n < \omega$, if $n \neq 2$ then there exists a complete theory T such that

$$I(T, \aleph_0) = n.$$

The next two examples will show that there exists a complete theory T such that

$$I(T, \aleph_0) = \aleph_0,$$

and likewise, that there exists a theory T such that

$$I(T, \aleph_0) = 2^{\aleph_0}.$$

Example 5.5. If T is the theory of algebraically closed fields of characteristic 0, then T is complete and

$$I(T, \aleph_0) = \aleph_0.$$

The proof of this claim is almost entirely algebraic, and thus, most of the details have been omitted (see [6]). If $\mathfrak{A}$ is an algebraically closed field of characteristic 0, then a transcendence basis of $\mathfrak{A}$ is a maximal set $X \subset A$ such that no n-tuple from X is the root of a nonzero polynomial in n variables over the rationals. It is a result in field theory, that $\mathfrak{A}$ has a transcendence basis and each transcendence basis of T has the same cardinality κ which we call the trascendence degree of $\mathfrak{A}$. An algebraically closed field with transcendence degree κ has cardinality $\kappa + \aleph_0$ and for each κ there is a unique model $\mathfrak{A}$ with transcendence degree κ. For each uncountable κ, if $\mathfrak{A}$ has size κ then the trascendence degree of $\mathfrak{A}$ is κ and thus, for each uncountable κ, there is a unique model $\mathfrak{A}$ of size κ. Thus, since there are no finite models of T, by Vaught's test, T is complete.

Similarly,

$$I(T, \aleph_0) = \aleph_0$$

as there is one model of T for each transcendence degree ranging from $0, 1, ..., \omega$.

It is interesting to note that since T has only countably many models, T must have a countably saturated model, and since T has a countably saturated model, T must also have an atomic model. Since the atomic model is prime and the countably saturated model is universal, we can see that the atomic model is the algebraically closed field of transcendence degree 0, and the countably saturated model of T is the algebraically closed field of transcendence degree ω.

Example 5.6. (The theory of countably many independent unary relations) Let $\mathcal{L}$ be the language with countably many unary relations $P_0, P_1, ...$ and consider the theory T consisting of the sentences

$$(\exists x)(P_{i_1}(x) \wedge ... \wedge P_{i_m}(x) \wedge \neg P_{j_1}(x) \wedge ... \wedge \neg P_{j_n}(x))$$

where $i_1, ..., i_m, j_1, ..., j_n$ are distinct.

If $\mathfrak{A}$ is a model of T and $n < \omega$, then for any $a, b \in \mathfrak{A}$ which satisfy the same atomic formulas containing at most the relations in $P_0(x), ..., P_n(x)$, it is easy to check that any formula $\varphi(x)$ containing at most the symbols $P_0, ..., P_n$, we have

$$\mathfrak{A} \models \varphi[a] \text{ iff } \mathfrak{A} \models \varphi[b].$$

Thus, any formula $\varphi(x)$ contains symbols in $P_0, ..., P_n$ for some n and thus, in any model $\mathfrak{A}$ of T we can find elements $a, b \in A$ such that

$$\mathfrak{A} \models \varphi[a] \wedge P_{n+1}(a) \text{ and } \mathfrak{A} \models \varphi[b] \wedge \neg P_{n+1}(b).$$

Thus, $\varphi(x)$ does not imply $P_{n+1}(x)$ and it does not imply $\neg P_{n+1}(x)$. Therefore, φ is not a complete formula so T is not atomic.

Then, T has no atomic models so by Theorem 4.15, T has no countably saturated model.

We can also check that two elements a and b realize the same type iff they satisfy exactly the same $P_n(x)$. This gives $2^{\aleph_0}$ possible types, by choosing a subset $X \subset \omega$ and choosing the type that says $P_n(x)$ holds precisely for the elements $n \in X$. If T has $2^{\aleph_0}$ different types, then

$$I(T, \aleph_0) = 2^{\aleph_0}.$$

If $\mathfrak{A}$ is a countable model of T then we can show that $\mathfrak{A}$ has an elementary extension $\mathfrak{B}$ of size $2^{\aleph_0}$ such that for any $X \subset \omega$ there are $2^{\aleph_0}$ many $b \in B$ satisfying

$$\mathfrak{B} \models P_n(b) \text{ if } n \in X \text{ and}$$
$$\mathfrak{B} \models \neg P_n(b) \text{ if } n \notin X.$$

Up to isomorphism, there is only one such model $\mathfrak{B}$. Any two models of T have isomorphic elementary extension, and from this we can show that they are elementarily equivalent. Then since T has no finite model, by Vaught's Test, T is complete.

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