SCHRAMM-LOEWNER EVOLUTION AS A UNIVERSAL SCALING LIMIT

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ABSTRACT. The Schramm-Loewner evolution is a random continuous curve that satisfies the scale invariance and domain Markov properties with a single parameter κ determining quickly the curve winds. First, we will introduce background topics necessary for understanding Schramm-Loewner evolution, including Brownian motion, stochastic calculus, conformal maps, and the Loewner differential equation. We will the explore the construction of the Schramm-Loewner evolution and discuss the proof that the scaling limit of the Loop Erased Random Walk is SLE with parameter $\kappa = 2$.

CONTENTS

1. INTRODUCTION

Many models in physics take the form of random walks on two-dimensional discrete lattices. Taking the limit of the random walks as the distance between lattice points goes to zero, known as the scaling limit, gives a probability measure on continuous curves in the plane. For several stochastic processes on discrete planar lattices, including self-avoiding walks, percolation, the critical Ising and FK-Ising model, uniform spanning and loop-erased random walks, the scaling limit of the process is known or conjectured to be the Schramm-Loewner Evolution (SLE). SLE is a family of probability measures on continuous curves generated by the

Loewner differential equation with a Brownian motion driving function and with a single parameter κ controlling the rate at which the curve turns. For these models, we will look at the SLE measure of continuous curves between two points in a domain.

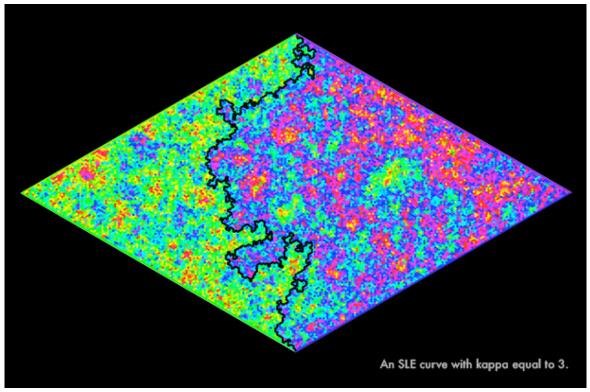

FIGURE 1. SLE_3 trace.
Retreived from [1].

All SLE_k probability measures share two key properties, the scale invariance and domain Markov properties. In fact, on the space over which SLE is a probability measure, $\mathcal{L}$, SLE uniquely satisfies the two properties. These properties and $\mathcal{L}$ are defined in Section 6. These properties are shared in the scaling limit by the physical models that approach SLE_κ, providing a conceptual connection between the various models. We will provide a short explanation of self-avoiding walks and percolations, then explore loop-erased random walks more closely in the rest of the paper.

1.1. **Self-Avoiding Walks.** A self-avoiding random walk on $A \subset \mathbb{Z}^2$ is a random walk starting at a point $z \in A$ and ending when it enters $\mathbb{Z}^2 \setminus A$, where nodes which have already been visited are avoided. We must be carefule to define this correctly. The simple way to do this would be to choose uniformly from unvisited adjacent points at each step of the process. However, this can lead to the process becoming trapped, as shown in Figure 2.

Instead, we must treat $\mathbb{Z}^2$ as a directed graph with edges in both directions between adjacent vertices. At each step, remove all edges pointing to the current vertex, then

- Remove all vertices with no out edges.
- Remove all vertices which are not path-connected to the boundary.
- If a vertex has only one out edge, which is also an in edge, remove both edges.

Repeat this process until no nodes or edges are removed on any step.

This process is conjectured to approach $\text{SLE}_{8/3}$ in it's scaling limit.

1.2. **Percolation.** Percolation models water being poured through a porous material. In the case relevant to SLE, we will model the porous material with a hexagoal

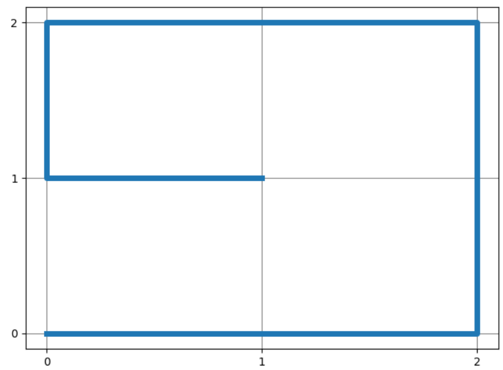

Figure 2. A self-avoiding walk can get trapped if not defined carefully.
Generated by the author.

lattice $\mathbb{T}$. For $p \in (0,1)$, each vertex in $\mathbb{T}$ will be open with probability p and otherwise closed. Two points $x, y \in \mathbb{T}$ are connected if there exists a path consisting only of open vertices connecting them. For $p_c = \frac{1}{2}$, if $p > p_c$, there almost surely exists an infinite set of connected vertices; if $p < p_c$, there almost surely exists no infinite set of connected vertices.

After embedding the hexagonal lattice in the plane and taking the limit of the length of the edges to zero, the critical percolation between two points in a connected domain approaches an SLE_6 curve between the two points.

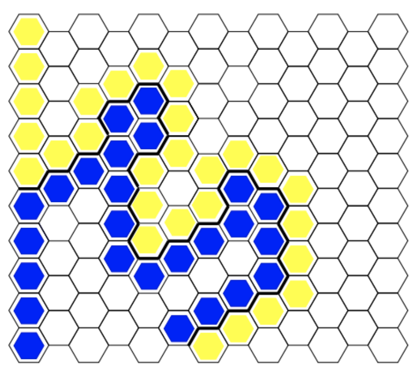

Figure 3. A percolation trace on a hexagonal lattice.
Retreived from [2].

1.3. **Main Results.** This paper will begin in Section 2 by defining martingales and markov processes, which are important tools used to understand SLE, as well as Brownian motion, which is used in the definition of SLE. In Section 3, we will discuss Itô Calculus, focusing on the Itô Lemma, Girsanov's Theorem, and Bessel Processes, all of which are necessary for our results about SLE. Section 4 will define conformal mappings and give the Riemann mapping theorem, which are necessary for defining the Loewner differential equation, as well as defining Green's function, which we will use in our result on the scaling limit of loop erased random walks. In Section 5, we will define the Loewner differential equation (LDE) and discuss the relationship between driving functions, Loewner chains, and curves. In Section 6, we give several more properties about SLE including scaling invariance and the domain Markov property, as well as proving SLE uniquely has these properties. We also give the Rhode-Schramm Theorem, which guarantees the existence of an SLE curve:

Theorem 1. *Let K_t the the random family of compact $\mathbb{H}$ hulls generated by the LDE with a Brownian motion driving function. Then almost surely there exists a curve $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ such that for each t, $H_t := \mathbb{H} \setminus K_t$ is unbounded component of $\mathbb{H} \setminus \gamma_t$.*

Finally, we will define the loop-erased random walk, and give an overview of the proof of the second main result:

Theorem 2. *For a domain A and points $a, b \in \partial A$, the LERW random variable on a domain A' and points $a', b' \in \partial A'$ converging respectively to $A, a,$ and b converges in the scaling limit to the SLE_2 random variable.*

2. Construction and Properties of Brownian Motion

SLE derives its randomness from a Brownian motion in the Loewner differential equation. Brownian motion describes a straightforward physical process: tracking the motion of a single partical bouncing off of a large number of other particles. It is named after Robert Brown, who described the motion of particles of pollen suspended on the surface of water. The existence of Brownian was proven much later by Wiener in 1923. For a more in-depth treatment of these topics, including ommitted proofs in the section, see [6] and [7].

2.1. Filtrations, Markov Processes, and Martingales.

Markov processes and Martingales are types of stochastic processes which place restrictions on future behavior of the process based on the current state of the process. To do so rigorously, especially for continuous time processes, we must introduce the filtrations. Conceptually, filtrations represents the "information" available to a stochastic process. For this section, we will assume all stochastic processes are continuous-time and real-valued.

Definitions 2.1. A **filtration** on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a family $\mathcal{F}_t, t \geq 0$ of σ-algebras with the property that $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ for all $s < t$.

A stochastic process X_t is **adapted** to a filtration $\mathcal{F}_t$ if for all $t \geq 0$, X_t is $\mathcal{F}_t$ measurable.

For a set of random variables $F = \{X_t\}$, the σ-algebra $\sigma(F)$ is the smallest σ-algebra such that each element of F is measurable. A common filtration we will use is $\mathcal{F}_t = \sigma(\{F_s\}_{s<t})$. Conceptually, this filtration represents the information from all the previous steps of the stochastic processes.

The first type of stochastic process we will look at are Markov processes. Markov processes have the property that the probability of a future state depends only on the current state, not on any of the previous steps.

Definition 2.2. A stochastic process X_t is a **Markov process** if for all $A \subset \mathbb{R}$ and $t > s \geq 0$,

$$P(X_t \in A|\sigma(\{X_u\}_{u\leq s})) = P(X_t \in A|X_u).$$

If we let S_n be a simple random walk on $\mathbb{Z}^2$ and define $X_n = S_{n+1} - S_n$, then X_n is a Markov process, since at any point in the integer lattice, X_n has $\frac{1}{4}$ chance of moving in any direction. However, if we define S_n to be the self-avoiding walk and X_n as before, then X_n is no longer a Markov process. Since it must avoid paths it previously created, its previous states do influence its future probabilities.

The next type of stochastic process is the martingale, which is a mathematical model of a fair game. The martingale has the property that at any time, the expectation of the martingale in the future is its current value.

Definition 2.3. A stochastic process X_t adapted to a filtration $\mathcal{F}_t$ is a **martingale** if

(1) $\mathbb{E}(|X_t|) < \infty$ for all $t \geq 0$
(2) $\mathbb{E}(X_t|\mathcal{F}_s) = X_s$ if $t > s$.

The simple random walk is again an example of a martingale, at all times the expectation for the future state is the current state. A stochastic process can be both a martingale and a Markov process, exclusively one or the other, or neither.

Stopping times are random variables which describe a condition when a stochastic process stops. For example, the amount of time it takes a random walk to leave a domain is a stopping time.

Definition 2.4. A random variable τ taking values in $[0, \infty]$ with a filtration $\mathcal{F}_t$ is called a **stopping time** if for all $t \in [0, \infty)$, the event $\{\tau \leq t\}$ is $\mathcal{F}_t$ measurable.

Stopping times are particularly useful for studying martingales because of the optional stopping theorem. If we consider the martingale as a game of chance that stops when some condition is satisfied, at which point we win the value of the martingale (or lose it if the value is negative), the optional stopping theorem tells the expected value of the martingale at the stopping time equals the expected value at the start.

Theorem 2.5. *(Optional Stopping Theorem) Suppose X_n is a martingale adapted to $\mathcal{F}_n$ and τ is a stopping time such that $\mathbb{P}(\tau < \infty) = 1$, $\mathbb{E}[|X_\tau|] < \infty$, and*

$$\lim_{n\to\infty} \mathbb{E}[X_n 1\{\tau > n\}] = 0.$$

Then $\mathbb{E}(X_\tau) = \mathbb{E}(X_0)$.

Proof. First, note that for all finite n,

$$\begin{aligned}
\mathbb{E}[X_{\tau\wedge n}] &= \mathbb{P}[\tau > n]\mathbb{E}[X_n] + \mathbb{P}[\tau \leq n]\mathbb{E}[X_\tau] \\
&= \mathbb{P}[\tau > n]\mathbb{E}[X_n] + \sum_{k=0}^{n} \mathbb{P}[\tau = k]\mathbb{E}[X_k] \\
&= \mathbb{P}[\tau > n]\mathbb{E}[X_0] + \sum_{k=0}^{n} \mathbb{P}[\tau = k]\mathbb{E}[X_0] \\
&= \mathbb{E}[X_0](\mathbb{P}[\tau > n] + \sum_{k=0}^{n} \mathbb{P}[\tau = k]) \\
&= \mathbb{E}[X_0]
\end{aligned}$$

Then for all $n \in \mathbb{N}$, $\mathbb{E}[X_0] = \mathbb{E}[X_{\tau\wedge n}] = \mathbb{E}[X_\tau] + \mathbb{E}[X_{\tau\wedge n} - X_\tau]$. Then it suffices to show that $\lim_{n\to\infty} \mathbb{E}[X_{\tau\wedge n} - X_\tau] = 0$. If $n \leq \tau$, then $X_{\tau\wedge n} = X_\tau$ so $X_{\tau\wedge n} - X_\tau = 0$, otherwise $X_{\tau\wedge n} - X_\tau = X_n - X_\tau$, so we have $X_{\tau\wedge n} - X_\tau = 1\{\tau > n\}X_n - X_\tau$. Taking the limit gives

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{E}[X_{\tau\wedge n} - X_\tau] &= \lim_{n\to\infty} \mathbb{E}[1\{\tau > n\}X_n - X_\tau] \\
&= \lim_{n\to\infty} \mathbb{E}[1\{\tau > n\}X_n] - \lim_{n\to\infty} \mathbb{E}[1\{\tau > n\}X_\tau] \\
&= 0 - X_\tau \lim_{n\to\infty} \mathbb{P}[\tau > n] \\
&= 0 - X_\tau \mathbb{P}[\tau = \infty] \\
&= 0
\end{aligned}$$

Hence, $\mathbb{E}[X_0] = \lim_{n\to\infty} \mathbb{E}[X_\tau] + \mathbb{E}[X_{\tau\wedge n} - X_\tau] = \mathbb{E}[X_\tau]$ as required. □

2.2. **Brownian Motion.** An important example of a stochastic process that is both Markovian and a martingale is Brownian motion. Brownian motion models the path of a single particle bouncing off many other particles. It was first observed by small pollen spores suspended on the surface water, colliding with each other. Brownian motion is a continuous random curve that has independent, normal increments at any scale.

Definition 2.6. A stochastic process B_t is a (linear) **Brownian motion** starting at $x \in \mathbb{R}$ if

(1) $B_0 = x$
(2) B_t has independent increments: for any times $0 \leq t_1 \leq t_2 \leq ... \leq t_n$, the random variables $(B_{t_n} - B_{t_{n-1}}), (B_{t_{n-1}} - B_{t_{n-2}}), \ldots, (B_{t_2} - B_{t_1})$ are independently distributed.
(3) For all $t \geq 0$ and $h > 0$, $B_{t+h} - B_t$ is normally distributed with expectation 0 and variance h.
(4) Almost surely, the function $t \mapsto B_t$ is continuous.

When Brownian motion was first discussed, it was not clear that a random curve with its properties was possible. Lévy introduced a construction of Brownian motion that satisfies the desired conditions.

Construction 2.7. Let $\mathcal{D}_n = \left\{\frac{k}{2^n} : k \in \mathbb{N}\right\}$ and $\mathcal{D} = \bigcup_{n=0}^\infty \mathcal{D}_n$. Define a collection of independent standard normal variables $\{Z_t : t \in \mathcal{D}\}$. We will define a stochastic process $B_t^{(k)}$ for $k \in \mathbb{N}, t \in [0, 1]$. Let

$$B_t^{(0)} = \begin{cases} 0 & t \leq \frac{1}{2} \\ Z_1 & t > \frac{1}{2} \end{cases}$$

For $k > 0$, we will first define $B_t^{(k)}$ on $\mathcal{D}_k$ as follows. For $d \in \mathcal{D}_{k-1}$, let $B_d^{(k)} = B_d^{(k-1)}$. For $d \in \mathcal{D}_k \setminus \mathcal{D}_{k-1}$, let

$$B_d^{(k)} = \frac{B_{d-2^{-k}}^{(k)} + B_{d+2^{-k}}^{(k)}}{2} + \frac{Z_d}{2^{(k+1)/2}}.$$

We linearly interpolate between the adjacent terms already defined, then add an independent random term, scaled appropriately. For $d \in [0, 1] \setminus \mathcal{D}_n$, let d' be the nearest term of $\mathcal{D}_n$ and define $B_d^{(k)} = B_{d'}^{(k)}$.

The Brownian motion B_t is defined as $B_t = \lim_{k\to\infty} B_t^{(k)}$.

To prove that this limit converges, we can almost surely find a uniform bound on the sequence, guaranteeing the limit converges. The next two results we give without proof. They follow from the construction of Brownian motion using independent standard normal variables.

Theorem 2.8. *The stochastic process B_t given by the Lévy construction is almost surely a Brownian motion and not differentiable.*

Proposition 2.9. *If B_t is a Brownian motion and $r > 0$, then $\frac{1}{r}B_{r^2t}$ is also a Brownian motion.*

We refer to the second proposition as Brownian scaling. A final theorem that will be useful in the discussion of SLE is about the uniqueness of Brownian motion.

Theorem 2.10. *Any continuous random curve which has independent increments and Brownian scaling is a Brownian motion.*

This theorem follows from the Lévy-Khintchine formula, which is beyond the scope of this paper.

3. Itô Calculus

This section develops Itô's calculus, which defines integration with respect to a Brownian motion. Itô calculus provides a powerful tool for providing estimates about SLE. These estimates depend on Bessel processes, which can be viewed as a Brownian motion, tilted by the value of a function. Girsanov's theorem is useful in defining and using Bessel processes, since it gives a new probability measures obtained by tilting the measure of the Brownian motion using a martingale. Both these applications rely on stochastic calculus, in particular the Itô integral and lemma.

3.1. Itô's Integral and Lemma.

Itô's integral $Z_t = \int_0^t A_s dB_s$ can be understood as a betting strategy where A_s are bets on the behavior of the Brownian motion B_s. We define the integral similarly to the Riemann integral: first, we define integration for simple processes, that is, processes which consists of a finite number of random variables, then we show that all continuous stochastic processes can be defined as the limit of a sequence of simple processes. Finally, we define the Itô's integral of a continuous process as the limit of the Itô's integrals of the converging simple processes.

Definition 3.1. A_t is a **simple stochastic process** if there exists a partition $0 = t_0 < t_1 < \cdots < t_n < \infty$ and random variables $Y_0, Y_1, \ldots, Y_n$ such that if $\tau \in [t_k, t_{k+1})$, $A_\tau = Y_\tau$. Then we define **Itô's integral** of A_t as

$$\int_0^\tau A_s dB_s = \sum_{i=0}^{k} Y_i[B_{t_{i+1}} - B_{t_i}] + Y_k[B_\tau - B_{t_k}]$$

Sometimes the integral of a simple process is referred to as a discrete stochastic integral. They can be used effectively to model discrete processes, like games or random walks on the integer lattice. The optional stopping theorem is a special case of the discrete stochastic integral where we stop betting when a certain condition, the stopping time, is reached. In general, the discrete stochastic integral is a martingale, as is the stochastic integral itself.

The next step in defining the continuous time Itô's integral is finding, for any random variable A_s, a sequence of simple random variables $A_s^{(k)}$ that converge to A_s.

Lemma 3.2. *Suppose A_t is a stochastic process with continuous paths, adapted to the filtration $\mathcal{F}_t$. Suppose there also exists $C < \infty$ such that almost surely $|A_t| < C$ for all t. Then there exists a sequence of simple stochastic processes $A_t^{(n)}$ such that for all t,*

$$\lim_{n\to\infty} \int_0^t \mathbb{E}[|A_s - A_s^{(n)}|^2]\ ds = 0.$$

A random variable A_t has continuous paths if the mapping $t \mapsto A_t$ is almost surely continuous. Similarly to with Riemann integration, we can define Itô's integral for random variables A_t where the set of discontinuities of $t \mapsto A_t$ is almost surely a null set. However, we will only be concerned with the continuous case in this paper.

Finally, we define Itô's integral for a process A_t as the limit of the integral for the simple processes $A_t^{(k)}$.

Definition 3.3. Suppose A_t is as in Lemma 3.2. Then there exists simple processes $A_t^{(n)}$ which converge to A_t as in Lemma 3.2. Then define **Itô's integral** as

$$\int_0^t A_s\ dB_s = \lim_{n\to\infty} \int_0^t A_s^{(n)}\ dB_s.$$

We can still state Itô's integral in differential form:

$$dZ_t = A_t dB_t.$$

The key result relating to Itô's integral is Itô's lemma, which gives a way to compute the value of Itô's integral, similar to the Fundamental Theorem of Calculus.

There are several different forms of Itô's lemma. A simple version states that if $f : \mathbb{R} \to \mathbb{R}$ is C^2 and B_t is a Brownian motion, then

$$f(B_t) = B_0 + \int_0^t f'(B_s)dB_s + \frac{1}{2}\int_0^t f''(B_s)ds'.$$

A more complicated version is given below:

Theorem 3.4. *Suppose $f(t,x)$ is a real-valued function that is C^1 in t and C^2 in x and $X_t = R_t dt + A_t dB_t$. Then*

$$df(t,X_t) = \partial_t f(t,X_t)dt + \partial_x f(t,X_t)dX_t + \frac{1}{2}\partial_{xx} f(t,X_t)d\langle X\rangle_t.$$

Where

$$\langle X\rangle_t = \lim_{n\to\infty} \sum_{j\le tn} \left[X\left(\frac{j}{n}\right) - X\left(\frac{j-1}{n}\right)\right]^2.$$

$\langle X\rangle_t$ is known as the quadratic variation of X_t. If $X_t = \int_0^t A_s dB_s$ then $\langle X\rangle_t = \int_0^t A_s^2 ds$. If we can find a simple form for $\langle X\rangle_t$, then Theorem 3.4 can be very helpful for evaluating $f(X_t)$, which we will use later to find the solution to the exponential stochastic differential equation (SDE).

The final important part of Itô's Calculus we will consider is when an Itô integral is a martingale. The condition we will require is that A_s be square integrable:

Proposition 3.5. *If A_s is a continuous or piecewise continuous stochastic process and $Z_t = \int_0^t A_s \, dB_s$, then if $\langle Z\rangle_t < \infty$ for all t, Z_t is a martingale.*

Itô's lemma has powerful applications used in the developement of SLE. The applications we will look at in depth are the Girsanov theorem and Bessel processes.

3.2. **Applications of Itô's Lemma.** Girsanov's theorem gives a sense of "weighting" a Brownian motion with a martingales. The martingale M_t must satisfy the exponential SDE

$$dM_t = A_t M_t dB_t, \quad M_0 = 1.$$

We can find M_t such that A_t satisfies the SDE using Itô's formula. We do so by applying Theorem 3.4 to $X_t = \int_0^t A_s \, dB_s - \frac{1}{2}\int_0^t A_s^2 \, ds$ and $f(t,x) = e^x$. Note that

(1) $dX_t = A_t dB_t - \frac{1}{2}A_t^2 dt$,
(2) $d\langle X\rangle_t = A_t^2 dt$,
(3) $\partial_t f(t,x) = 0$, and
(4) $f(t,x) = \partial_x f(t,x) = \partial_{xx} f(t,x) = e^x$.

Then by Theorem 3.4, we have

$$\begin{aligned} de^{X_t} &= \partial_t f(t,X_t) + \partial_x f(t,X_t)dX_t + \frac{1}{2}\partial_{xx} f(t,X_t)d\langle X\rangle_t \\ &= e^{X_t}(A_t dB_t - \frac{1}{2}A_t^2 dt) + \frac{1}{2}e^{X_t}A_t^2 dt \\ &= e^{X_t}A_t dB_t. \end{aligned}$$

Thus, by setting $M_t = e^{X_t}$ we get the desired result. For Girsanov's theorem, we will take as an assumption that M_t is a non-negative martingale.

Theorem 3.6. ***(Girsanov's Theorem)*** *Suppose M_t is a non-negative martingale on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which satisfies the exponential SDE given above. Define $\mathbb{P}^*(V) = \mathbb{E}[1_V M_t]$. If*

$$W_t = B_t - \int_0^t A_s ds$$

then with respect to $\mathbb{P}^*$, W_t is a standard Brownian motion.

With respect to $\mathbb{P}$, W_t is a Brownian motion with a drift of A_t in $\mathbb{P}$.

Definition 3.7. The **Bessel Process** with parameter a is the solution to the SDE

$$dX_t = \frac{a}{X_t}dt + dB_t.$$

We will consider the Bessel process up to a time $T = \inf\{t : X_t = 0\}$, since at that time dX_t isn't well-defined, so the solution X_t is itself undefined. However, with conditions on a, the solution X_t almost surely exists everywhere.

Proposition 3.8. *If $a > \frac{1}{2}$, then $\mathbb{P}\{T = \infty\} = 1$. If $a < \frac{1}{2}$, $\mathbb{P}\{T < \infty\} = 1$.*

In Section 6 we will discuss the connection between Bessel processes and SLE, which makes Itô's calculus an important tool for making estimates about SLE.

We will move away from stochastic processes to discuss the theory of conformal maps. Conformal mapping theory gives background to the Loewner Differential Equation. We will return to stochastic processes and the properties developed here in Section 6, when we discuss SLE.

4. Conformal Mapping Theory

A conformal map $f : D \to f(D)$ is a map which is holomorphic and injective. The Riemann mapping theorem guarantees conformal mapping exist between most domains. The relationship between compact $\mathbb{H}$-hulls and conformal mappings from their complement's are the basis of Loewner differential equation. For a more thorough treatment of this and the following two sections, including ommitted proofs, see [5].

We will begin by defining some key terms or symbols used throughout this section and giving the general statement of the Riemann mapping theorem.

Notations 4.1.
(1) The half-plane $\mathbb{H} = \{a + ib \in \mathbb{C} : b > 0\}$.
(2) The unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.
(3) A domain D is an open, connected set.
(4) D is simply connected if $\mathbb{C} \setminus D$ is also connected.
(5) A point $z \in \partial D$ is a regular point if a Brownian motion starting at z almost surely immediately leaves D.
(6) A set K is a compact $\mathbb{H}$-hull is K is bounded and $\mathbb{H}\setminus K$ is a simply connected domain.
(7) $\hat{\mathbb{C}}$ is the Riemann sphere: $\mathbb{C} \cup \{\infty\}$

Theorem 4.2. *Suppose D is a simply connected domain of $\mathbb{C}$ containing the origin. Then there exists a conformal transformation $f : D \to \mathbb{D}$ with $f(0) = 0, f'(0) > 0$.*

As we will be largely working with domains that do not contain the origin for the Loewner differential equation, we need something beyond Theorem 4.2. The following corollary gives conditions for uniqueness of the conformal transformation.

Corollary 4.3. *Let D be a proper, simply connected domain and let $w \in D$. Then there exists a unique conformal transformation $\phi : D \to \mathbb{D}$ such that $\phi(w) = 0$ and $\arg \phi'(w) = 0$.*

Proof. By the Riemann mapping theorem, there exists a conformal transformation $\phi_0 : D \to \mathbb{D}$. Set $v = \phi_0(w)$ and $\theta = -\arg \phi_0'(w)$. Let $f(z) = e^{i\theta} \frac{z-v}{1-\overline{v}z}$ and define $\phi = f \circ \phi_0$. The $\phi(w) = f(v) = 0$ and

$$\arg \phi'(w) = \arg f(v) + \arg \phi_0(w) = \theta - \theta = 0.$$

Suppose there exists a function φ which also satisfies $\varphi(w) = -\arg \varphi'(w) = 0$. Then $f = \varphi \circ \phi^{-1}$ and its inverse f^{-1} are conformal maps from $\mathbb{D}$ to $\mathbb{D}$ with $f(0) = f^{-1}(0) = 0$ and $\arg f'(0) = \arg {f^{-1}}'(0) = 0$. By the Schwarz lemma, we have

$$|z| = |f^{-1}(f(z))| \leq |f(z)| \leq |z|.$$

It follows that $|f(z)| = |z|$, so applying the Schwarz lemma again, we have $f(z) = \alpha z$ with $|\alpha| = 1$. Then $f'(z) = \alpha$, so $\arg \alpha = 0$ and thus $\alpha = 1$. Hence, f is the identity map, so $\varphi = \phi$. Thus, the map ϕ exists and is unique. □

With certain conditions we can guarantee the form and uniquneness of the conformal transformation between specific types of domains. The following proposition defines a conformal transformation $g_K : \mathbb{H} \setminus K \to \mathbb{H}$ for any compact $\mathbb{H}$-hull K, which will be crucial for the Loewner differential equation.

Theorem 4.4. *Let K be a compact $\mathbb{H}$-hull. There exists a unique conformal transformation $g_K : \mathbb{H} \setminus K \to \mathbb{H}$ such that for $z \in \mathbb{H} \setminus K$, $g_K(z) - z \to 0$ as $|z| \to \infty$ and*

$g_K(z) - z$ is uniformly bounded. Furthermore, there exists $a_K \in \mathbb{R}$ such that g_K is the form

$$g_K(z) = z + \frac{a_K}{z} + O(|z|^{-2}).$$

We will use the notation g_K for a conformal transformation of the form given in Theorem 4.4 through the rest of the paper. Since g_K is unique, this is a well-defined injection from compact $\mathbb{H}$-hulls to conformal transformations. The condition that $g_K(z) - z \to 0$ is called hydrodynamic renormalization.

We need the following lemmas to prove Theorem 4.4. The first we will give without proof. We will provide a definition before the lemma.

Definition 4.5. A **Möbius transformation** is a function f from $\mathbb{C} \cup \infty$ of the form $f(z) = \frac{az+b}{cz+d}$ where $ad - bc \neq 0$.

Lemma 4.6. *Möbius transformations are conformal.*

Before we give the next lemma, we define

$$D^0 \subset \mathbb{R} = \{x \in \mathbb{R} : \text{ there is a } \mathbb{H}\text{-neighborhood of } x \text{ in } D\ \}.$$

Lemma 4.7. *Let $D \subset \mathbb{H}$ be a simply connected domain, $I \subset D^0$ be open, and $x \in I$. Then there exists a unique conformal transformation $\phi : D \to \mathbb{H}$ which extends to a homeomorphism $D \cup I \to \mathbb{H} \cup (-1, 1)$ taking x to 0.*

Proof. Define $D^* = D \cup I \cup \overline{D}$ and $\mathbb{H}^* = \mathbb{H} \cup (-1, 1) \cup \overline{\mathbb{H}}$. These are both simply connected, so by Corollary 4.3, there exists a conformal map $\phi^* : D^* \to \mathbb{H}^*$ with $\phi^*(x) = 0$ and $\arg(\phi^{*\prime})(x) = 0$. Define the reflection $\overline{\varphi(z)} = \phi^*(\overline{z})$. Then $\varphi(z)$ is a conformal transformation from $D^* \to \mathbb{H}^*$ with $\varphi(0) = \varphi'(0) = 0$, so by the uniqueness of ϕ^*, $\varphi = \phi^*$, so ϕ^* is reflection invariant. If $y \in \mathbb{R}$, then $\overline{y} = y$, so $\phi^*(y) = \phi^*(\overline{y}) = \overline{\phi^*(y)}$ and thus $\phi^*(y) \in \mathbb{R}$. It follows that $\phi^*(I) \subset (-1, 1)$. By similar reasoning ${\phi^*}^{-1}(-1, 1) \subset I$, so $\phi^*(I) = (-1, 1)$. Then $\phi^*(D)$ is connected and disjoint from $(-1, 1)$. Since $\arg(\phi^{*\prime})(x) = 0$, $\phi^{*\prime}(x) = \alpha \in \mathbb{R}^+$, since ϕ^* is analytic, in a neighborhood of x, $\phi^*(x + yi) = \alpha yi + O((yi)^2)$. If $y > 0$ is sufficiently small, then $x + yi \in D$ since $x \in I$, so $\phi^*(x + yi) \approx \alpha yi > 0$ and thus $\phi^*(x + yi) \in \mathbb{H}$. It follows that $\phi^*(D) \subset \mathbb{H}$. By similar argument with ${\phi^*}^{-1}$, we get $\phi^*(D) = \mathbb{H}$. Then let ϕ be the restriction of ϕ^* to $D \cup I$. Since ϕ^* was unique, ϕ is unique and has the desired properties.

□

Now, to prove Theorem 4.4. We will use some of the same notation as in Lemma 4.3, including D^0 and D^*.

Proof. Let $D = \{z : -z^{-1} \in \mathbb{H} \setminus K\}$. As K is bounded, $\mathbb{H} \setminus K$ is unbounded and, by definition, open and simply connected, which means D is open, connected, and has 0 in its boundary. Choose an open $I \subset D^0$, where D^0 is defined as in Lemma 4.5, such that $0 \in I$. By Lemma 4.3, there exists a conformal transformation $\phi : D \to \mathbb{H}$ which extends to a conformal transformation ϕ^* on D_I^* with the properties given in Lemma 4.3. Since ϕ^* maps I into $\mathbb{R}$, the coefficients of the Taylor series must be real around 0. So as $z \to 0$,

$$\phi^*(z) = az + bz^2 + cz^3 + O(|z|^4).$$

where $a > 0$, since $\phi^{*\prime}(0) > 0$. Define g_K on $\mathbb{H} \setminus K$ as

$$g_K(z) = -a\phi(-z^{-1})^{-1} - \frac{b}{a}.$$

It remains to check that g_K is a conformal transformation which maps $\mathbb{H} \setminus K$ to $\mathbb{H}$, it is of the desired form at ∞, and it is unique.

To see that g_K is conformal and maps $\mathbb{H} \setminus K$ to $\mathbb{H}$, note that it is can be written as $f \circ \phi \circ h$, where $f(z) = -\frac{bz+a^2}{az} = -\frac{a}{z} - \frac{b}{a}$ and $h(z) = \frac{-1}{z}$. Since $a \neq 0$, $-a^3 \neq 0$, so f is a Möbius transformation onto its image and by Lemma 4.6 is conformal. Similarly, $h(z)$ is a Möbius transformation, so it is conformal onto its image. ϕ is conformal by construction, and thus g_K is conformal. By construction, ϕ maps from $D = \{z : -z^{-1} \in \mathbb{H} \setminus K\}$ to $\mathbb{H}$. Clearly h maps $\mathbb{H} \setminus K$ to D. We will first show $f(\mathbb{H}) \subset \mathbb{H}$. For $x + yi \in \mathbb{H}$,

$$\begin{aligned} f(x+yi) &= \frac{-a}{x+yi} - \frac{b}{a} \\ &= \frac{-a(x-yi)}{x^2+y^2} - \frac{b}{a} \\ &= -\frac{ax}{x^2+y^2} - \frac{b}{a} + \frac{ay}{x^2+y^2} i \end{aligned}$$

As $a, y, (x^2+y^2) > 0$, $\frac{ay}{x^2+y^2} > 0$ which means $-\frac{ax}{x^2+y^2} - \frac{b}{a} + \frac{ay}{x^2+y^2} i \in \mathbb{H}$ and thus $f(x+yi) \in \mathbb{H}$. It remains to prove $\mathbb{H} \subset f(\mathbb{H})$. Let $c + di \in \mathbb{H}$. Let

$$x = -\frac{a^2(ac+b)}{a^2(c^2+d^2) + 2abc + b^2}, \quad y = \frac{a^3 d}{a^2(c^2+d^2) + 2abcb^2}.$$

Then $f(x+yi) = c + di$, so $\mathbb{H} \subset f(\mathbb{H})$ and thus $\mathbb{H} = f(\mathbb{H})$. Therefore, g_K is conformal and takes $\mathbb{H} \setminus K$ to $\mathbb{H}$.

We will prove

$$\lim_{z \to \infty} g_K(z) = z + \left(\frac{b}{a}\right)^2 \frac{1}{z} - \frac{c}{a}\frac{1}{z} + O(|z|^{-2}),$$

giving the desired expansion at ∞ by setting $a_K = \frac{b^2 - ac}{a^2}$. as $z \to \infty$, $-z^{-1} \to 0$, so we can use the expansion of ϕ given earlier: $\phi(z) = az + bz^2 + cz^3 + O(|z|^4)$. Then we have

$$g_K(z) = \frac{-a}{-az^{-1} + bz^{-2} - cz^3 + O(|z|^4)} - \frac{b}{a} = \frac{az^3}{az^2 - bz + c} - \frac{b}{a}.$$

The key observation here is that

$$\frac{az^3}{az^2 - bz + c} = z + \frac{b}{a} + \frac{(b^2 - ac)z - bc}{a^2z^2 - abz + c}.$$

Substituting that into the previous expression for g_K gives

$$g_K(z) = z + \frac{(b^2 - ac)z - bc}{a^2z^2 - abz + c}.$$

At ∞,

$$\frac{(b^2 - ac)z - bc}{a^2z^2 - abz + c} \to \frac{b^2 - ac}{a^2 z} - \frac{bc}{a^2z^2} = \frac{a_K}{z} + O(|z|^{-2}).$$

Finally, we have $g_K(z) \to z + \frac{a_K}{z} + O(|z|^{-2})$ as desired.

It remains to prove uniqueness. We will use the following lemma.

Lemma 4.8. *Let $\phi : \mathbb{H} \to \mathbb{H}$ be a conformal transformation. If $\phi(\infty) = \infty$, then for some $\mu, \sigma \in \mathbb{R}$ with $\sigma > 0$, $\phi(z) = \sigma z + \mu$.*

Suppose $g : \mathbb{H} \setminus K \to \mathbb{H}$ is another conformal transformation with the properties of g_K, then $f = g \circ g_K^{-1}$ is a conformal transformation from $\mathbb{H}$ to $\mathbb{H}$ satisfying $f(\infty) = \infty$. Then by Lemma 4.6, we have $f(z) = \sigma z + \mu$ for some $\sigma > 0, \mu$. It follows that $f(z) = z$, so $g = g_K$ and thus g_K is unique. □

The value a_K gives a notion of the size of the K. It is defined for compact $\mathbb{H}$-hulls and is called the half-plane capacity of K: $\mathrm{hcap}(K)$. Another way to define hcap is to use Brownian motions:

Theorem 4.9.
$$\mathrm{hcap}(K) = \lim_{y\to\infty} y\mathbb{E}_{iy}(Im(B_{T(K)}))$$

Where $T(K)$ is the first time the Brownian motion B_t enters K and $\mathbb{E}_{iy}$ indicates the expectation for a Brownian motion B_t originating at iy.

We will not use this definition going forward, but it is an interesting connection between Brownian motion and conformal maps.

Another, more straightforward, notion of the size of a compact $\mathbb{H}$-hull K is its radius, that is, the radius of the smallest ball centered on the real axis that contains K.

Definition 4.10. The **radius** of a compact $\mathbb{H}$-hull K is the defined as
$$\mathrm{rad}(K) = \inf\{r > 0 : K \subset r\mathbb{D} + x, x \in \mathbb{R}\}.$$

Some properties of hcap and rad we will find useful going forward are as follows:

Proposition 4.11. *Let $K = K_s \subset K_t$ be compact $\mathbb{H}$-hulls. Then:*

1. $\mathrm{hcap}(K_s) \le \mathrm{hcap}(K_t)$,
2. $\mathrm{hcap}(K_t) = \mathrm{hcap}(K_s) + \mathrm{hcap}(g_{K_s}(K_t \setminus K_s))$,
3. $\mathrm{hcap}(K_t) \le \mathrm{rad}(K_t)^2$,
4. $\mathrm{hcap}(K) = 0$ *if and only if* $K = \emptyset$,
5. *For all* $z \in \mathbb{H} \setminus K$, $|g_K(z) - z| \le 3\,\mathrm{rad}(K)$.
6. *Let* $K' = K_s \cup g_{K_s}^{-1}(K_t)$. *Then* $g'_K = g_{K_t} \circ g_{K_s}$.
7. *There exists a constant $C < \infty$ such that for all $r \in (0, \infty)$ and all $U \in \mathbb{R}$, for any compact $\mathbb{H}$-hull $K \subset r\overline{\mathbb{D}} + U$,*
$$\left| g_K(z) - z - \frac{a_K}{z - U} \right| \le \frac{Cra_K}{|z - U|^2}, \quad |z - U| \ge 2r.$$
8. *For* $r \in (0, \infty)$, $g_{rK}(z) = rg_K(z/r)$.
9. *For* $x \in \mathbb{R}$, $g_{K+x}(z) = g_K(z - x) + x$.

In some parts of Loewner theory, it is useful to change how "fast" a family of compact $\mathbb{H}$ hulls K_t runs. We do this by defining a new family $K'_t = K_{\alpha(t)}$, where $\alpha : [0, \infty) \to [0, \infty)$ is strictly increasing and surjective.

Definition 4.12. A family of compact $\mathbb{H}$-hulls K_t has the **standard parametrization** if $\mathrm{hcap}(K_t) = 2t$ for all $t \ge 0$.

5. The Loewner Differential Equation

We will discuss the Loewner differential equation (LDE) next. The LDE establishes a connection in the form of a differential equation between conformal maps, the continuous curves they map to, and the compact $\mathbb{H}$-hulls they map from.

The LDE stems from a correspondence between families of compact $\mathbb{H}$-hulls K_t satisfying the local growth property discussed below, families of conformal transformations $g_t = g_{K_t} : \mathbb{H} \setminus K \to \mathbb{H}$, and continuous curves U_t generated by taking the image of instantaneous increments of K_t, mapped by g_t. This correspondence has two key properties: g_t satisfies the LDE

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}$$

and g_t exists up to the time $T(z) = \inf\{t \geq 0 : z \in K_t\}$ and if $T(z) < \infty$,

$$\lim_{t \to T(z)} g_t(z) - U_t = 0.$$

We will begin with the local growth property:

Definition 5.1. A family of compact $\mathbb{H}$-hulls K_t satisfies the **local growth property** if

(1) $K_s \subsetneq K_t$ when $s \leq t$.
(2) Set $K_{s,t} = g_{K_s}(K_t \setminus K_s)$. Then $\lim_{h\to 0} \operatorname{rad}(K_{t,t+h}) = 0$.

Note that the subset in the first property is strict. As t increases, points are added to K_t. The two conditions are often given separately. A family of hulls with the first condition is referred to as an increasing family of hulls and the local growth property consists solely of the second condition. For the purposes of this paper, we never consider the two conditions separately, se we will refer to them together as the local growth property.

There is a connection between families of compact $\mathbb{H}$-hulls satisfying the local growth property and continuous curves, given in the following proposition.

Proposition 5.2. *Let K_t be a family of compact $\mathbb{H}$-hulls satisfying the local growth property. Then*

(1) $\lim_{h\to 0} K_{t+h} = K_t$,
(2) The map $t \mapsto \operatorname{hcap}(K_t)$ is continuous and strictly increasing,
(3) for all $t \geq 0$, there exists a unique $U_t \in \mathbb{R}$ such that $U_t \in \bigcap_{h>0} \overline{K_{t,t+h}}$, and
(4) The map $t \mapsto U_t$ is continuous.

We will use the properties given in Proposition 4.9 throughout this proof.

Proof. We will prove each part of the proposition separately.

(1) Let $K_{t+} = \lim_{h\to 0} g_{K_t}(K_{t+h} \setminus K_t)$. For all $t \geq 0$ and $h > 0$ we have
$$\operatorname{hcap}(K_{t+h}) = \operatorname{hcap}(K_t) + \operatorname{hcap}(K_{t+h,t})$$
and $\operatorname{hcap}(K_{t+}) \leq \operatorname{hcap}(K_{t,t+h}) \leq \operatorname{rad}(K_{t,t+h})^2$. Since $\operatorname{rad}(K_{t,t+h}) \to 0$ as $h \to 0$, we have $\operatorname{hcap}(K_{t+}) = 0$. It follows that $K_{t+} = \emptyset$, so $\lim_{h\to 0} K_{t,t+h} = \emptyset$ and thus $\lim_{h\to 0} K_{t+h} = K_t$.
(2) Since $K_t \subsetneq K_{t+h}$, $\operatorname{hcap}(K_{t,t+h}) > 0$ and thus $t \mapsto \operatorname{hcap}(K_{t,t+h})$ is continuous and strictly increasing.
(3) For fixed $t \geq 0$, $\overline{K_{t,t+h}}$ are compact and descending, so their intersection contains a single point U_t.

(4) For $h > 0$, choose $z \in K_{t+2h} \setminus K_{t+h}$ and set $w = g_{K_t}(z)$ and $w' = g_{K_{t+h}}(z)$. Then $w \in K_{t,t+2h}$ and $w' \in K_{t+h,t+2h}$. Since $U_t \in g_{K_t}(K_{t+2h} \setminus K_t)$ and $w \in g_{K_t}(K_{t+2h} \setminus K_t)$, $|U_t - w| \leq 2\,\mathrm{rad}(K_{t,t+2h})$. Similarly, $|U_{t+h} - w'| \leq 2\,\mathrm{rad}(K_{t+h,t+2h})$. Furthermore, $g_{K_{t+h}} = g_{K_{t,t+h}} \circ g_{K_t}$, so $w' = g_{K_{t,t+h}}(w)$. Then we have $|w - w'| \leq 3\,\mathrm{rad}(K_{t,t+h})$. Combining these we see

$$\begin{aligned}|U_{t+h} - U_t| &\leq |U_t - w| + |w - w'| + |w' - U_{t+h}| \\ &\leq 2\,\mathrm{rad}(K_{t,t+2h}) + 2\,\mathrm{rad}(K_{t+h,t+2h}) + 3\,\mathrm{rad}(K_{t,t+h})\end{aligned}$$

Since

$$\lim_{h\to 0} 2\,\mathrm{rad}(K_{t,t+2h}) + 2\,\mathrm{rad}(K_{t+h,t+2h}) + 3\,\mathrm{rad}(K_{t,t+h}) = 0,$$

$$\lim_{h\to 0} U_{t+h} - U_t = 0,$$

and thus U_t is continuous.

□

We call the continuous curve U_t the driving function for K_t. The following proposition gives that we can reparametrize K_t to get the standard parametrization, while preserving the local growth property and driving function.

Proposition 5.3. *Suppose K_t is a family of compact $\mathbb{H}$-hulls with the local growth property, with driving function U_t. Let $f(t) = \frac{\mathrm{hcap}(K_t)}{2}$ and let $\tau(t) = f^{-1}(t)$. Then $K_{\tau(t)}$ has the local growth property, the standard parametrization, and has driving function $U_{\tau(t)}$.*

This gives a bijective association between families of compact curves K_t with the local growth properties and their driving functions U_t. We will add a third part of the relationship, families of conformal maps $g_t = g_{K_t} : \mathbb{H} \setminus K_t \to \mathbb{H}$. These families are the solutions to the LDE.

Theorem 5.4. *Let K_t be a family of compact $\mathbb{H}$-hulls, with the local growth property and standard parametrization. Let U_t be the associated driving function. Set $g_t = g_{K_t}$ and $T(z) = \inf\{t \in [0,\infty] : z \in K_t\}$. Then for all $z \in \mathbb{H}$ and for all $t < T(z)$, g_t is differentiable and satisfies the LDE:*

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z.$$

Furthermore, if $T(z) < \infty$, then as $t \to T(z)$, $g_t(z) \to U_t$.

Proof. Fix $z \in \mathbb{H}$ and let $0 \leq s < t < T(z)$. Set $z_t = g_t(z)$. Using the standard parametrization and Proposition 4.9.2, we have

$$\mathrm{hcap}(K_{s,t}) = \mathrm{hcap}(K_t) - \mathrm{hcap}(K_2) = 2(t - s).$$

Additionally, from Proposition 4.9.6, we have $g_{K_t}(z) = g_{K_{s,t}} \circ g_{K_s}(z)$, so $g_{K_{s,t}}(z_s) = z_t$. Since $U_s \in K_{s,t}$ by definition, we have $K_{s,t} \subset U_s + 2\,\mathrm{rad}(K_{s,t})\overline{\mathbb{D}}$. Then by Proposition 4.9.5

$$|z_t - z_s| \leq 3\,\mathrm{rad}(K_{s,t})$$

so by taking $|t - s| \to 0$, by the local growth property $3\,\mathrm{rad}(K_{s,t}) \to 0$ and thus $z_t \to z_s$, so z_t is continuous.

Furthermore, since $z_s \neq U_s$ by taking t sufficiently close to s we can guarantee $|z_s - U_s| \geq 4\,\mathrm{rad}(K_{s,t})$. Then by Proposition 4.9.7 with $K = K_{s,t}$, we get

$$\left| z_t - z_s - \frac{2(t-s)}{z_s - U_s} \right| \leq \frac{4C \operatorname{rad}(K_{s,t})(t-s)}{|z-U|^2}$$

Since $t \neq s$, we can divide by $|t-s|$, then taking the limit as $t \to s$, we see that

$$\lim_{t \to s} \left| \frac{g_t(z) - g_s(z)}{t-s} - \frac{2}{z_s - U_s} \right| \leq \lim_{t \to s} \frac{4C \operatorname{rad}(K_{s,t})}{|z-U|^2} = 0$$

Where the last equality follows from the local growth property. Then $\partial_t g_t(z) = \frac{2}{z_t - U_t}$.

Finally, if $T(z) < \infty$, then for $s < T(z)$, set $t = T(z) - s$. Then $z \in K_t \setminus K_s$, so $z_s \in K_{s,t}$ and thus, since $K_{s,t} \subset U_s + 2\operatorname{rad}(K_{s,t})\overline{\mathbb{D}}$, $|z_s - U_s| \leq 2\operatorname{rad}(K_{s,t})$. Then using the local growth property, we have $|z_s - U_s| \to 0$ as $s \to U(z)$. □

We have completed the correspondence between families of compact $\mathbb{H}$-hulls with the local growth property, driving functions, and Loewner chains. However, so far we have done so starting with the compact $\mathbb{H}$-hulls. The next question we will investigate is whether we can find the same correspondence starting from the driving function or from Loewner chains. The Loewner chains case is, in fact, trivial, since the family of compact $\mathbb{H}$-hulls is the complement of the domains of the conformal maps in the Loewner chain and once we have the family of compact $\mathbb{H}$-hulls, we can use 5.2 to generate the driving function. Starting with driving functions is a little more complicated. For any $z \in \mathbb{C} \setminus \{U_0\}$, we can solve the LDE with driving function U_t by setting

$$g_t(z) = z + \int_0^t \frac{2}{g_s(z) - U_s}\, ds.$$

We define the Loewner chain up to a time

$$T(z) = \{t \geq 0 : |g_s(z) - U_s| > 0 \text{ for all } 0 \leq s < t\}.$$

W

For fixed $t \geq 0$, set $K_t = \{z \in \mathbb{C} : T(z) \leq t\}$. We will generally refer to g_t and K_t restricted to $\mathbb{H}$, but for some proofs it will be more useful to consider a different subset of $\mathbb{C}$, often $(0, \infty)$, which we will refer to as the Loewner flow on $\mathbb{R}$.

Our final step is to prove K_t is the expected family of compact $\mathbb{H}$-hulls.

Theorem 5.5. *For all t, K_t is a compact $\mathbb{H}$-hull which satisfies the local growth property with standard parametrization. Additionally, the driving function generated by K_t is U_t and $g_{K_t} = g_t$.*

A final result useful in the discussion of SLE is as follows:

Proposition 5.6. *Let $g_t(x)$ be the Loewner chain on $\mathbb{R}$. For all $x \in \mathbb{R}$, $x \in \overline{K}_t$ if and only if $T(x) \leq t$.*

This can be proven using the reflection invariant conformal isomorphism given in Lemma 4.5, but will be ommited for brevity.

The following example demonstrates solves an LDE with constant driving function and describes the family of compact $\mathbb{H}$ hulls it generates.

Example 5.7. The solution to the Loewner equation with $U_t = c$ is

$$g_t(z) = \sqrt{c^2 - 2cz + 4t + z^2} + c.$$

This can be confirmed by taking the derivative:

$$\partial_t g_t(z) = \frac{2}{\sqrt{c^2 - 2cz + 4t + z^2}} = \frac{2}{g_t(z) - c}.$$

We can compute that $K_t = \{c + i\sqrt{4s} : s \leq t\}$, by finding the roots to $\sqrt{c^2 - 2cz + 4t + z^2}$.

Now we will add randomness to the LDE by setting the driving funtion to be a Brownian motion. By Theorem 5.5, we can guarantee a solution to this LDE, which is the Schramm-Loewner Evolution (SLE), a probability distribution on the three-part association of compact $\mathbb{H}$-hulls, driving functions, and conformal maps discussed earlier. We will discuss two key facts about SLE: that it is almost surely generated by a continuous curve and that it uniquely satisfies the scale invariance and domain markov properties.

6. The Schramm-Loewner Evolution

The Schramm-Loewner Evolution is defined as the solution to the LDE with with a Brownian motion with variance κ as the driving function.

Definition 6.1. Suppose $\kappa > 0$ and $U_t = -B_t$ is a standard one-dimensional Brownian motion. Let g_t denote the solution to

$$\partial_t g_t(z) = \frac{2/\kappa}{g_t(z) - U_t}, \quad g_0(z) = z.$$

Then g_t is called the **Schramm-Loewner evolution** with parameter κ from 0 to ∞ in $\mathbb{H}$.

By Theorem 5.5, we have a family of compact $\mathbb{H}$-hulls with the local growth property which generate SLE_κ. However, we can further restrict these compact $\mathbb{H}$-hulls to specifically be continuous curves. We will now prove that a curve generates SLE_κ. We mean this in the following sense:

Definition 6.2. A continuous curve γ_t **generates** a family of compact $\mathbb{H}$-hulls K_t if for all times t, $H_t = \mathbb{H} \setminus K_t$ is the unbounded component of $\mathbb{H} \setminus \gamma_{[0,t]}$.

The following theorem, often known as the Rhode-Schramm theorem, proves that a curve generates SLE_κ.

Theorem 6.3. *Let K_t be an SLE_κ family of compact $\mathbb{H}$-hulls for some $\kappa \in [0, \infty)$. Let g_t, U_t the associated Loewner chain and driving function. The map $g_t^{-1} : \mathbb{H} \to H_t$ almost surely extends continuously to $\overline{\mathbb{H}}$ for all $t \geq 0$. Furthermore, if we set $\gamma_t = g_t^{-1}(U_t)$, then almost surely γ_t is continuous and generates K_t.*

We would like to set $\gamma(t) = \lim_{\delta \downarrow 0} g_t^{-1}(B_t + i\delta)$, a heuristic which we will use in our discussion of the loop erased random walk, but it is complicated to prove this limit exists or the resulting function γ_t is continuous. This was originally proven in the $\kappa \neq 8$ case by Rhode and Schramm. The $\kappa = 8$ case is more difficult, but was later proven by Lawler, Schramm, and Werner.

The effect of κ on the SLE_κ curve is to increase how quickly the curve "winds", in a sense. Higher values of κ cause the curve to "wind" faster. We discuss this further in Propisition 6.6.

SLE_κ gives random variables for Loewner chains, hulls, and curves and each $\kappa \geq 0$. We will specify which SLE_κ variable we refer to unless it is clear from the context.

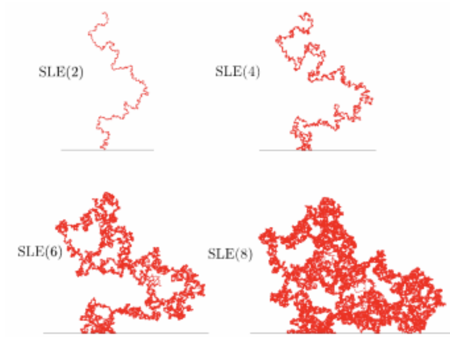

FIGURE 4. Various SLE_κ curves for different values of κ.
Retreived from [3].

We will first look at the the Scale Invariance and Domain Markov Properties, which SLE_κ uniquely posseses. First, we will define the properties and prove SLE_κ posses them.

Proposition 6.4. *Let K_t be the SLE_κ family of compact $\mathbb{H}$ hulls satisfying the local growth property and g_t be the SLE_κ family of conformal maps.*

(1) Scale Invariance: For $\alpha > 0$, let
$$g_t^*(z) = \alpha^{-1/2} g_{\alpha t}(\sqrt{\alpha} t).$$
Then $g_t^* = g_t$.

(2) Domain Markov Property: For $t_0 > 0$, let
$$g_t^*(z) = g_{t+t_0} \circ g_{t_0}^{-1}(z + U_{t_0}) - U_{t_0}.$$
Then for all $t \geq 0$, $g_t^* = g_t$.

This proof involves two straightforward computations.

Proof. (1) We will prove that for all $z \in \mathbb{H}$, $g_t^*(z)$ satisfies the same LDE as $g_t(z)$ with the same initial value:
$$\begin{aligned}
\partial_t g_t^*(z) &= \sqrt{\alpha}\partial_t g_{t'}(\sqrt{\alpha z}) \\
&= \sqrt{\alpha}\frac{2}{g_{\alpha t}(\sqrt{\alpha} z) - U_{\alpha t}} \\
&= \sqrt{\alpha}\frac{2}{\sqrt{\alpha} g_t^*(z) - \sqrt{\alpha} U_t} \\
&= \frac{2}{g_t^*(z) - U_t}
\end{aligned}$$
Where the final equality follows from the scaling property of Brownian motion. Hence, $g_t^*(z)$ satisfies the LDE with the same driving function as $g_t(z)$. Additionally,
$$g_0^*(z) = \alpha^{-1/2}\sqrt{\alpha} = z = g_0(z),$$
so the two have the same initial value. It follows that they are equal.

(2) Will prove the second property similarly. Set $t_0 \geq 0$. Let $z \in \mathbb{H}$ and $z^* = g_{t_0}^{-1}(z + U_{t_0})$ Then:

$$\begin{aligned}\partial_t g_t^*(z) &= \partial_t g_{t+t_0}(z^*)\\ &= \frac{2}{g_{t+t_0}(z^*) - U_{t+t_0}}\\ &= \frac{2}{g_t^*(z) - U_{t+t_0} + U_{t_0}}\\ &= \frac{2}{g_t^*(z) - U_t}\end{aligned}$$

Where the final equality follows the Markov property of Brownian motion. Hence, $g_t^*(z)$ satisfies the LDE with the same driving function as $g_t(z)$. Additionally, $g_0^*(z) = g_{t_0}(g_{t_0}^{-1}(z + U_{t_0})) - U_{t_0} = z + U_{t_0} - U_{t_0} = z$, so the two have the same initial value. It follows that they are equal.

□

We will refer to the set of compact $\mathbb{H}$-hulls with the local growth property as $\mathcal{L}$ and consider both of the above properties as properties of random variables on $\mathcal{L}$. Note that the scaling property for a random $K_t \in \mathcal{L}$ is that for $\lambda > 0$, $K_t^\lambda = \lambda K_{\lambda^{-2}t}$ follows the same distribution as K_t. The domain Markov property gives that for $s \geq 0$ $K_t^{(s)} = g_{K_s}(K_{s+t} \setminus K_s) - U_s$, $K_t^{(s)}$ follows the same distribution as K_t. Our next step will be to prove that any random variable on $\mathcal{L}$ which satisfies the two given properties is SLE_κ.

Theorem 6.5. *Let K_t be a random variable on $\mathcal{L}$. K_t is SLE_κ if and only if it satisfies scale invariance and the domain Markov Property.*

Proof. We have already proven that SLE hulls satisfy scale invariance and the domain Markov property. It remains to prove that if K_t satisfies the scale invariance and domain Markov property then it is SLE, that is, its Loewner chain satisfies the LDE with a Brownian motion driving function. Suppose K_t satisfies the two properties. Let $\lambda \in (0, \infty)$, for any time $t' = \lambda^{-2}t$, we can find the driving function $U_{t'}^\lambda$ for K_t^λ with the following computation

$$\begin{aligned}U_t^\lambda &= \bigcap_{h>0} \overline{g_{\lambda K_t^\lambda}(\lambda(K_{t+h}^\lambda \setminus K_t^\lambda))}\\ &= \bigcap_{h'>0} \lambda\overline{g_{K_{t'}}(K_{t'+h'} \setminus K_{t'})}\\ &= \lambda \bigcap_{h'>0} g_{K_{t'}}(K_{t'+h'} \setminus K_{st}))\\ &= \lambda U_{t'}\\ &= \lambda U_{\lambda^{-2}t}\end{aligned}$$

Where $h' = \lambda^{-2}h$. The second equality follows from Proposition 4.9.8. Since $K_t^\lambda = K_t$, we have $U_t^\lambda = U_t$ and thus $\lambda U_{\lambda^{-2}t} = U_t$. Next, for $s \in [0, \infty)$ we can compute the driving function $U_t^{(s)}$ of $K_t^{(s)}$. Note that for $z \in K_t$

$$\begin{aligned} g_{K_t^{(s)}}(z_t) &= g_{K_{s,t+s}-U_s}(z_t) \\ &= g_{K_{s,t+s}}(z_t + U_s) - U_s \end{aligned}$$

Therefore,

$$\begin{aligned} U_t^{(s)} &= \bigcap_{h>0} \overline{g_{K_{s,t+s}}(K_{s,t+s+h} \setminus K_{s,t+s})} - U_s \\ &= \bigcap_{h>0} \overline{g_{K_{t+s}}(K_{t+s+h} \setminus K_{t+s})} - U_s \\ &= U_{t+s} - U_s \end{aligned}$$

Since $K_t^{(s)} = K_t$, we have $U_t = U_{t+s} - U_s$. It follows that U_t has Brownian scaling and independent increments, so it is a Brownian motion. Then since K_t has a Brownian motion driving function, it is an SLE. $\square$

This unique property of SLE_κ gives us an idea of when we should expect random walks on a discrete lattice to converge in their scaling limit to SLE_κ.

Before we move on to random walks, we will first connect SLE to Bessel processes. Let $Z_t(z) = g_t(z) - U_t$. Then, fixing $z \in \mathbb{H}$,

$$dZ_t = \frac{2/\kappa}{Z_t} dt + dB_t.$$

It follows that Z_t is a Bessel process with parameter $a = \frac{2}{\kappa}$. This is an important tool for making estimates about SLE.

The following section moves on to investigating SLE as a scaling limit of a loop-erased random walk. The random walk is in this case defined on a domain D, with start and end points respectively $z, w \in \partial D$. SLE, as currently defined, exists in the upper $\mathbb{H}$ plane. We will transform it to a probability distribution, called the chordal SLE, on domains in the following way:

Definition 6.6. Let D be a domain with $z, w \in \partial D$. Then the **chordal** SLE_κ on D is a random curve γ_t from z to w defined as $g(\gamma_t')$, where g is a conformal mapping from $\mathbb{H}$ to D with $g(0) = z$ and $g(\infty) = w$ and γ' is an SLE_κ curve in $\mathbb{H}$.

From here, we will discuss the proof that the Loop Erased Random Walk in a domain D from z to w scales to the chordal SLE_2 from z to w.

7. SLE_2 is the Scaling Limit of Loop Erased Random Walk

SLE connects a number of random processes, discussed in the introduction. Schramm first introduced SLE in 2000 as a scaling limit of the loop erased random walk and the uniform spanning tree, with different variances in the Brownian motion driving function. We will give a brief overview of a proof of the scaling limit of the loop erased random walk given by Lawler and Viklund in 2018. In this paper, we will not prove the relationship and bounds given in the proof for the sake brevity. The proofs in full can be found in [8].

We begin by rigourously defining the loop erased random walk.

Definitions 7.1. A **loop erased random walk** is a random walk in a domain $D \subset \mathbb{Z}^2$ starting at $z \in \partial D$ with loops removed in chronological order. Specifically, if $\eta = [\eta_0, \eta_1, ..., \eta_n]$ is a random walk, we create the loop erased random walk $\eta' = [\eta_{i(0)}, \eta_{i(1)}, \dots, \eta_{i(k)}]$ as a subsequence of η as follows:

(1) Let c be the current index. Set $c = 0$ and $i(c) = 0$.
(2) Let $l = \max\{n \geq j \geq i(c), \eta_j = \eta_{i(c)}\}$.
(3) If $l < n$, set $i(c+1) = l + 1$ then set c to $c + 1$ and return to step 2.
(4) If $l = n$, set $k = c$ and stop.

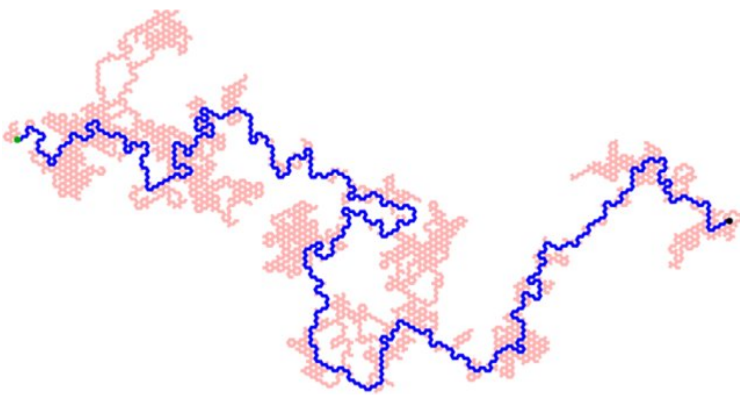

Figure 5. A loop erased random walk on the hexagonal lattice. The light red path is the random walk before loops are erased, the blue path is the loop erased random walk. Retreived from [4].

Before we state the main result, we will define a number of quantities. First off, let $D \subset \mathbb{C}$ be the domain over which we will examine SLE and LERW curves. We let $f', b' \in \partial D$ be the points the curves respectively begin and end at. We consider the integer lattice $\mathbb{Z}^2$, scaled by N^{-1} for $N \in \mathbb{N}$ embedded in the complex plane. Let $A = (N^{-1}\mathbb{Z}^2) \cap D$ be the simply connected set of lattice points in D and define D_A be the subset of $\mathbb{C}$ bounded by ∂A. Let $a, b \in A$ be the points nearest a', b'. Let $\eta \subset A$ be the LERW curve in A from a to b. We parametrize η by taking a conformal map $F : D_A \to \mathbb{H}$ taking a to 0 and b to ∞ and reparametrizing such that $\text{hcap}(F(\eta(t))) = 2t$. Define $R = 4|(F^{-1})'(2i)|$. We are now ready to state the theorem.

Theorem 7.2. *There exists $p_0 > 0$ and for each $p \in (p_0, 1]$ a $q > 0$ such that for the choice of domains and boundary points D, a, b, there exists $N_0 \in \mathbb{N}$ such the followinng hold. Take η as defined above. Choose F such that $R \geq N^p$ for some $N \geq N_0$. Then for each $N \geq N_0$ there exists a coupling of η with a chordal SLE_2 path $\gamma \subset D_A$ from a to b, parametrized in the same way as η. As $N \to \infty$ with probability approaching one the greatest distance between γ and η approaches zero.*

This implies that the LERW converges to SLE_2 because as we take $p < 1$, as $N \to \infty$, we can take $R \to \infty$, which means there is a very large probabilty the distance between η and γ, measured by ρ, is very small. As $N \to \infty$, it can be proven that γ converges to the SLE_2 curve γ' on D, parametrized appropriately.

Thus, as $N \to \infty$, η approaches an SLE_2 curve almost surely, the scaling limit of η is SLE_2.

The key element of the proof is the coupling between the SLE_2. We do so by defining a discrete analog to the SLE_2 process using sections of the LERW path. We start with a subset of the lattice A with points $a, b \in \partial D_A$ as described above and a conformal map $F : D_A \to \mathbb{H}$ with $F(a) = 0, F(b) = \infty$. We also define an LERW curve η as above and let $\eta_j = \eta[0, j]$ for all $j > 0$ and define $A_j = A \setminus \eta_j$ and $D_j = D_{A_j}$. Additionally, we have a constant $h = R^{-2u/3}$, where R is as given above and u is a postive but unspecified constant. Since $u > 0$, as $R \to \infty$, $h \to 0$.

We will define a sequence $m_0, m_1, \dots$ with associated conformal transformations F_{m_n} and $t_{m_n}, r_{m_n} \geq 0$. We will additionally define a sequence of compact $\mathbb{H}$ hulls K^n with associated conformal transformations g^n and g_n and numbers Δ_n for $n \geq 1$. We do so inductively. We let $m_0 = 0$, $t_{m_0} = r_{m_0} = 0$ and $F_0 = F$. Then for $j = 0, 1, \dots$ and $n = 1, 2 \dots$, we define

$$K_j^n = F_{m_{n-1}}(D_{m_{n-1}} \setminus D_{m_{n-1+j}}).$$

We then let

$$\Delta_n = \min\{j \geq 0 : \text{hcap}(K_j^n) \geq h \text{ or } \operatorname{diam}(K_j^n) \geq h\}.$$

We use Δ_n to define $m_n = m_{n-1} + \Delta_n$ and $K^n = K^n_{\Delta n}$. Then we define

$$t_{m_n} = t_{m_{n-1}} + \text{hcap}(K^n), \qquad r_{m_n} = r_{m_{n-1}} + \operatorname{diam}(K^n).$$

Let $g^n : \mathbb{H} \setminus K_n \to \mathbb{H}$ be the conformal transformation with $g^n(z) - z = o(1)$ and set $F_{m_n} = g^n \circ F_{m_{n_1}}$ and $g_n = g^n \circ g^{n-1} \circ ... g^1$. We will only consider this function over the subset of its domain where it is well-defined. Thus finishes our inductive step. We stop induction until n_0, when

$$r_{m_n} \geq 3/2 \qquad \text{or} \qquad t_{m_n} \geq 3/2.$$

We define a_{m_n} as the midpoint of the last edge in $\eta_{t_{m_n}}$ and define a discrete "Loewner process" $U_n = F_{m_n}(a_{m_n}) \in \mathbb{H}$. This process is instrumental to the coupling because it is "close" to a Brownian motion. More specifically, for a standard Brownian motion W_t in the half plane, we can find a sequence of stopping times $\{\tau_n\}$ and a constant $c < \infty$ which, with probability approaching 1 as $R \to \infty$, the following holds:

For $n = 0, 1, ..., n_0$,

(1) $|W_{\tau_n} - U_n| < ch^{1/10}$
(2) $|\tau_n - nh| < ch^{1/5}$
(3) $\max_{\tau_{n-1} \leq t \leq \tau_n} |W_t - W_{\tau_{n-1}}| \leq ch^{2/5}$
(4) $\max_{t \leq \tau_n} \max_{t - h^{1/5} \leq s \leq t} |W_t - W_s| \leq ch^{1/2}$

Intuitively, the "Loewner process" is very close at times n to being a Brownian motion at times τ_n and τ_n is very close to being nh. For times between τ_{n-1} and τ_n, W_t is close to $W_{\tau_{n-1}}$ and for short sections of times, W_t has limited variation.

Note that as $R \to \infty, h \to 0$, so these bounds approach 0. We can now relate this to the SLE_2 curve. Let γ be the SLE_2 curve in $\mathbb{H}$ and $g_t^{\text{SLE}}(z)$ be the associated Loewner chain. Define

$$F_n^{\text{SLE}}(z) = (g_{\tau_n}^{\text{SLE}} \circ F)(z) - W_{\tau_n}$$

and

$$F_n^{\text{LERW}}(z) = (g_n \circ F)(z) - U_n.$$

We can prove there exists $c < \infty$ such that the following holds. For R sufficiently large, with probability approaching 1 as R goes to ∞, for all $z \in A$ with $\operatorname{Im} F_n^{\text{SLE}}(z) \geq h^{1/80}$, we have

$$|F_n^{\text{LERW}}(z) - F_n^{\text{SLE}}(z)| \leq ch^{1/15}$$

Additionally, if $z \in \mathbb{H}$, then

$$|f_n^{\text{LERW}}(z) - f_{\tau_n}^{\text{SLE}}(z)| \leq ch^{1/15}.$$

Where $f_n^{\text{LERW}} = g_n^{-1}$ and $f_{\tau_n}^{\text{SLE}} = (g_{\tau_n}^{\text{SLE}})^{-1}$.

We will also $\delta = h^{1/80}$ and define $u_n \in \mathbb{H}$ as the midpoint of the smallest intervale containing $g^n(\partial K^n)$ and $z_n = f_n^{\text{LERW}}(u_n + i\delta)$. We define scaled variables

$$N^{-1}a_n = \check{a}_n, N^{-1}z_n = \check{z}_n, F(Nz) = \check{F}(z), \text{ and } \check{\gamma} = \check{F}^{-1}(\gamma).$$

The paper proves the following bounds: with probability approaching one as $N \to \infty$, $|\check{a}_n - \check{z}_n|$ and $|\check{z}_n - \check{F}^{-1} \circ f_{\tau_n}^{\text{SLE}}(W_{\tau_n} + i\delta)|$ approaches zero as $N \to \infty$. Additionally, as discussed in the previous section, we expect $g_{\tau_n}^{-1}(W_{\tau_n} + i\delta) \to \gamma(\tau_n)$ as $N \to \infty$, since $\delta \to 0$ as $N \to \infty$. It follows that $|\check{\gamma}(\tau_n) - \check{F}^{-1} \circ f_{\tau_n}^{\text{SLE}}(W_{\tau_n} + i\delta)| \to 0$. Combining these gives that with probability approaching one as $N \to \infty$,

$$|\check{a}_n - \check{\gamma}(\tau_n)| \leq |\check{a}_n - \check{z}_n| + |\check{z}_n - \check{F}^{-1} \circ f_{\tau_n}^{\text{SLE}}(W_{\tau_n} + i\delta)| + |\check{\gamma}(\tau_n) - \check{F}^{-1} \circ f_{\tau_n}^{\text{SLE}}(W_{\tau_n} + i\delta)| \to 0.$$

Intuitively $\check{a}_n$ and $\check{\gamma}(\tau_n)$ are the endpoints of "steps" along respectively the LERW and chordal SLE_2 paths. As $N \to \infty$, the length of these steps approaches zero, so $\check{a}_n$ and $\check{\gamma}(\tau_n)$ approach the curves η and γ themselves. A rigorous proof is given in the paper, but using this heuristic we see that as $N \to \infty$, we expect with probability approaching one the greatest distances between η and γ to approach zero. Thus, the scaling limit of LERW paths is chordal SLE_2.

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References

[1] https://www.quantamagazine.org/a-unified-theory-of-randomness-20160802
[2] https://arxiv.org/pdf/math/0604487.pdf
[3] https://indico.in2p3.fr/event/12461/contributions/9694/attachments/8106/10086/Wu.pdf
[4] https://www.researchgate.net/publication/335290876_Fractal_dimension_of_critical_curves_in_the_On-symmetric_phi4-model_and_crossover_exponent_at_6-loop_order_Loop-erased_random_walks_self-avoiding_walks_Ising_XY_and_Heisenberg_models/figures
[5] Norris, James; Berestycki, Nathanaël. "Lectures on Schramm-Loewner Evolution". (16 January, 2016).
Retreived from http://www.statslab.cam.ac.uk/ james/Lectures/sle.pdf.
[6] Roch, Sebastian. "Lecture Notes on Measure-theoretic Probability". (2010).
Retreived from http://www.math.wisc.edu/ roch/grad-prob/.
[7] Durrett, Rick. *Probability Theory and Examples, Fourth Edition.* Cambridge Series in Statistical and Probabilistic Mathematics. (2010).
[8] Lawler, G. F. and Viklund, F., "The Loewner difference equation and convergence of loop-erased random walk", *arXiv e-prints.* (2018).
Retreived from https://arxiv.org/abs/1611.01406.