THE SCHWARZIAN DERIVATIVE IN ONE-DIMENSIONAL DYNAMICS

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Abstract. We review elementary definitions of topological dynamics and introduce the Schwarzian derivative. In particular, we examine properties of maps with Schwarzian derivative everywhere negative, and we show, following Singer, that if such a map admits n critical points, then it has at most $n+2$ attracting orbits.

Contents

1. Introduction

In dynamics, we study the behavior of a system under iteration. We are particularly interested in asymptotic behavior—that is, whether points approach a fixed value or set of values under iteration of the system. We focus here on systems modeled by continuous maps $f : \mathbb{R} \to \mathbb{R}$.

For any sufficiently smooth map f, we can define a function called the Schwarzian derivative of f, which was introduced to one-dimensional dynamics in 1978 by David Singer. The Schwarzian is useful in dynamics because we can classify maps by the *sign* of their Schwarzian derivative. In particular, the dynamics of maps with *negative* Schwarzian are well understood and serve as a model for all of one-dimensional dynamics. Indeed, any piecewise monotonic system can be modeled by a polynomial with negative Schwarzian derivative [2].

Before expanding upon the dynamical role of the Schwarzian, we first review, in §2, the basic definitions of one-dimensional dynamics. Then, in §3, we introduce the Schwarzian derivative and study, in particular, maps with negative Schwarzian derivative. Finally, in §4, we show that maps with negative Schwarzian have, at most, two more attracting orbits than critical points, following [6] and [3].

2. Basic Definitions

In this paper, we will study the dynamics of continuous maps $f : \mathbb{R} \to \mathbb{R}$. The behavior of any point $x \in \mathbb{R}$ under iteration of f is given by the sequence

$x, f(x), f(f(x)), f(f(f(x))), \ldots$, which is called the *orbit* of x. We denote the points in this orbit $x, f(x), f^2(x), f^3(x)$, and so on.

The simplest orbits to understand are those of *fixed points* and *periodic points*. A point p is called a *fixed point* for f if $f(p) = p$. Note that this implies $f^n(p) = p$ for all $n \in \mathbb{N}$. Similarly, a point q is called a *periodic point* of period m for f if $f^m(q) = q$. If m is the least positive integer with $f^m(q) = q$, then m is called the *prime period* of q. By convention, when we say q is periodic of period m, we assume that m is the prime period of q. Note that q is also a fixed point for f^m and that fixed points are periodic of period one.

We can graphically interpret fixed points for f as intersections of the graph of f with the line $y = x$, which we call *the diagonal*. Periodic points of period m are then intersections of f^m with the diagonal.

We can start to say more about the orbits of *non-periodic* points by studying the local behavior of f near fixed and periodic points. We can examine this local behavior using the derivative of f, which leads us to the following definitions.

Definition 2.1. A periodic point $p \in \mathbb{R}$ of period m for f is *hyperbolic* if $|(f^m)'(p)| \neq 1$. The point p is called a *hyperbolic attractor* if $|(f^m)'(p)| < 1$ or a *hyperbolic repellor* if $|(f^m)'(p)| > 1$.

We will discuss *non-hyperbolic attractors* and *repellors*, that is, points p with $|(f^m)'(p)| = 1$, at the end of this section. Note that when we say "p is an *attractor*" or "p is a *repellor*," the point p can be either hyperbolic or non-hyperbolic. We may also use the terms *attracting point* and *repelling point* instead of attractor and repellor.

We now give some intuition for Definition 2.1 via an example.

Example 2.2. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = 3x^3/2 - x/2$. The graph of g intersects the diagonal at three points, $x = -1, 0,$ and 1, which are the fixed points of g (see Fig. 1). Computing $g'(x) = 9x^2/2 - 1/2$, we see that $g'(0) = -1/2$ and $g'(\pm 1) = 4$. By Definition 2.1, the point $x = 0$ is a hyperbolic attractor, and the points $x = \pm 1$ are hyperbolic repellors.

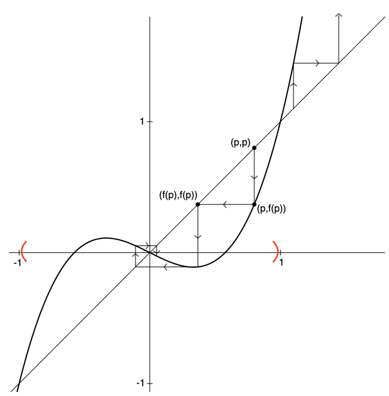


FIGURE 1. The graph of $g(x) = 3x^3/2 - x/2$ from Examples 2.2 and 2.5. The boundary of $W^s(0)$ is marked in red. Arrows indicate graphical analysis of g.

To see this "attraction" and "repulsion" empirically, we can compute the first few iterates in the orbits of $x = -0.1$ and $x = 1.01$. Notice that the iterates of $x = -0.1$ decrease in magnitude approximately by a factor of $g'(0) = 1/2$ after each iteration. Similarly, the distance $|g^k(1.01) - 1|$ between the fixed point $x = 1$ and the iterates of $x = 1.01$ increases approximately by a factor of $g'(1) = 4$ with each iteration.

More generally, if a point x is close enough to a fixed hyperbolic attractor p for a map f, then $f^n(x)$ approaches p as n becomes large. Similarly, if x is sufficiently close to a fixed hyperbolic repellor q, then x is "pushed away" from q under iteration of f; that is, there exist $\epsilon, N > 0$ such that $|x - q| < \epsilon$ implies $\epsilon < |f^N(x) - q|$. We show this rigorously in Propositions 2.3 and 2.4, both of which apply readily to hyperbolic periodic points of period m for f by replacing f with f^m.

Proposition 2.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable. If $p \in \mathbb{R}$ is a fixed hyperbolic attractor for f, then there is an open interval U containing p such that $\lim_{n\to\infty} |f^n(x) - p| = 0$ for all $x \in U$.*

Proof. The point p is a fixed hyperbolic attractor, so we know $|f'(p)| < 1$. In particular, there exists $\lambda \in \mathbb{R}$ such that $|f'(p)| < \lambda < 1$. By continuity of f', there is an open interval U containing p such that $|f'(x)| < \lambda$ for all x in U. Now, fix $x \in U$. We need to show that the sequence of distances $\{|f^n(x) - p|\}_{n\in\mathbb{N}}$ converges to zero. We do this via induction on n.

For the base case, we apply the Mean Value Theorem to the interval with endpoints x and p and find, for some $\xi \in U$ between x and p,

$$|f(x) - p| = |f(x) - f(p)| = f'(\xi)|x - p| < \lambda|x - p|.$$

Notice that U contains $f(x)$ since we have $|f(x) - p| < |x - p|$.

Now, assume $|f^k(x) - p| < \lambda^k|x - p|$ and $f^k(x) \in U$ for some k. Applying the Mean Value Theorem to the interval between $f^k(x)$ and p, we find

$$|f^{k+1}(x) - p| < \lambda|f^k(x) - p| < \lambda^{k+1}|x - p|,$$

where the second inequality follows from the inductive hypothesis. As before, the interval U contains $f^{k+1}(x)$ as we have $|f^{k+1}(x) - p| < |f^k(x) - p|$. We therefore have $|f^n(x) - p| < \lambda^n|x - p|$ for all n. Finally, since $|\lambda| < 1$, we conclude

$$\lim_{n\to\infty} |f^n(x) - p| \leq \lim_{n\to\infty} \lambda^n|x - p| = 0.$$

□

Now, we take a similar approach to formalize the intuition for fixed hyperbolic repellors.

Proposition 2.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable. If $p \in \mathbb{R}$ is a fixed hyperbolic repellor for f, then there is an open interval V containing p such that, for each $x \in V \setminus \{p\}$, there is $N > 0$ such that $f^N(x) \notin V$.*

Proof. For $\epsilon > 0$, define the interval $V := (p - \epsilon, p + \epsilon)$. Choose ϵ and $\nu > \lambda > 1$ so that $\nu > |f'(x)| > \lambda > 1$ for all $x \in V \setminus \{p\}$. We can do this because f' is bounded and continuous on V. Now, fix $x \in V$. By the argument from the proof of Proposition 2.3, we find that if $f^{n-1}(x) \in V$, then

$$\nu^n|x - p| > |f^n(x) - p| > \lambda^n|x - p|$$

holds for each $n \in \mathbb{N}$. Now, since $\lambda, \nu > 1$, we may choose M to be the least positive integer that satisfies $\lambda^M|x-p| > \epsilon$ and K to be the greatest positive integer that satisfies $\nu^K|x-p| < \epsilon$. By our choice of M, we have $f^M(x) \notin V$ but perhaps not $f^{M-1}(x) \in V$. However, our choice of K implies $f^k(x) \in V$ for all $k \leq K$, since

$$|f^k(x) - p| < \nu^k|x-p| \leq \nu^K|x-p| < \epsilon.$$

It follows that there is some $K < N \leq M$ such that $f^{N-1}(x) \in V$ and $f^N(x) \notin V$, which completes the proof. □

Propositions 2.3 and 2.4 show that hyperbolic points control the dynamics of the points near them. In particular, if a point p is a hyperbolic m-periodic attractor for f, then there is a neighborhood U of p whose points are all *forward asymptotic* to p under f^m; that is, we have $f^{mn}(x) \to p$ as $n \to \infty$ for all x in U. Such a set U is called a *local stable set* of p and is denoted $W^s_{loc}(p)$. We then can define *the stable set* of p, denoted $W^s(p)$, as the set of points forward asymptotic to p, or equivalently as the set of points mapped to $W^s_{loc}(p)$ under iteration of f^m.

We also define the *stable set of the orbit* of a periodic attractor p as the union of the stable sets of all points in the orbit of p. This union contains every point whose orbit "approaches" the orbit of p. More precisely, if a point x belongs to the stable set of the orbit of p, then the sequence of iterates of x becomes arbitrarily close to the sequence of iterates of p, up to a change of indices.

We now show that stable sets are open, which we will use in §4. To see this, consider a local stable set of a hyperbolic attractor p of period m. From the proof of Proposition 2.3, we know that any open neighborhood U of p such that $|(f^m)'(x)| < 1$ for all x in U is a local stable set of p. Since all points in the stable set of p are mapped into U under iteration of f^m, the stable set of p is the union of all preimages of U under iteration of f^m. That is,

$$W^s(p) = \bigcup_{k=1}^{\infty} \{x | f^{km}(x) \in U\}.$$

Finally, since preimages of open sets under continuous maps are open, the stable set $W^s(p)$ is open.

We now give two examples to make the definition of the stable set more concrete.

Example 2.5. Recall the map $g(x) = 3x^3/2 - x/2$ from Example 2.2. As we saw, g has a fixed hyperbolic attractor at $x = 0$ and two fixed hyperbolic repellors at $x = \pm 1$. We can visualize the iterates of any point by drawing a series of lines that extend vertically to the graph of g from the diagonal and then extend horizontally from the graph of g back to the diagonal (see Figure 1). This path begins at some point (p, p) and then travels vertically to $(p, f(p))$, horizontally to $(f(p), f(p))$, and then vertically to $(f(p), f^2(p))$. Repeating this process, we can find any iterate of p on the graph of g. This powerful technique is called *graphical analysis*, and, using it, we find that the stable set of the fixed point $x = 0$ is $W^s(0) = (-1, 1)$. Note that $W^s(0)$ consists of a single open interval that contains the fixed point $x = 0$. In general, stable sets may consist of several disconnected components, as we will now see.

Example 2.6. Consider $h : \mathbb{R} \to \mathbb{R}$ given by $h(x) = x^2 - 1$. The map h has one periodic orbit of period two, consisting of the points $p_1 = -1$ and $p_2 = 0$. This orbit is an attracting orbit, as $(h^2)'(0) = (h^2)'(-1) = 0$. We can use graphical

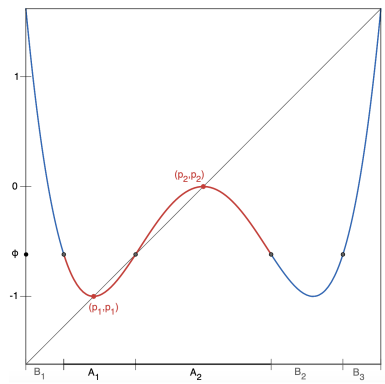


FIGURE 2. The components of $W^s(p_1 \cup p_2)$ that contain (red) or do not contain (blue) the periodic points $p_1 = -1$ and $p_2 = 0$ for $h(x) = x^2 - 1$, shown on the graph of h^2 (see Example 2.6). Points in grey are not in $W^s(p_1 \cup p_2)$.

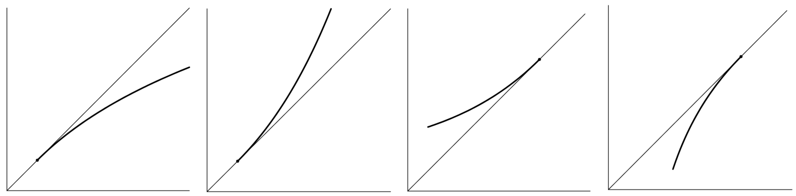
analysis on the graph of h^2 to find the stable set $W^s(p_1 \cup p_2)$ of this orbit. We find that $W^s(p_1 \cup p_2)$ consists of several separate components, only two of which contain either p_1 or p_2. These two components of $W^s(p_1 \cup p_2)$ are labeled A_1 and A_2, respectively, and are depicted in Figure 2. The boundary points of A_1 and A_2 are the repelling fixed point $\phi = (1 - \sqrt{5})/2$ and points mapped to ϕ under h^2, none of which are asymptotic to p_1 or p_2.

We conclude with a discussion of non-hyperbolic periodic points, that is, periodic points p of period m with $|(f^m)'(p)| = 1$. Unlike hyperbolic points, which attract or repel on both sides, non-hyperbolic points may attract or repel on only one side or on both sides. These four possibilities are depicted in Figure 3 for the case when $(f^m)'(p) = 1$. A non-hyperbolic attractor which attracts on only one side is called *one-sided*. A *two-sided attractor* may be either a non-hyperbolic attractor that attracts on both sides or a hyperbolic attractor. The broad terms “attractor” and “attracting point” may denote a one-sided attractor or a two-sided attractor.

Local dynamical behavior near non-hyperbolic points is analogous to local behavior near hyperbolic points. For example, if p is a one-sided m-periodic attractor that attracts on the right, then the inequality $|f^m(x) - p| < |x - p|$ holds for $x > p$ sufficiently close to p. If p were a two-sided non-hyperbolic attractor, then the inequality would hold also for $x < p$ sufficiently close to p. Non-hyperbolic attractors also have local stable sets. When p is two-sided, $W^s_{loc}(p)$ is an open interval $(p - \epsilon, p + \epsilon)$, which implies $W^s(p)$ is open as in the hyperbolic case. When p is one-sided, $W^s_{loc}(p)$ is a half-closed, half-open interval given either by $(p - \epsilon, p]$ or $[p, p + \epsilon)$, depending on whether p attracts on the left or the right. In the one-sided case, $W^s(p)$ need not be open.

3. The Schwarzian Derivative

Our focus in this section is on the Schwarzian derivative and, in particular, on maps with everywhere negative Schwarzian derivative.



(A) (B) (C) (D)

FIGURE 3. Weak repellors and attractors. The point in (A) attracts on the right; (B) repels on the right; (C) attracts on the left; (D) repels on the left. These designations can be verified with graphical analysis.

Definition 3.1. Let $f : \mathbb{R} \to \mathbb{R}$ be a C^3 map* such that $f' \not\equiv 0$ (which we will assume for the remainder of this paper). The *Schwarzian derivative* of f at $x \in \mathbb{R}$ is defined as

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2.$$

We write $Sf < 0$ if we have $Sf(x) < 0$ for all x in the domain of Sf.

Maps with negative Schwarzian derivative are fairly common. For example, we have $S(\sin x) = -1 - \frac{3}{2}(\tan x)^2 < 0$ and $S(x^2 - 1) = -3/2x^2 < 0$. Indeed, any polynomial P with distinct, real roots has $SP < 0$ (see [3], §1.11).

The Schwarzian derivative is useful in the study of one-dimensional dynamical systems because its sign is preserved under composition.

Proposition 3.2. *Take $f, g \in C^3(\mathbb{R})$ with $Sf < 0$ and $Sg < 0$. Then $S(f \circ g) < 0$.*

Proof. Let us first compute

$$(f \circ g)'' = (g' \cdot (f' \circ g))' = (g')^2 \cdot (f'' \circ g) + g'' \cdot (f' \circ g)$$

and

$$(f \circ g)''' = (g')^3 \cdot (f''' \circ g) + 3g'g'' \cdot (f'' \circ g) + g''' \cdot (f' \circ g).$$

Now we have

$$\begin{aligned} S(f \circ g) &= (g')^2 \frac{f''' \circ g}{f' \circ g} + 3g'' \frac{f'' \circ g}{f' \circ g} + \frac{g'''}{g'} - \frac{3}{2}\left(g' \frac{f'' \circ g}{f' \circ g} + \frac{g''}{g'}\right)^2 \\ &= (g')^2 \left[\frac{f''' \circ g}{f' \circ g} - \frac{3}{2}\left(\frac{f'' \circ g}{f' \circ g}\right)^2\right] + \left[\frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2\right] \\ &= (g')^2 \cdot (Sf \circ g) + Sg < 0. \end{aligned}$$

□

Remark 3.3. An immediate corollary of this result is that if $Sf < 0$, then $Sf^n < 0$ for all $n > 0$. This is the reason we can use the Schwarzian in dynamics.

For the remainder of this section, we will discuss analytic properties of maps with negative Schwarzian, particularly those related to critical points. By Lemma 3.2, we know that the iterates of maps with negative Schwarzian inherit these analytic properties.

*A map $f : \mathbb{R} \to \mathbb{R}$ is of class $C^3(\mathbb{R})$ if f', f'', and f''' exist and are continuous.

Lemma 3.4. *Take $f \in C^3(\mathbb{R})$ such that $Sf < 0$. If f' has a critical point $c \in \mathbb{R}$, then $f'(c)$ is neither a positive local minimum value nor a negative local maximum value for f'. In particular, if c is a local minimum point for f' on $\mathbb{R}$, then $f'(c) \leq 0$, and if c is a local maximum point for f' on $\mathbb{R}$, then $f'(c) \geq 0$.*

Proof. Since we have $f''(c) = 0$, the Schwarzian of f at c is

$$Sf(c) = \frac{f'''(c)}{f'(c)} < 0.$$

Hence $f'''(c)$ and $f'(c)$ must have opposite signs. By the Second Derivative Test, $f'(c)$ is neither a negative local maximum value nor a positive local minimum value for f'. □

Remark 3.5. Though Lemma 3.4 concerns critical points of f', the lemma is also relevant to critical points of f. To see this, suppose f' admits a local minimum point y on an open set U and has $f'(x) > 0$ for some $x \in U$. Then, by the Intermediate Value Theorem, f has at least one critical point between x and y, since $f'(y) \leq 0$. The following corollary is a useful reformulation of this fact.

Corollary 3.6. *Let $f \in C^3(\mathbb{R})$ satisfy $Sf < 0$. If there are points $x_1 < x_2 < x_3$ such that $0 < f'(x_2) \leq f'(x_1) = f'(x_3)$, then f admits a critical point in (x_1, x_3).*

Proof. By Remark 3.5, it suffices to show that f' admits a local minimum point on (x_1, x_3) since we already have $f'(x_3) > 0$. We will consider separately the cases when $f'(x_2) = f'(x_3)$ and when $f'(x_2) < f'(x_3)$.

First, consider the case when $f'(x_1) = f'(x_2) = f'(x_3) =: C$. Define $I_1 := [x_1, x_2]$. By continuity, f' must attain a maximum and a minimum value on I_1. Moreover, we cannot have $f' \equiv C$ on $[x_1, x_3] \supset I_1$, as that would imply $Sf \equiv 0$, which contradicts $Sf < 0$. It follows that either the maximum value or the minimum value of f' on I_1 is not equal to C. The same is true of f' on $I_2 := [x_2, x_3]$. Now we can take $\xi \in I_1$ and $\eta \in I_2$ to be extremum points of f' that are not equal to C. Moreover, this means that ξ and η are not equal to x_1, x_2, or x_3. If at least one of ξ and η is a local minimum point, then we are done. Suppose ξ and η are both local maximum points. By continuity, f' must attain a maximum and a minimum value on $[\xi, \eta]$. We know that x_2 is strictly between ξ and η and that $f'(x_2)$ is smaller than both $f'(\xi)$ and $f'(\eta)$. Hence f' has a local minimum point in $(\xi, \eta) \subset (x_1, x_3)$.

The second case is when we have $f'(x_2) < f'(x_1) = f'(x_3)$. Since the minimum value of f' on $[x_1, x_3]$ is smaller than $f'(x_1) = f'(x_3)$ and is therefore attained on (x_1, x_3), it follows that f' has a local minimum point in (x_1, x_3). □

Remark 3.7. Corollary 3.6 also holds for $f'(x_2) < 0$, since by the Intermediate Value Theorem, there exists $x' \in (x_1, x_3)$ such that $f'(x') = 0$. Hence x' is a critical point of f.

We will use Corollary 3.6 many times in the next section. We conclude our discussion of the Schwarzian with an example that alludes to the relationship between critical points and stable sets for maps that have negative Schwarzian. This relationship is at the heart of Singer's Theorem.

Example 3.8. Consider the family of functions $F_\mu(x) = \mu x(1 - x)$, for $2 < \mu < 3$. We compute $F'_\mu(x) = \mu(1 - 2x)$ and $F''_\mu(x) = -2\mu$. Hence we have

$$SF_\mu(x) = -\frac{6}{(1 - 2x)^2} < 0.$$

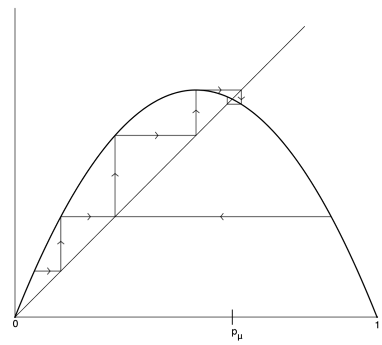


FIGURE 4. Graphical analysis of $F_\mu(x) = \mu x(1-x)$ for $2 < \mu < 3$.

Now, note that F_μ has fixed points at $x = 0$ and $p_\mu := (\mu - 1)/\mu$. Both fixed points are hyperbolic: the point $x = 0$ is repelling with $F'_\mu(0) = \mu > 1$, and p_μ is attracting with $|F'_\mu(p_\mu)| = |2 - \mu| < 1$. One can check using graphical analysis that the stable set of p_μ is the interval $(0, 1)$ (see Fig. 4). It follows that F_μ admits no other periodic points in that interval, for those periodic points would not be asymptotic to p_μ and therefore could not be in the stable set of p_μ. In particular, the orbit of p_μ is the only attracting orbit of F_μ, and the only critical point of F_μ, $x = 1/2$, lies in the stable set of p_μ. We shall see in the next section why this must be so.

4. Singer's Theorem

The goal of this section is to prove the following theorem due to Singer.

Theorem 4.1 (Singer)**.** *If $f : \mathbb{R} \to \mathbb{R}$ has $Sf < 0$, then the stable set of every attracting orbit for f either contains a critical point of f or is unbounded.*

We will also prove the following corollary of Theorem 4.1.

Corollary 4.2. *Suppose $f : \mathbb{R} \to \mathbb{R}$ has $Sf < 0$. If f has n critical points, then f admits at most $n + 2$ attracting orbits.*

To prove the theorem, we will first define the semi-local stable set of a periodic attractor p of period m. We will next use Corollary 3.6 to show that if the semi-local stable set of p is bounded, then it contains a critical point of f^m. We will relate this critical point of f^m to a critical point of f by considering the whole orbit of p. To prove the corollary, we will show that f may have up to two attractors that have unbounded stable sets. From now on, $f : \mathbb{R} \to \mathbb{R}$ will denote a C^3 map that has negative Schwarzian derivative.

Before embarking on the proof of Singer's Theorem, we show that periodic points of the same period for f are isolated, which ensures the (semi-)local stable set is defined for all periodic attractors of f.

Lemma 4.3. *If $f : \mathbb{R} \to \mathbb{R}$ has $Sf < 0$, then periodic points of the same period for f are isolated.*

Proof. We will consider two cases. First, we show that f cannot have an interval of periodic points of the same period. Indeed, if there were an interval of points that were all m-periodic for f, then f^m would be identically equal to x, and Sf^m would be identically equal to 0 on that interval, which contradicts $Sf < 0$.

Now we show that fixed points for f^m cannot be dense in $\mathbb{R}$. For the sake of contradiction, suppose f^m admits a convergent sequence of fixed points x_n, and assume, without loss of generality, that this sequence increases monotonically. Consider four points $x_i < x_{i+1} < x_{i+2} < x_{i+3}$. Since each of these is fixed, we have $f^m(x_{j+1}) - f^m(x_j) = x_{j+1} - x_j$ for $i \leq j \leq i+2$. Hence, by the Mean Value Theorem, there are points $x_j < \xi_j < x_{j+1}$ such that $(f^m)'(\xi_j) = 1$, for $i \leq j \leq i+2$. Now, by Proposition 3.2, we can apply Corollary 3.6 to the points $\xi_i < \xi_{i+1} < \xi_{i+2}$, and we find that f^m must admit a critical point in (ξ_i, ξ_{i+2}). We can now define a sequence $\{c_n\}_{n=0}^{\infty}$ such that c_n is a critical point of f^m and $\xi_{3n} < c_n < \xi_{3n+2}$ for each n. By construction, the points x_n, ξ_n, and c_n all converge to the same limit point. However, since we have $(f^m)'(\xi_n) = 1$ and $(f^m)'(c_n) = 0$ for all n, it follows that f' is not continuous at this limit point. This contradicts the assumption that f is C^3. □

Remark. The proof of Lemma 4.3 highlights the relationship between critical points and fixed points for maps with negative Schwarzian. In particular, we showed that whenever there is a sequence of fixed points for f, there is also a sequence of critical points. The key step was finding three points to which we could apply Corollary 3.6. In the proof of Singer's Theorem, we will see how the structure of stable sets of periodic attractors generates a similar three points.

Now that we know periodic points of the same period are isolated, we can consider neighborhoods of periodic points that do not contain periodic points of the same period. In particular, we can consider a local stable set of a periodic attractor, which allows us to define the attractor's semi-local stable set. Informally, the semi-local stable set of an attractor is the largest possible local stable set of that attractor.

We define the semi-local stable set separately for two-sided and one-sided attractors. If p is a two-sided m-periodic attractor, then the *semi-local stable set* of p, denoted $W^s_{sl}(p)$, is the largest connected component of $W^s(p)$ that contains p. Moreover, $W^s_{sl}(p)$ is an open interval (a, b) since it shares its boundary with $W^s(p)$, which is open. Recall that $W^s(p)$, the stable set of p, is the set of points x satisfying $\lim_{n\to\infty} f^{mn}(x) = p$ and that the stable set of the orbit of p is the union of the stable sets $W^s(f^i(p))$ of all iterates $f^i(p)$ in the orbit of p. The *semi-local stable set of the orbit* of p is defined similarly as a union of the sets $W^s_{sl}(f^i(p))$.

If p is a one-sided periodic attractor which attracts on the right (or left), then we define $W^s_{sl}(p)$ to be the largest half-closed, half-open interval $[p, b)$ (or $(a, p]$) such that all points in $W^s_{sl}(p)$ are asymptotic to p.

Example 4.4. Consider $h(x) = x^2 - 1$ from Example 2.6, and recall that $x = 0$ is a hyperbolic attractor of period two for h. The semi-local stable set $W^s_{sl}(0)$ is the set we called A_2 (see Fig. 2).

We are at last equipped to prove Singer's Theorem.

Proof of Theorem 4.1. Assume $f : \mathbb{R} \to \mathbb{R}$ has $Sf < 0$. We need to show that the stable set of any attracting orbit for f either contains a critical point of f or is unbounded. An equivalent conclusion is that if the stable set of an attracting orbit for f is bounded, then it contains a critical point of f. We will actually go further and show that this conclusion holds for an attracting orbit's semi-local stable set. Note that if the stable set of an attracting orbit is bounded then so is the semi-local stable set. Hence, by showing that bounded semi-local stable sets must contain a critical point of f, we show that the same is true for bounded stable sets.

First, we will consider an m-periodic attractor p with a bounded semi-local stable set. Then, we will use Corollary 3.6 to show that $W^s_{sl}(p)$ contains a critical point of f^m. We will conclude that if $W^s_{sl}(p)$ contains a critical point of f^m, then the stable set of the orbit of p contains a critical point of f.

We will deal separately with the cases when p is two-sided and when p is one-sided. In the two-sided case, we will apply Corollary 3.6 to the two boundary points of $W^s_{sl}(p)$ and to p itself. In the one-sided case, p is a boundary point of $W^s_{sl}(p)$, so we will modify this scheme slightly. In both cases, we will find that $W^s_{sl}(p)$ contains a critical point of f^m and deduce that the stable set of the orbit contains a critical point of f.

Let us begin by assuming p is a two-sided periodic attractor of period m for f with bounded semi-local stable set $W^s_{sl}(p) := (a, b)$. Define $g = f^m$. Now, since the endpoints a and b are not elements of $W^s_{sl}(p)$, the points $g(a)$ and $g(b)$ are also not elements of $W^s_{sl}(p)$; otherwise, a and b would be asymptotic to p. On the other hand, the points $g(a)$ and $g(b)$ cannot lie outside the closed interval $[a, b]$. Otherwise, we would have $g([a, b]) \supsetneq [a, b]$, which would imply that $W^s_{sl}(p)$ is not the largest interval of $W^s(p)$ containing p. We know $g([a, b])$ is an interval because g is continuous.

We have shown that there are three cases for the values of $g(a)$ and $g(b)$:

1. $g(a) = a$ and $g(b) = b$;
2. $g(a) = b$ and $g(b) = a$;
3. $g(a) = g(b) = a$ or b.

In each case, g admits at least one critical point in the interval (a, b). To see this, we consider each case individually. In the first case, we find, by the Mean Value Theorem, that there is α in (a, p) such that $g'(\alpha) = 1$, as $g(p) - g(a) = a - p$. We also find β in (p, b) such that $g'(\beta) = 1$ by the same reasoning. Recalling that $g'(p) \le 1$, we now have $g'(p) \le g'(\alpha) = g'(\beta)$. By Corollary 3.6, we conclude that g admits a critical point in $(\alpha, \beta) \subset (a, b)$. The second case can be reduced to the first by replacing g with g^2, since we have $g^2(a) = a$, $g^2(b) = b$, and $|(g^2)'(p)| < 1$. In the third case, the interval (a, b) contains a critical point of g by Rolle's Theorem, as $g(b) - g(a) = 0$.

We have shown that $W^s_{sl}(p) = (a, b)$ contains a critical point of $g = f^m$. Now we must show that the semi-local stable set of the orbit of p contains a critical point of f. If p is fixed, then we have $g = f$, which means the critical point of g in (a, b) is a critical point of f. So we are done. For p periodic of period m, however, we have found only a critical point of f^m, and more work is required to find a critical point of f.

Let c be this critical point of f^m in $W^s_{sl}(p)$. By the chain rule, there exists $0 \le i \le m - 1$ such that $f^i(c)$ is a critical point of f. We now show that $f^i(c)$ is an element of the stable set of the orbit of p; that is, there exists $0 \le j \le m - 1$ such that $f^i(c) \in W^s_{sl}(f^j(p))$. First, note that we have $f(W^s_{sl}(f^k(p)) \subseteq W^s_{sl}(f^{k+1}(p))$ for all $0 \le k \le m - 1$. Otherwise, there would be $y \in W^s_{sl}(f^k(p))$ such that $f(y)$ is a boundary point of $W^s_{sl}(f^{k+1}(p))$, which is impossible since y is asymptotic to $f^k(p)$.

This implies $f^i(c) \in W^s_{sl}(f^i(p))$ since we have $c \in W^s_{sl}(p)$. We have shown $f^i(c)$ is a critical point of f that is in the semi-local stable set of the orbit of p. This concludes the case when p is a two-sided attractor.

Now let p be a one-sided periodic attractor of period m. First, consider the case when p attracts on the right, and suppose $W^s_{sl}(p)$ is a bounded half-closed, half-open interval $[p, b)$. We will now show that $[p, b)$ contains a critical point of f^m and, using the arguments above, conclude that the semi-local stable set of the orbit of p contains a critical point of f. As before, define $g = f^m$. Assume without loss of generality that $(f^m)'(p) = 1$ (if not, let $g = f^{2m}$). Now consider the possible values of $g(b)$. First, we cannot have $g(b) \in (p, b)$. Otherwise, there would be $x \notin [p, b)$ close enough to b such that $g(x) \in (p, b)$, but x cannot be asymptotic to p. We also cannot have $g(b) \notin [p, b]$, for then there would be $y \notin [p, b]$ close enough to $g(b)$ so that $g^{-1}(y) \in [p, b)$, but y cannot be not asymptotic to p. Here, we used continuity of g^{-1}. Hence, we have shown that either $g(b) = p$ or $g(b) = b$. As before, we now show for both cases that $[p, b)$ contains a critical point of g.

For the case when $g(b) = p$, the map g admits a critical point in $[p, b)$ by Rolle's Theorem, as $g(b) - g(p) = 0$. For the case when $g(b) = b$, we find, by the Mean Value Theorem, that there exists η between p and b such that $g'(\eta) = 1$, since $g(b) - g(p) = b - p$. Now, since p attracts on the right, there must be a point x close enough to p so that $p < g(x) < x < \eta$ (see Fig. 3a). Then, by the Mean Value Theorem, there is some point ξ between p and x that satisfies

$$g'(\xi) = \frac{g(x) - p}{x - p} < 1.$$

Recall that one-sided attractors are non-hyperbolic, so we have $g'(p) = 1$. Now, since we have $p < \xi < \eta$ such that $g'(\xi) < g'(p) = g'(\eta)$, Corollary 3.6 implies that g admits a critical point in $(p, \eta) \subset [p, b)$. This shows that $W^s_{sl}(p)$ contains a critical point of $g = f^m$, and, by the reasoning used in the two-sided case, there must be a critical point of f in the stable set of the orbit of p.

This argument applies similarly to the case when p attracts on the left instead of on the right. The only difference is that we modify $W^s_{sl}(p)$ to be some interval $(a, p]$, and then, in the second possibility for the value of $g(a)$, we choose a point x that has $x < g(x) < p$. This concludes the case when p is a one-sided attractor and completes the proof. $\square$

We have shown that if p is an m-periodic attractor for f such that $Sf < 0$ and $W^s_{sl}(p)$ is bounded, then the semi-local stable set of the orbit of p contains at least one critical point of f. In particular, if f admits n critical points, then f has at most n attracting orbits whose semi-local stable sets are bounded, since two stable sets cannot attract the same critical point.

We now prove Corollary 4.2 by showing that f has no more than two attracting orbits that have unbounded stable sets. Moreover, these unbounded stable sets need not contain a critical point of f.

Proof of Corollary 4.2. If f had three attracting *points*—let alone three attracting *orbits*—with unbounded stable sets, then two of their stable sets would overlap, which is impossible, as a point cannot be asymptotic to two different attractors. We will see in Example 4.5 that the two allowed unbounded semi-local sets need not contain a critical point of f. $\square$

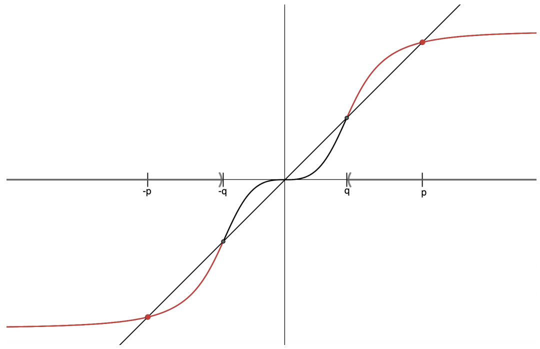


FIGURE 5. The graph of $A(x) = \frac{7}{5}\arctan(x^3)$ from Example 4.5. The stable sets (red) of the points p and $-p$ are unbounded.

Notice that we showed f may have no more than two attracting points (not attracting orbits) that have unbounded stable sets. This means f can only have two attracting orbits with unbounded stable sets when both orbits are fixed points for f. Moreover, attracting orbits of period $m > 2$ must attract a critical point, since at least $m-2$ of the points in that orbit have bounded stable sets. We showed in the proof of Theorem 4.1 that if a periodic attractor has a bounded stable set, then its orbit attracts a critical point of f.

The case when f has two more attracting orbits than critical points is achieved in the following example.

Example 4.5. Consider $A(x) = \frac{7}{5}\arctan(x^3)$, whose graph is depicted in Figure 5. We have

$$S(x^3) = -\frac{4}{x^2} < 0 \quad \text{and} \quad S(\arctan x) = -\frac{2}{(1+x^2)^2} < 0.$$

Hence we have $SA(x) < 0$ by Proposition 3.2. Now, the map A has three attracting fixed points but only one critical point, which happens to be the attracting fixed point $x = 0$. Note that the other two attracting fixed points, labeled p and $-p$ in Figure 5, have unbounded (semi-local) stable sets, which are the intervals (q, ∞) and $(-\infty, -q)$, respectively, as shown in the figure. Moreover, neither stable set contains the critical point $x = 0$.

Though we have now seen that for maps with negative Schwarzian, the existence of an attracting periodic orbit of period greater than two implies the existence of a critical point, the converse does not hold. We demonstrate this in the following example.

Example 4.6. Consider the quadratic map $F_4(x) = 4x(1-x)$, which has $SF_4 < 0$ and admits a single critical point $x = 1/2$. First, note that if F_4 had an attracting periodic orbit, then its stable set would be bounded in the interval $[0, 1]$, since we have $F_4^n(x) \to -\infty$ for $|x| > 1$. It follows that any attracting orbit of F_4 would attract the critical point $x = 1/2$, by Theorem 4.1. The point $F_4^2(1/2) = 0$, however, is a repelling fixed point for F_4, which means the critical point $x = 1/2$ is not contained in the stable set of an attracting orbit. The only possibility, then, is that F_4 does not have any attracting periodic orbits. Even stranger is the fact that periodic repellors for F_4 are dense in $[0, 1]$. See [3, §1.11] for a proof of this result which uses many of the properties presented here of maps with negative Schwarzian.

We conclude by showing that the assumption of negative Schwarzian derivative is necessary for Theorem 4.1 to hold.

Example 4.7. Consider the map $h : \mathbb{R} \to \mathbb{R}$ given by $h(x) = \frac{1}{2}\sin x + x$. Note that h' is bounded below by $1/2$, so h is strictly increasing and admits no critical points. At the same time, h has infinitely many attracting fixed points. In particular, we have $h(\pi + 2\pi n) = \pi + 2\pi n$ and $h'(\pi + 2\pi n) = 1/2$ for all integers n. Hence, the number of critical points of h does not bound the number of attracting orbits for h. This does not contradict Theorem 4.1, however, as Sh is not everywhere negative. In particular, we compute

$$Sh(x) = \frac{\cos^2 x - 4\cos x - 3}{2(\cos x + 2)^2}$$

and find $Sh(\pi + 2\pi n) = 1 > 0$ for every n. Indeed, f' attains the positive local minimum value of $1/2$ infinitely often.

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