AN INTRODUCTION TO GROMOV'S h-PRINCIPLE

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ABSTRACT. The twentieth century saw great advances in the theory of immersions and differential equations spearheaded by mathematicians such as Stephen Smale. Beginning with his PhD thesis in 1969, Mikhail Gromov began creating the theory of the h-priciple which serves to generalize and unify the results proved by others and to further generate new ones. This paper is devoted to introducing the reader to the basic notions, constructions, and results of this theory, by illustrating them through various examples, particualrly the Whitney Graustein theorem.

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1. INTRODUCTION

In 1937, Hassler Whitney published an unassuming paper entitled "On regular closed curves in the plane," which explores the regular homotopy classes of immersions of circles in the plane. This was an early sign of the great topological advances in the theory of immersions to come in the latter half of the twentieth century spearheaded by mathematicians such as René Thom and Stephen Smale and John Nash. In the 1970s Mikhail Gromov, motivated by the work of these mathematicians and what he saw as a common thread between them, began to develop his theory of the h-principle. The goal of this paper is to motivate and develop the theory which undergirds Gromov's h-principle, primarily through progressive abstraction from and then an ultimate return to the Whitney Graustein Theorem.

Note that this paper assumes a fundamental understanding of differential topology, and a firm grasp on the fundamentals of analysis although the proofs are kept to a minimum of technicality.

2. MOTIVATING THE h-PRINCIPLE

Throughout the mid 20th century, mathematicians began gaining interest in the theory of immersions and the homotopies between them. Two primary examples are the work of René Thom and Stephen Smale, both of whom were instrumental in developing the theory of immersions. It was this theory which Mikhail Gromov sought to generalize in his 1986 book *Partial Differential Relations*, in which he developed the idea of the h-principle. However, Thom and Smale's work was foreshadowed by Whitney's 1937 proof of the Whitney-Graustien theorem, which categorizes immersions of S^1 in the plane. This first section presents this example to contextualize and motivate the need for a greater generalization.

Definition 2.1 (Regular Homotopy). Two immersions $f, g : S^1 \to \mathbb{R}^2$ are **regularly homotopic** if there exists a homotopy $H : S^1 \times [0,1] \to \mathbb{R}^2$ such that $H(\cdot, 0) = f$, $H(\cdot, 1) = g$ and for every $x \in (0,1)$, $H(\cdot, x)$ is an immersion.

Definition 2.2. (Winding Number) Given an immersion $f : S^1 \to \mathbb{R}^2$ such that $||f'|| = 1$, the **winding number** γ is the degree of $f' : S^1 \to S^1 \subset \mathbb{R}^2$, i.e. the equivalence class of f' when viewed as an element of the fundamental homotopy group $\pi_1(S^1) \cong \mathbb{Z}$.

Theorem 2.3 (Whitney-Graustein). *There exists a regular homotopy between two immersions $f_0, f_1 : S^1 \to \mathbb{R}^2$ if and only if their winding numbers are equal.*

Proof. The forward implication is trivial (and not particularly relevant), so we will only prove the reverse implication.

To begin, assume WLOG that the total arc length of the image of both immersions is 1 and that $||f_0'|| = ||f_1'|| = 1$, as any immersion can be reparametrized by arc length so that the tangent vectors are of unit length.

Since $\gamma(f_0) = \gamma(f_1)$, we know f_0' and f_1' are homotopic by definition 2.2 as they correspond to the same element of the fundamental group $\pi_1(S^1)$. Thus, there exists a homotopy $h : [0,1] \times S^1 \to S^1$ which connects f_0' and f_1', so that each h_u is not constant (if $\gamma = 0$ the homotopy can be trivially perturbed to make h_u not constant). From here, it remains to show that h_u can be properly integrated to make a regular homotopy.

For a given function $\alpha : S^1 \to \mathbb{R}^2 \backslash 0$ to be the derivative of an immersion of S^1 into $\mathbb{R}^2$, it must be that, $\int_{S^1} \alpha(t)dt = (0,0) \in \mathbb{R}^2$, so that the integral $\int_0^\theta \alpha(t)dt$ creates a closed loop when θ returns to 0 on S^1. Therefore, to ensure h_u satisfies this property, we define a new homotopy $h^* : [0,1] \times S^1 \to \mathbb{R}^2$

$$
(2.4) \qquad h_u^* = h_u - \int_{S^1} h_u(t)dt
$$

where the subtraction operation is the usual vector subtraction in $\mathbb{R}^2$, considering the image of S^1 under h_u to be the unit circle centered at the origin in $\mathbb{R}^2$. This guarantees that $\int_{S^1} h_u^* dt = (0,0)$ (assuming S^1 is paramatrized with $[0,1)$). Moreover, by the Cauchy-Schwarz Inequality we know that

$$
(2.5) \qquad \left\| \int_{S^1} h_u(t)dt \right\|^2 \leq \int_{S^1} ||h_u(t)||^2 \, dt
$$

and since h_u is not constant, the inequality is strict, and since $||h_u(t)||^2 = 1$ we have that

$$(2.6) \qquad \left\| \int_{S^1} h_u(t)dt \right\|^2 < 1$$

thus $\int_{S^1} h_u(t)dt$ lies in the interior of $S^1 \subset \mathbb{R}^2$, so $\|h_u^*\| > 0$. Finally, the function $f : [0,1] \times S^1 \to \mathbb{R}^2$, defined by

$$(2.7) \qquad f_u(t) = f_0(0) + u(f_1(0) - f_0(0)) + \int_0^t h_u^*(\theta)d\theta$$

forms the desired regular homotopy between f_0 and f_1, because $h_0^* = f_0'$, $h_1^* = f_1'$, and $||h_u^*|| > 0$. $\qquad \square$

Although correct and explicitly constructive, this proof fails to provide much insight into the underlying structures and patterns of regular homotopy, making it somewhat unsatisfying. This proof is also inherently tied to the particular structure of S^1 and $\mathbb{R}^2$ as it relies heavily on integration and the Euclidean norm. Thom and Smale's work was similarly domain-specific, although all dealt with homotopies of immersions. The goal (or at least one goal) of the h-principle is to create a unified theory of such homotopies to better understand their underlying structure and prove a wide variety of further results. The following sections will develop this theory show how it may be applied, at which point we will return to Whitney's 1937 theorem and re-prove it with the new theory.

3. Jets and Holonomy

3.1. **Jets over $\mathbb{R}^n$.** From the statement and proof of Theorem 2.3, it is clear that central to the analysis of regular homotopy is not only the analysis of functions but their derivatives (particularly since being an immersion is a condition on the first derivative). Thus, the first step in developing the h-principle is to consolidate all the information about a function and its relevant derivatives so that they can be jointly analyzed. It is for this purpose that we define the jet of a function.

Definition 3.1. Given a (smooth) function $f : \mathbb{R}^n \to \mathbb{R}^q$, its r-**jet** is the function given by

$$J_f^r(x) = (f(x), f'(x), ..., f^{(r)}(x))$$

where $f^{(k)}(x)$ consists of all partial derivatives up to order k of the function at the point x.

For example given a function $f : \mathbb{R}^2 \to \mathbb{R}^2$, its 1-jet would be given by

$$(3.2) \qquad J_f^1(x) = (f_x(x,y), f_y(x,y), \partial_x f_x(x,y), \partial_x f_y(x,y), \partial_y f_x(x,y), \partial_y f_y(x,y)).$$

where f_x and f_y denote the function's x and y components respectively. From this example it is clear that it would be natural to consider $J_f^1(x)$ as a point of $\mathbb{R}^2 \times \mathbb{R}^4 = \mathbb{R}^8$. Therefore, we can consider any r-jet as a function from $\mathbb{R}^n$ to some higher dimensional euclidean space. This leads us to the following definition.

Definition 3.3. The r-jet bundle $J^r(\mathbb{R}^n, \mathbb{R}^q)$ is defined as

$$J^r(\mathbb{R}^n, \mathbb{R}^q) = \mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}^{d_1} \times ... \times \mathbb{R}^{d_r}$$

where $d_k = \frac{(n+k-1)!}{(n-1)!k!}$ is the number of partial derivatives of order k, assuming the function is smooth.

Given this definition, we can consider the jet of a function $f : \mathbb{R}^n \to \mathbb{R}^q$ to be a map which sends each point $x \in \mathbb{R}^n$ to the point $(x, J_f^r(x)) \in J^r(\mathbb{R}^n, \mathbb{R}^q)$. However, this construction hints at a better approach. Rather than consider f, we can consider its **graph** $\Gamma : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^q$, which is the unique map such that $\Gamma(x) = (x, f(x))$. More generally we can consider Γ to be a section of the trivial fibration (i.e. projection) $\mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^n$. Therefore, rather than consider functions from $\mathbb{R}^n$ to $\mathbb{R}^q$, we will consider sections of the trivial fibration, so that the theory can later be extended to general fibrations.

So, given a section $f : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^q$ its jet can be considered a map

$$(3.4) \qquad\qquad J_f^r : \mathbb{R}^n \to J^r(\mathbb{R}^n, \mathbb{R}^q)$$

from the base space ($\mathbb{R}^n$) to the jet bundle.

3.2. Jets over Manifolds.
As it stands, the constructions thus far do not entirely suffice when analyzing problems involving arbitrary manifolds. Therefore, it serves to further extend the definition of jets to arbitrary sections of manifold fibrations.

Consider two smooth manifolds V and X of dimension n and $n + q$ respectively, along with an arbitrary section $f : V \to X$ of a smooth fibration $p : X \to V$. Given any point $v \in V$, we can consider a local section of $f_v : Op\, v \to X$, along with a local chart $\phi : Op\, v \to \mathbb{R}^n$, where $Op\, v$ is an open neighborhood around v. Moreover, there exists an open subset $U \subset X$ with $U \ni f_v(v)$, such that there exists a local trivialization $\tau : U \to \mathbb{R}^n \times \mathbb{R}^q$. This is summarized in the following diagram:

$$
\begin{array}{ccc}
U & \xrightarrow{\ \tau\ } & \mathbb{R}^n \times \mathbb{R}^q \\
{\scriptstyle p}\big\downarrow\big\uparrow{\scriptstyle f_v} & & \\
Op\, v & \xrightarrow{\ \phi\ } & \mathbb{R}^n
\end{array}
$$

This allows us to associate to each point $v \in V$ a function $F : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^q$, defined by $F = \tau \circ f_v \circ \phi^{-1}$.

Definition 3.5 (r-tangency)**.** Two sections $f, g : Op\, v \to X$ are called r**-tangent** if $f(v) = g(v)$ and there exist an open neighborhood $U \ni f(v)$ such that for a local trivialization $\tau : U \to \mathbb{R}^n \times \mathbb{R}^q$

$$J_{\tau \circ f \circ \phi^{-1}}^r(v) = J_{\tau \circ g \circ \phi^{-1}}^r(v)$$

It happens that if two functions are r-tangent relative to one local trivialization, they are r-tangent relative to any other trivialization. Furthermore, r-tangency forms an equivalence relation on the space of sections from $Op\, v$ to X, meaning we can speak of r-tangency equivalence classes. This leads us to the following definition.

Definition 3.6 (Jets of Manifolds)**.** Given a section $f : Op\, v \to X$ its jet $J_f^r(v)$ is the r-tangency class of r-jets at the point v.

Definition 3.7 (Jet Bundle of a Fibration)**.** The jet bundle $X^{(r)}$ of an arbitrary fibration $p : X \to V$ is

$$X^{(r)} = V \times \mathrm{Sec}(X)/\sim$$

where $\sim$ is r-tangency equivalence and $\mathrm{Sec}(X)$ is the space of sections of the fibration.

One can come to an equivalent definition to (3.7) by gluing together the jet spaces of the local trivializations according to the charts of V, which is a much

more intuitive construction, but harder to portray succinctly. A full discussion of this can be found in Section 6.2 of D.J. Saunders's *The Geometry of Jet Bundles.*

3.3. **Holonomy.** This view of jets as sections of the jet bundle begs us to consider arbitrary sections of the jet bundle. In particular, not all sections of the jet bundle are jets of sections from V to X. For example consider the section $f : \mathbb{R} \to J^1(\mathbb{R}, \mathbb{R})$, such that

$$(3.8) \qquad f(x) = (x, x^2, 0).$$

This is a perfectly good section of $J^1(\mathbb{R}, \mathbb{R})$ since, $\pi_1 \circ f = id_\mathbb{R}$ (where π_1 is projection to the first coordinate), but there exists no function on the reals such that $f(x) = x^2$ and $f'(x) = 0$. Therefore, we must distinguish between sections that derive from functions and those that do not.

Definition 3.9. A section $F : V \to J^r(V, W)$ is **holonomic** if there exists a section $f : V \to V \times W$ such that $J_f^r = F$.

This notion of holonomy will allow us to rephrase questions of functions and their C^r homotopies in terms of the existence of particular holonomic sections of the jet space. More concretely, we will seek to understand the relationship between the space of sections of the jet space Sec $X^{(r)}$, and the subspace of holonomic sections Hol $X^{(r)}$. Fortunately, in a sense to be defined, Hol $X^{(r)}$ is "dense" in Sec $X^{(r)}$.

4. HOLONOMIC APPROXIMATION

4.1. **The Holonomic Approximation Theorems.** Given the existence of both holonomic and non-holonomic sections, the natural question becomes how they are related. The Holonomic Approximation Theorems describe the nature of this relationship, which is that given any section of the jet bundle, there exists a holonomic section that approximates it arbitrarily well over a perturbed domain. To rigorously address the "smallness" of this perturbation, we need the following notion.

Definition 4.1. A diffeotopy $h : [0, 1] \times V \to V$ such that $h_0 = id_V$ is called δ-**small** if, for all $t \in [0, 1]$ [1],

$$\mathrm{dist}(h_t(v), v) < \delta$$

In this paper, I refer to a diffemorphism h as δ-small if there exists a δ-small diffeotopy such that $h_1 = h$.

Equipped with this notion of smallness, we can state the core result of the h-principle.

Theorem 4.2 (Holonomic Approximation Theorem)**.** *Let $A \subset V$ be a polyhedron [2] of positive codimension and*

$$F : Op\, A \to X^{(r)}$$

be an arbitrary section. Then for arbitrarily small $\delta, \varepsilon > 0$, there exists a δ-small diffeomorphism (diffeotopic to id_V)

$$h : V \to V$$

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[1]This is the distance induced by a fixed Riemannian metric on the base space.

[2]The formal definition of a polyhedron is a space homeomorphic to the geometric realization of a simplicial complex and can thus be triangulated, but such constructions are far beyond the scope of this paper, so we will only really consider the case of low dimensional manifolds.

and a holonomic section

$$\tilde{F} : Op\, h(A) \to X^{(r)}$$

such that

$$dist(\tilde{F}(v), F(v)) < \varepsilon$$

for all $v \in Op\, h(A)$ (assuming $Op\, h(A) \subset Op\, A$).

This first result is itself already very powerful; however, it only addresses a single section, but we are more commonly concerned with families (particularly homotopies) of sections. As such, a somewhat more useful theorem is the following.

Theorem 4.3 (Parametric Holonomic Approximation Theorem)**.** *Let $A \subset V$ be a polyhedron of positive codimension and*

$$F_z : Op\, A \to X^{(r)}$$

be an arbitrary family of sections parametrized by $z \in I^m$, for $m \in \mathbb{N}$, such that F_z is holonomic for all $z \in Op\, \partial I^m$. Then for arbitrarily small $\delta, \varepsilon > 0$, there exists a family of δ-small diffeomorphisms (diffeotopic to id_V)

$$h_z : V \to V$$

and a family of holonomic sections

$$\tilde{F}_z : Op\, h_z(A) \to X^{(r)}$$

such that for all $z \in Op\, \partial I^m$, $h_z = id_V$ and $\tilde{F}_z = F_z$, and

$$dist(\tilde{F}(v), F(v)) < \varepsilon$$

for all $v \in Op\, h_z(A)$ and $z \in I_m$ (assuming $Op\, h_z(A) \subset Op\, A$).

These two theorems, the latter in particular, are the technical core of the *h*-principle and are the theorems on which almost all proofs of *h*-principles ultimately rely. Unfortunately, a complete proof of these theorems is beyond the scope of this paper; however, rather than leave them unjustified, I will prove a narrow sub-case to demonstrate the fundamental technique of perturbation, which is at the heart of these two theorems.

4.2. **Justifying the HATs.** The primary insight of the HATs is that any section of the jet bundle is holonomic at a point. That is to say, for every section of the jet bundle $F : V \to X^{(r)}$ and point $v \in V$, there exists a function $f_v : V \to X$ such that $J_f^r(v) = F(v)$ since one can always construct a function (e.g., a Taylor polynomial) which has arbitrary derivatives at one particular point. The HATs show that these approximations can be smoothly connected. The goal of the following proof is to illustrate how these local approximations can be "strung together" so that their jets approximate F, thereby rendering the holonomic section guaranteed by Theorem 4.1.

Theorem 4.4. *Theorem 4.2 holds for $A = I \times \{0\}$, $V = \mathbb{R}^2$ and $X^{(r)} = J^r(\mathbb{R}^2, \mathbb{R})$.*

Proof. Let the section $F : Op\, A \to J^r(\mathbb{R}^2, \mathbb{R})$ be represented as

$$F = (F^0, F^{(1,0)}, F^{(0,1)}, F^{(2,0)}, \cdots).$$

Let $U_\delta(t) \subset \mathbb{R}^2$ be a square neighborhood of width δ around the point $(t, 0)$.

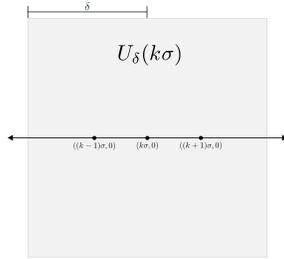FIGURE 1. The set $U_\delta(k\sigma)$ (grey) relative to $(k\sigma, 0)$ and its neighbors.

Futhermore, for $t \in I$, let $f_t : U_\delta(t, 0) \to \mathbb{R}$ be the polynomial given by

$$(4.5) \qquad f_t(x, y) = F^0(t, 0) + \sum_{a+b \leq r} F^{(a,b)}(t, 0) \frac{(x-t)^a}{a!} \frac{y^b}{b!}$$

so that its jet locally approximates F, that is to say

$$\frac{\partial^a}{\partial^a x} \frac{\partial^b}{\partial^b y} f_t(t, 0) = F^{(a,b)}(t, 0)$$
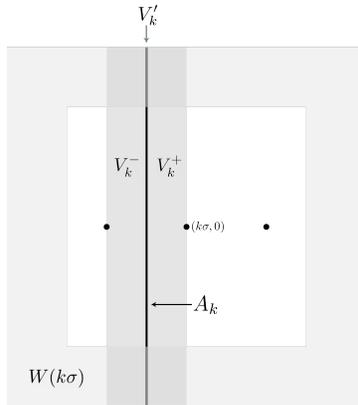
and so $J_f^r(t, 0) = F(t, 0)$.

FIGURE 2. The sets as defined in (4.6)-(4.10)

Now, pick $N \in \mathbb{N}$ such that $\sigma = \frac{1}{N} \ll \delta$, let γ be such that $0 < \gamma < \delta$, for $k \in \{1, ..., N\}$ let $c_k = k\sigma - \sigma/2$, and let

$$(4.6) \qquad A_k = \overline{U_\gamma(c_k)} \cap (\{c_k\} \times \mathbb{R})$$

$$(4.7) \qquad V_k^- = U_\delta(k\sigma) \cap (((k-1)\sigma, c_k] \times \mathbb{R})$$

$$(4.8) \qquad V_k^+ = U_\delta(k\sigma) \cap ([c_k, k\sigma) \times \mathbb{R})$$

$$(4.9) \qquad V_k = V_k^- \cup V_k^+$$

$$(4.10) \qquad V_k' = V_k^- \cap V_k^+.$$
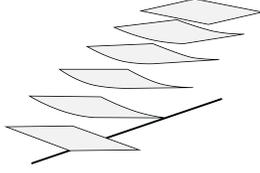
as shown in Figure 2.

FIGURE 3. An example family $f_{k\sigma} : U_\delta(k\sigma) \to \mathbb{R}$ for $N = 5$ (grey), and $I \times \{0\} \subset \mathbb{R}^2$ (black) [3]

For each k we can consider the local approximations $f_{k\sigma}$ to be a family of functions parametrized by k, an example of which is given in Figure 3. This family of functions satisfies the interpolation property stated in Lemma 4.13. Therefore there exists a family of functions f_t^τ which satisfy all the conditions ensured by the interpolation property. Using the existence of such a family, we define

$$(4.11) \qquad \tilde{f}(x) = \begin{cases} f_{k\sigma}^0(x) & x \in V_k^- \\ f_{k\sigma}^\sigma(x) & x \in V_k^+ \end{cases}$$

which is smooth on $V_k \backslash A_k$ because $V_k' \backslash A_k \subset W(k\sigma)$ ($W(t) = U_\delta(t) \backslash \overline{U_\gamma(t)}$), is the only place where the two parts of the piecewise function meet. Since $V_k \cap V_{k+1} = \emptyset$, we also have that $\tilde{f}$ is smooth on $\bigcup_{k=1}^{N}(V_k \backslash A_k)$.
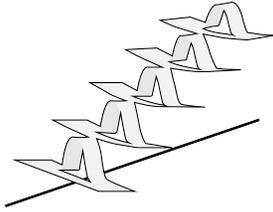

FIGURE 4. $\tilde{f}|_\Omega$ (grey) and $I \times \{0\}$ (black)

Furthermore, by part (d) of the interpolation property, we know that $f_{(k-1)\sigma}^\sigma$ and $f_{k\sigma}^0$ agree on an open neighborhood of $(k\sigma, 0)$. Therefore, $\tilde{f}$ is also well defined on $\bigcup_{k=1}^{N} Op(k\sigma, 0)$, where $Op(k\sigma, 0)$ is sufficiently small. Thus, $\tilde{f}$ is well defined on

$$(4.12) \qquad \Omega = \bigcup_{k=1}^{N}(V_k \backslash A_k) \cup \bigcup_{k=1}^{N} Op(k\sigma, 0).$$

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[3]This particular example is generated by $F(t,0) = (t, -\frac{1}{2}\cos(\frac{\pi}{2}t), 0, 0, \sin(\pi t))$

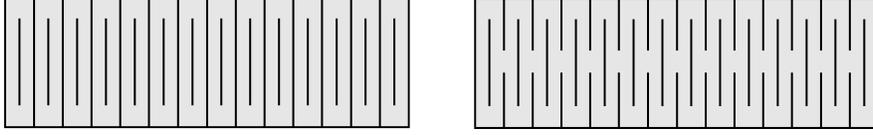as illustrated in Figure 4, with Ω itself depicted in Figure 5.

FIGURE 5. $\bigcup_{k=1}^{N}(V_k \backslash A_k)$ (left) and Ω (right) in grey with black boundaries excluded.

Now, define the diffeotopy $h_u : \mathbb{R}^2 \to \mathbb{R}^2$ so that $h_1(I \times \{0\}) \subset \Omega$. Then pick $Op\, h_1(I \times \{0\})$ so that it is entirely contained in Ω as shown in Figure 6.

$$h_1(I \times \{0\})$$
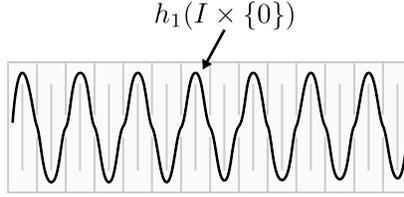

FIGURE 6. The image of $I \times \{0\}$ under h_1.

Thus, Theorem 4.4 is proved with $\tilde{F} = J_{\tilde{f}}^r : Op\, h_1(I \times \{0\}) \to \mathbb{R}$ and $h = h_1$, with the smaller than ϵ distance being ensured by part (c) of the interpolation property. $\square$

Lemma 4.13 (Interpolation Property). *A family of functions $f_t : U_\delta(t) \to \mathbb{R}$ constructed as in the proof of Theorem 4.4 satisfies the interpolation property. That is to say, given a $\sigma = \frac{1}{N} \ll \delta$, there exists another family*

$$f_t^\tau : U_\delta(t) \to \mathbb{R}, \ t \in [0,1], \ \tau \in [0,\sigma]$$

such that

    a) $J_{f_t^0}^r = J_{f_t}^r$ *for all t.*
    b) $J_{f_t^\tau}^r|_{W(t)} = J_{f_t}^r|_{W(t)}$ *for all t and τ*
    c) $|J_{f_t^\tau}^r - J_{f_t}^r| < \varepsilon$ *for all t and τ*
    d) $J_{f_t^\tau}^r|_{Op(t,0)} = J_{f_{t+\tau}}^r|_{Op(t,0)}$ *for all t and τ*

where $W(t) = U_\delta(t) \backslash \overline{U_\gamma(t)}$.

Although a proof that $f_{k\sigma}$ satisfies the interpolation property is beyond the scope of this paper, it is fundamentally a more intricate version of the statement that, given a smooth function defined on $(-\infty, 0)$ and another on $(1, \infty)$, one can find a function defined on $[0, 1]$ which connects them so that the composite of the three functions is smooth. Once one has this in mind, the task of the lemma is to construct the multi-dimensional parametric analog of such a connecting function.

4.3. **Applications.** With the holonomic approximation theorems somewhat justi-
fied, we can begin to apply them to various problems that illustrate the philosophy
of the h-principle. Two primary examples that illustrate the theorems' use are the
cone eversion and the sphere eversion. I will do the former in somewhat gritty detail
to illustrate the mechanics involved in proving the existence of an h-principle, and
the latter I will do more abstractly to illustrate the elegance of the technique.

A first interesting result is the cone eversion, that is to say, flipping a cone upside
down while only allowing points to move vertically. Intuitively, one expects that
any homotopy between an upward slope and a downward slope must pass through
a flat slope at some point. However, the following theorem defies this intuition
by demonstrating that, at least in two dimensions, one can evert the cone without
having a gradient of 0 at any point on the cone at any stage of the homotopy

Theorem 4.14 (Cone Eversion). *There exists a homotopy of continuously differen-*
tiable functions $f : [0,1] \times V \to \mathbb{R}$ such that $f_0(x,y) = x^2 + y^2$, $f_1(x,y) = -x^2 - y^2$
and $\nabla f_u(x,y) \neq (0,0)$ for all $(x,y) \in V$ where $V = \{(x,y) \mid \delta^2 < x^2 + y^2 < 4\} \subset \mathbb{R}^2$,
where $\delta > 0$ is arbitrarily small.

Proof. Begin by considering the jets of f_0 and f_1, taking $J^1(V,\mathbb{R}) = V \times \mathbb{R} \times \mathbb{C}$ for
notational simplicity

$$\begin{aligned}
(4.15) \qquad J^1_{f_0}(x,y) &= (x,y,x^2+y^2,2x+2yi) \\
J^1_{f_1}(x,y) &= (x,y,-x^2-y^2,-2x-2yi).
\end{aligned}$$

We can construct a formal homotopy of sections $F : [0,1] \times Op\, A \to J^1(V,\mathbb{R})$ -
where A is the unit circle of radius 1 and $Op\, A = V$ - defined by

$$(4.16) \qquad F_u^*(x,y) = (x,y,(1-2u)(x^2+y^2),(2x+2yi)e^{i\pi u})$$

so that $F_0^* = J^1_{f_0}$ and $F_1^* = J^1_{f_1}$, and the derivative component is never 0. However,
this section is only holonomic on ∂I and not on $Op\, \partial I$, so we define a trivially
perturbed homotopy F given by

$$(4.17) \qquad F_u(x,y) = \begin{cases} F_0^*(x,y) & u \in [0,\frac{1}{4}) \\ F_{\frac{1}{2}u+\frac{1}{4}}^*(x,y) & u \in [\frac{1}{4},\frac{3}{4}] \\ F_1^*(x,y) & u \in (\frac{3}{4},1] \end{cases}$$

which is holonomic on $[0,\frac{1}{4}) \cup (\frac{3}{4},1] = Op\, \partial I$.

From here we apply the Parametric Holonomic Approximation Theorem (4.3),
which guarantees the existence of a family of diffeomorphisms diffeotopic to id_V
and a family of holonomic sections $\tilde{F}_u : Op\, h_u(A) \to J^1(V,\mathbb{R})$ which connect $J^1_{f_0}$
and $J^1_{f_1}$, and $\operatorname{dist}(\tilde{F}(v),F(v)) < \varepsilon$ for an arbitrarily small $\varepsilon > 0$. This second
condition ensures that $\nabla \tilde{f}_u \neq 0$ $(\tilde{F}_u = J^1_{\tilde{f}_u})$ since its jet is arbitrarily close to $\tilde{F}_u$
whose gradient component is never 0.

Finally, since h_u is diffeotopic to id_V, there exists a family of isotopies $\phi_u :$
$[0,1] \times V \to V$ (with $\phi_u^0 = id_V$) such that $\phi_u^1(V) = Op\, h_u(A)$. Therefore $g_u = \tilde{f}_u \circ \phi_u^1$
is a homotopy which connects f_0 and f_1 such that $\nabla g_u \neq 0$. [4]

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[4]For those curious as to the exact nature of such a homotopy, the follow-
ing resources will be of interest: http://www.personal.psu.edu/sot2/prints/ever.pdf,
https://jonathanevans27.wordpress.com/2014/10/12/cone-eversion/

To prove Smale's Sphere eversion, we require some preliminary notions. Define the thickened sphere $\Sigma = \{\vec{x} \in \mathbb{R}^3 \mid 1 - \delta \leq \|x\| \leq 1 + \delta\}$, along with its inclusion $i_\Sigma : \Sigma \to \mathbb{R}^3$. Furthermore, define the orientation-preserving inversion map inv : $\mathbb{R}^3 \backslash \{0\} \to \mathbb{R}^3 \backslash \{0\}$ by

$$\mathrm{inv}(x, y, z) = \frac{(x, y, -z)}{\|(x, y, z)\|^2}.$$

This function effectively turns the sphere inside out, just as one might to do an inflatable ball with a hole in its side. The following theorem shows that, if one allows self-intersection, the usual immersion of the sphere can be smoothly turned inside out through immersions.

Theorem 4.18 (Sphere Eversion). *There exists a homotopy of immersions between the standard embedding of the thickened sphere $f_0 = i_\Sigma$ and its inverted embedding $f_1 = \mathrm{inv} \circ i_V$.*

Proof. Since f_0 and f_1 are immersions which induce the same orientation, their differentials df_0, df_1 can be viewed as elements of $SO(3) \cong \mathbb{RP}^3$, their jets

$$J^1_{f_0}, J^1_{f_1} : \Sigma \to J^1(\Sigma, \mathbb{R}^3)$$

can be viewed as maps $\Sigma \to \mathbb{R}^3 \times SO(3)$, and since $\pi_2(SO(3)) = 0$, df_0 and df_1 are homotopic as maps to $SO(3)$. Thus, there exists a homotopy of sections $F_u : \sigma \to J^1(\Sigma, \mathbb{R}^3)$, with $F_0 = J^1_{f_0}$ and $F_1 = J^1_{f_1}$ which can be assumed holonomic for $u \in Op\,\partial I$ such that the derivative component of F_u always has full rank. Applying the Parametric Holonomic Approximation Theorem with the base polyhedron being S^2, we know there exists an ε-approximate family of holonomic sections

$$\tilde{F}_u : Op\,h_u(S^2) \to J^1(\Sigma, \mathbb{R}^3)$$

and a family of δ-small diffeomorphisms h_u which connects F_0 and F_1. One can choose ε small enough so that $\tilde{f}_u$ forms a regular homotopy. Finally, as with Theorem 5.5 we can compose $\tilde{f}_u$ with a retraction $\phi_u : \Sigma \to Op\,h_u(S^2)$, which completes the regular homotopy $g_u : \Sigma \to \mathbb{R}^3$ defined by $g_u = \tilde{f}_u \circ \phi_u$. $\qquad\square$

5. Differential Relations

With the HATs and some preliminary applications established, the concept of the h-principle is almost entirely developed, save the final abstraction: differential relations. It should be clear that differential notions are at the core of the theory thus-far developed since it centers around the theory of jets. However, one may think that the two applications presented seem fundamentally topological, but this section will show how they can be rephrased in an almost purely differential language.

Definition 5.1. A **differential relation** is any collection of ordinary or partial differential equations or inequalties.

Under this definition, every differential relation $\mathcal{R}$ for order r imposed on sections $f : V \to X$ defines a subset of the jet bundle $X^{(r)}$ through the conditions it imposes on each coordinate. For example, consider the differential relation $f'(x) = f(x) + x$ on a function $f : \mathbb{R} \to \mathbb{R}$ (where the section we consider is its graph $f : \mathbb{R} \to \mathbb{R}^2$). Comparing this to the 1-jet of f, $J^1_f(x) = (x, f(x), f'(x))$, we see that we can extend the relation to any point $(x, y, z) \in J^1(\mathbb{R}, \mathbb{R})$, by viewing it as the algebraic relation

$z = y + x$. This equation then defines a subset of the jet bundle. This rephrasing turns questions about differential relations into algebraic (or at least topological) questions about the jet bundle and its sections.

Definition 5.2. A differential relation $\mathcal{R}$ is **open** or **closed** if it is an open or closed subset of the jet bundle respectively.

The distinction between open and closed differential relations is vital, as it influences the ability of sections of the jet bundle to be perturbed by remaining in the same subset.

Definition 5.3. A **formal solution** to a differential relation $\mathcal{R}$ is a section of the jet bundle F such that $F(V) \subset \mathcal{R}$

This definition creates a sub-distinction of the formal-holonomic dichotomy for differential relations; however, we refer to holonomic sections whose image is entirely contained in $\mathcal{R}$ as *real solutions*.

This language of differential relations is the most natural way to phrase questions of the existence of h-principles since it simplifies questions of homotopies that satisfy particular conditions to sections of the jet bundle contained in subsets of the jet bundle.

6. The Homotopy Principle

Using the language of differential relations, we can finally state the h-principle.

Definition 6.1 (Homotopy Principle)**.** A differential relation $\mathcal{R}$ satisfies the h-**principle** if every formal solution of $\mathcal{R}$ is homotopic through formal solutions of $\mathcal{R}$ to a genuine solution of $\mathcal{R}$.

Put more intuitively, a differential relation $\mathcal{R}$ satisfies the h-principle if every section of the jet bundle whose image is contained in $\mathcal{R}$ can be perturbed into a holonomic section which is still contained in $\mathcal{R}$. Just as with the HATs, we also have a somewhat more practical notion of a parametric h-principle.

Definition 6.2. A differential relation $\mathcal{R}$ satisfies the **one-parametric h-principle** if every family of formal solutions F_t which joins two reals solutions F_0 and F_1 can be deformed into a family of real solutions while keeping the same endpoints.

This notion of the h-principle is a powerful tool to prove results across mathematics, particularly in topology and in the theory of differential equations. Showing that a particular differential relation satisfies an h-principle can be valuable information when explicit solutions are hard or even impossible to construct.

Finally, we return to the Whitney-Graustein theorem, which we are now equipped to rephrase in the language of the h-principle and prove using Theorem 4.3. In this theorem the differential relation in question is that of being an immersion, i.e., $\mathcal{R}_{\mathrm{imm}} = \{(x, f, f') \in J^1(S^1, \mathbb{R}^2) \mid f' \neq 0\}$.

Theorem 6.3 (Whitney-Graustien Theorem)**.** *There exists a formal homotopy between real solutions of $\mathcal{R}_{\mathrm{imm}}$ if their winding numbers are equal, and the one-parametric h-principle holds for $\mathcal{R}_{\mathrm{imm}} \subset J^1(S^1, \mathbb{R}^2)$.*[5]

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[5]For simplicity, this statement excludes the forward implication of Theorem 2.3 since it is trivial and is not fruitfully expressed in the language of the h-principle.

Proof. Consider two immersions $f_0, f_1 : S^1 \to \mathbb{R}^2$ with equal winding numbers, and assume WLOG that $\|f'\| = 1$ and that their lengths are both 1. Since their winding numbers are equal we know that f_0' and f_1' are homotopic as endomorphisms on S^1. Call a homotopy which connects them $h_u : S^1 \to S^1$. We further define the formal section $F_u(x) = (x, (1 - u)f_0 + uf_1, h_u(x)) \subset J^1(S^1, \mathbb{R}^2)$, and since $\|h_u\| = 1$, we have that $F_u(S^1) \subset \mathcal{R}_{\mathrm{imm}}$.

To apply the PHAT we must first define a "thickening" of the two endpoint immersions $\bar{f}_0, \bar{f}_1 : K \to \mathbb{R}^2$ where $K = S^1 \times (-\delta, \delta)$ and δ is arbitrarily small. From this we can construct a formal homotopy $F_u : K \to J^1(K, \mathbb{R}^2)$ between $J^1_{\bar{f}_0}$ and $J^1_{\bar{f}_1}$ as in the previous paragraph, such that $F_u(K) \subset \mathcal{R}_{\mathrm{imm}}$ for all $u \in [0, 1]$. Now, since K is open set around S^1, which is a polygon, the Parametric Holonomic Approximation Theorem guarantees the existence of a family of arbitrarily small diffeotopies $h : K \to K$, and a family of holonomic sections $\tilde{F}_u : Op\, h(S^1) \to J^1(K, \mathbb{R}^2)$ connecting $J^1_{\bar{f}_0}$ and $J^1_{\bar{f}_1}$.

Finally, we know that $\mathcal{R}_{\mathrm{imm}} \subset J^1(S^1, \mathbb{R}^2)$ is an open differential relation, since its complement relation $(f' = 0)$ is closed. Moreover, the PHAT guarantees that $\mathrm{dist}(\tilde{F}, F)$ is arbitrarily small on $Op\, h(S^1)$, and therefore $J^1_{\tilde{f}_u \circ h}$ and F are similarly small, with $\tilde{F}_u = J^1_{\tilde{f}_u}$, when viewed as sections of $J^1(S^1, \mathbb{R}^2)$. Therefore, as $\mathcal{R}_{\mathrm{imm}}$ is open, we can pick $\tilde{F}$ close enough to F so that $J^1_{\tilde{f}_u \circ h}$ lies entirely in $\mathcal{R}_{\mathrm{imm}}$. Thus, $J^1_{\tilde{f}_u \circ h}$ is a family of holonomic sections connecting f_0 and f_1 lying entirely in $\mathcal{R}_{\mathrm{imm}}$. $\qquad\square$

Acknowledgments

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