AN INTRODUCTION TO CATEGORY THEORY AND THE YONEDA LEMMA

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ABSTRACT. We begin this introduction to category theory with definitions of categories, functors, and natural transformations. We provide many examples of each construct and discuss interesting relations between them. We proceed to prove the Yoneda Lemma, a central concept in category theory, and motivate its significance. We conclude with some results and applications of the Yoneda Lemma.

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INTRODUCTION

Category theory is an interdisciplinary field of mathematics which takes on a new perspective to understanding mathematical phenomena. Unlike most other branches of mathematics, category theory is rather uninterested in the objects being considered themselves. Instead, it focuses on the relations between objects of the same type and objects of different types. Its abstract and broad nature allows it to reach into and connect several different branches of mathematics: algebra, geometry, topology, analysis, etc.

A central theme of category theory is abstraction, understanding objects by generalizing rather than focusing on them individually. Similar to taxonomy, category theory offers a way for mathematical concepts to be abstracted and unified. What makes category theory more than just an organizational system, however, is its ability to generate information about these abstract objects by studying their relations to each other. This ability comes from what Emily Riehl calls "arguably the most important result in category theory" [4], the Yoneda Lemma. The Yoneda Lemma allows us to formally define an object by its relations to other objects, which is central to the relation-oriented perspective taken by category theory. Thus, this

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introductory paper to category theory provides the Yoneda Lemma as a stepping stone which must be understood before grappling with more complex concepts in category theory.

This paper assumes basic knowledge of some fundamental algebraic objects, such as rings, groups, modules over rings, etc. The format of this paper, how it presents the introductory material in category theory, loosely follows the structure of [1].

1. Categories

Definition 1.1. A *category* C consists of two main pieces of data:

(1) a collection of objects, which we denote $Ob(C)$; and
(2) a collection of morphisms, which we denote Mor_C.

such that

- for any $X, Y \in Ob(C)$, Mor_C contains the set of all morphisms $X \to Y$, denoted $Mor_C(X, Y)$;
- for every $X \in Ob(C)$, there exists a two-sided identity morphism $\mathbb{1}_X \in Mor_C(X, X)$;
- morphisms can be composed with each other associatively: for any three objects $X, Y, Z \in Ob(C)$, if $f \in Mor_C(X, Y)$ and $g \in Mor_C(Y, Z)$, then there exists a morphism $g \circ f \in Mor_C(X, Z)$.

As stated above, the composition of morphisms is associative and every object's identity morphism is two-sided. Thus for any $X, Y, Z, W \in Ob(C)$,

$$h \circ (g \circ f) = (h \circ g) \circ f \text{ and } f \circ \mathbb{1}_X = f, \mathbb{1}_Y \circ f = f,$$

where $f \in Mor_C(X, Y)$, $g \in Mor_C(Y, Z)$, $h \in Mor_C(Z, W)$.

Categories are the basic object of category theory; they are usually named by the objects contained within them. As we will see, practically every mathematical object or structure seen in a college algebra course can be formalized into a category.

Examples 1.2. The following are a few fundamental categories, taken from [1]:

(1) The category of sets, Set, with sets as objects and functions as morphisms;
(2) The category of groups, Grp, with groups as objects and group homomorphisms as morphisms. The term "morphism" was taken from groups, rings, and fields;
(3) Similar to Grp, the category of rings, $Ring$, with rings (unital and associative by definition, but not necessarily commutative) as objects and ring homomorphisms as morphisms, and the category of fields, $Field$, with fields and field homomorphisms as objects and morphisms respectively;
(4) The categories of left and right R-modules, ${}_RMod$ and Mod_R respectively, with module homomorphisms as morphisms, for some $R \in Ring$;
(5) The single element category (also known as a monoid), $*$, where $Ob(*)$ is a single object. Let G be a group; then $*_G$ is defined as a category with a single object $*$ where $Mor(*, *) = G$; in other words, the morphisms of $*_G$ are the elements of G, and composition is simply multiplication of these elements:

$$Mor(*, *) \times Mor(*, *) \to Mor(*, *), (f, g) \mapsto fg.$$

Although all the previous examples have morphisms which are functions, this example demonstrates that a category's morphisms are not necessarily functions.

To capture the nested nature of some mathematical objects, like how abelian groups are a subset of groups, categories have subcategories.

Definition 1.3. Let C be a category. A category S is called a *subcategory* of C if

(1) $Ob(S) \subset Ob(C)$ and $Mor_S \subset Mor_C$;
(2) for all $A \in Ob(S)$, $1_A \in Mor_S$; and
(3) for all $f : A \to B, g : B \to C \in Mor_S$, $g \circ f \in Mor_S$.

A subcategory S of C is *full* if for all $A, B \in S$, $Mor_S(A, B) = Mor_C(A, B)$, in which case specifying just the objects in S is sufficient to distinguish it.

Remark 1.4. All fields are rings, so if $k \in Ring$ satisfies the field axioms, then k-modules are also k-vector spaces; thus ${}_kMod = Mod_k = kVect$, which is the category of k-vector spaces with linear maps as morphisms. Similarly, if $k = \mathbb{Z}$, then we have the category of $\mathbb{Z}$-modules, which we know are abelian groups, forming the category *Ab*. *Ab* is a full subcategory of *Grp*, with abelian groups as objects.

If we focus on the basic category structure, one may ask: since categories are defined by objects and morphisms, what would happen if we created a new category with the same objects, but all the morphisms were "backwards"?

Definition 1.5. Let C be a category. The *opposite category* C^{op} is a category which has the same objects as C, but for all $f \in Mor_C(X, Y)$, there exists an $f^{op} \in Mor_{C^{op}}(Y, X)$ such that for any $x \in X$, $y \in Y$,

$$\text{if } f(x) = y \text{ then } f^{op}(y) = x.$$

Colloquially, C^{op} is the same category as C except its morphisms are "going in the opposite direction." This means that the identity morphism for every object $X \in C$ remains the same ($1_X^{op} : X \to X$) and the composition of morphisms is as follows: if $f \in Mor_C(A, B)$ and $g \in Mor_C(B, C)$, such that $f \circ g \in Mor_C(A, C)$, then $g^{op} \circ f^{op} \in Mor_{C^{op}}(C, A)$. Note that $(C^{op})^{op} = C$.

2. Functors

Although category theory is named after the categories themselves, it is actually more focused on interactions between them; namely, functors. Functors are a way to pass from one category to another, mapping objects to objects and morphisms to morphisms as seen in the following definition:

Definition 2.1. A *functor* $F : C \to D$ takes $A \in Ob(C)$ to $F(A) \in Ob(D)$ for all $A \in Ob(C)$. There are two different types (variance) of functors, covariant and contravariant, which differ in their mapping of morphisms:

- *Covariant* functors map $f \in Mor_C(A, B)$ to $F(f) \in Mor_D(FA, FB)$, such that $F(g \circ f) = Fg \circ Ff$;
- *Contravariant* functors map $f \in Mor_C(A, B)$ to $F(f) \in Mor_D(FB, FA)$, such that $F(g \circ f) = Ff \circ Fg$. Contravariant functors can be thought of as covariant functors from $C^{op} \to D$.

Both covariant and contravariant functors satisfy $F(\mathbb{1}_A) = \mathbb{1}_{FA}$.

Remark 2.2. A functor mapping from a category to itself, such as $F : C \to C$, is called an *endofunctor*.

Remark 2.3. With the introduction of functors, we must also introduce *commutative diagrams*. A diagram of the form

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow[i]{} & D \end{array}$$

is said to *commute* if $g \circ f = i \circ h$. More colloquially, a diagram commutes when there are multiple paths from one object to another such that the map along one path is equal to the map along the others. Commutative diagrams are used very frequently in category theory for diagram chasing and provide a visualization in cases where one may easily get lost.

In the same way that we think of morphisms of mathematical objects as maps which preserve all the underlying structure of an object, functors are the maps for categories which preserve the integrity of their category structure: objects, morphisms, compositions, and identities. Like morphisms of other mathematical objects, functors have some sort of notion of injectivity, surjectivity, and bijectivity on morphisms.

Definition 2.4. Let F be a functor from $C \to D$.

- F is *faithful* if for every $A, B \in C$ and $f, g \in Mor_C(A, B)$, $Ff = Fg \implies f = g$;
- F is *full* if for every $h \in Mor_D(FA, FB)$ ($Mor_D(FB, FA)$ if F is contravariant), there exists some $f \in Mor_C(A, B)$ such that $h = Ff$;
- F is *fully faithful* if it's both faithful and full.

A simple and telling example of a functor are forgetful functors:

Example 2.5. Forgetful functors are functors which "forget" or drop some of the input's structure or properties. One such forgetful functor, denoted $\Phi : C \to Set$, is a covariant functor which essentially gets rid of the structure of C, reducing its objects to sets and its morphisms to functions in Set. Every $\Phi(A)$ will contain the same elements as $A \in C$, and its functions will still map the same way as $Mor_C(A, -)$ element-wise; the only difference is that $\Phi(A) \in Set$. Sets are a very loose mathematical object, so many categories have a forgetful functor, such as Grp, $Ring$, and $Field$. The forgetful functor Φ is faithful, but not full (proof is left as exercise to the reader).

A particularly interesting functor is the free functor, which deals with the free group. What makes the free functor particularly interesting is its relation to the forgetful functor, which will be discussed in Section 3. A specific example of a free functor is the free group functor.

Example 2.6. The free group functor is a functor $F : Set \to Grp$ that sends a set S to the free group on S. A *free group* $F(S)$ is a group whose elements are finite "words" composed from the "alphabet" S and its inverse, S^{-1}. Words are of the form $w_1^{\epsilon_1} \dots w_n^{\epsilon_n}$ for some $n \in \mathbb{N}$, where $w_i \in S$ and $\epsilon_i = \pm 1$. The free group has the following equivalence relation:

$$\text{for any } s \in S, ss^{-1} \sim s^{-1}s \sim \mathbb{O}$$

where $\mathbb{O}$ is the empty word. The free group's operation is concatenation, with the empty word as the identity and an inverse $w_n^{-\epsilon_n} \dots w_1^{-\epsilon_1}$ for every $w_1^{\epsilon_1} \dots w_n^{\epsilon_n} \in F(S)$. For all $s \in S$, $F(s)$ is a distinct one-letter word $[s]$, so longer words in $F(S)$ are formed exclusively through concatenation. Consider a function $f : S \to S'$ in Set where for every $s \in S$ there exists a $t \in S'$ such that $f(s) = t$; then $F(f) : F(S) \to F(S')$ where every $s_1^{\epsilon_1} \dots s_n^{\epsilon_n} \in F(S)$ is mapped to $t_1^{\epsilon_1} \dots t_n^{\epsilon_n} \in F(S')$. The free group functor is faithful, but not full (proof is left as exercise to the reader).

Arguably the most important functor, particularly for the Yoneda lemma, is the hom functor.

Example 2.7. Let C be some category. For all objects $A \in Ob(C)$, we consider the following two functors, called hom functors:

$$h_A : C \to Set \text{ and } h^A : C \to Set.$$

h_A is a covariant functor which maps every $X \in Ob(C)$ to $Mor_C(A, X)$ and maps each morphism $f : X \to Y$ (for some $Y \in Ob(C)$) to the function

$$h_A(f) : Mor_C(A, X) \to Mor_C(A, Y),$$

which can be thought of as a collection of morphisms $f \circ g$ for each $g \in Mor_C(A, X)$, where $f \circ g : A \to X \to Y$.

On the other hand, h^A is a contravariant functor which maps every $X \in Ob(C)$ to $Mor_C(X, A)$ and maps each morphism $h : X \to Y$ to the function

$$h^A(h) : Mor_C(Y, A) \to Mor_C(X, A),$$

which, in parallel to $h_A(f)$, can be thought of as a collection of morphisms $g \circ h$ for each $g \in Mor_C(Y, A)$, where $g \circ h : X \to Y \to A$.

Remark 2.8. Unsurprisingly, these two functors are related. Consider some category C with hom functors, and take $X, X', Y, Y' \in Ob(C)$. For any pair of morphisms $i : Y \to Y'$ and $k : X' \to X$, the following diagram commutes:

$$\begin{array}{ccc} Mor_C(X, Y) & \xrightarrow{h^Y(k)} & Mor_C(X', Y) \\ {\scriptstyle h_X(i)}\downarrow & & \downarrow{\scriptstyle h_{X'}(i)} \\ Mor_C(X, Y') & \xrightarrow[h^{Y'}(k)]{} & Mor_C(X', Y'). \end{array}$$

Every function $j : X \to Y$ will be mapped to $i \circ j \circ k$ through either path:

$$(h_{X'}(i) \circ h^Y(k))(j) = h_{X'}(i)(j \circ k) = i \circ j \circ k$$

and

$$(h^{Y'}(k) \circ h_X(i))(j) = h^{Y'}(k)(i \circ j) = i \circ j \circ k.$$

This implies that the hom functor $Hom(-, -)$ (which, unlike h_A or h^A, isn't defined by a fixed object) is a bifunctor $C^{op} \times C \to Set$ [4]. A *bifunctor* is a functor whose domain is the product of two categories. The bifunctor $Hom(-, -)$ is simply a functor which takes two objects in C to the set of morphisms between them.

3. Natural Transformations

We can tack on another layer of abstraction to categories and functors with natural transformations. Natural transformations can be thought of as morphisms of functors.

Definition 3.1. Let F, G be functors $C \to D$ (where C, D are arbitrary categories) of the same variance: covariant or contravariant. A *natural transformation* $\varphi : F \to G$ is a collection of morphisms $\{\varphi_A : FA \to GA \mid A \in C\}$ such that for any $f : A \to B$ in Mor_C, depending on the variance of F and G, the following diagram commutes:

$$\begin{array}{ccc} FA & \xrightarrow{\varphi_A} & GA \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ FB & \xrightarrow[\varphi_B]{} & GB \end{array}$$

if F, G are covariant, and

$$\begin{array}{ccc} FA & \xrightarrow{\varphi_A} & GA \\ {\scriptstyle Ff}\uparrow & & \uparrow{\scriptstyle Gf} \\ FB & \xrightarrow[\varphi_B]{} & GB \end{array}$$

if F, G are contravariant. The set of natural transformations between F and G is denoted $Nat(F, G)$.

Remark 3.2. If φ_A is an isomorphism for every $A \in C$, then $F \cong G$. We say that F and G are isomorphic or *naturally equivalent*, and φ is a natural isomorphism or a natural equivalence.

A commonly-used example of natural transformations is with the single-element category [2].

Example 3.3. Recall from Example 1.2(5) that a group G can be used to define a category $*_G$ with a morphisms $* \to *$ for every $g \in G$. We can define a functor $*_G \to Set$, which maps $*$ to a set X and every morphism g to a function $f_g : X \to X$, $f_g(x) = g(x)$. Functors of this type encode a left group action of G on X, which begs the question of natural transformations between these functors. Let $A, B : *_G \to Set$ be functors with $A : *_G \to X$ and $B : *_G \to Y$ ($X, Y \in Set$). Then the natural transformation $\mu : A \to B$ consists of just one function $\mu : X \to Y$ such that $\mu(g(x)) = g(\mu(x))$ for every $x \in X$, $g \in G$, which can be demonstrated in the following commutative diagram:

$$\begin{array}{ccc} x & \xrightarrow{\mu} & \mu(x) \\ {\scriptstyle f_g}\downarrow & & \downarrow{\scriptstyle f_g} \\ g(x) & \xrightarrow[\mu]{} & \mu(g(x)) = g(\mu(x)) \end{array}$$

where $\mu(x), \mu(g(x)) \in Y$. This natural transformation μ is called a *G-equivariant map*.

An interesting example of a natural transformation from [4] deals with the construction of the opposite group.

Example 3.4. Let G be a group with the operation $*$. We can define its *opposite group* G^{op} as the group with the same set as G (for every $g \in G$, $g \in G^{op}$) and the operation $*^{op}$ such that $a *^{op} b = b * a$. Colloquially, the opposite group G^{op} is the same group with its operations "turned around" (so if G is abelian, then $G = G^{op}$). Constructing the opposite group from a group G can be seen as a covariant endofunctor $(-)^{op} : Grp \to Grp$, where a group homomorphism $f : G \to H$ is mapped to $f^{op} : G^{op} \to H^{op}$ such that $f^{op}(x) = f(x)$.

We can form a natural isomorphism between this opposite functor $(-)^{op} : Grp \to Grp$ and the identity functor $\mathbb{1}_{Grp} : Grp \to Grp$. We define the natural transformation ν as the collection of maps $\nu_G : G \to G^{op}$ for every group G which sends $g \in G$ to $g^{-1} \in G^{op}$. This natural transformation is a group homomorphism with inverse $\nu_{G^{op}}$ such that for any $a, b \in G$,

$$(a * b)^{-1} = b^{-1} * a^{-1} = a^{-1} *^{op} b^{-1} \text{ and } (a^{-1})^{-1} = a.$$

Thus, for any group homomorphism $f : G \to H$, the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{\nu_G} & G^{op} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^{op}} \\ H & \xrightarrow[\nu_H]{} & H^{op} \end{array}$$

since $\nu_H \circ \mathbb{1}_{Grp}(f(g)) = \nu_H(h) = h^{-1} = f^{op}(g^{-1}) = f^{op} \circ \nu_G(g)$ for $h = f(g)$.

Another interesting natural transformation is the following example from [1], dealing with left R-modules:

Example 3.5. Let R be a ring and consider the left R-modules ${}_RMod$. There exists a natural equivalence between h_R and the identity functor $\mathbb{1}_{{}_RMod} : {}_RMod \to {}_RMod$. We know h_R can be defined as a covariant functor mapping from ${}_RMod$ to the set of functions between R and ${}_RMod$. For every $M \in {}_RMod$, this map can be seen as mapping $M \to R \cdot M$, which emulates the left R-module structure; thus h_R can be seen as a map ${}_RMod \to {}_RMod$. Consider the natural transformation $\varphi : Mor_R(R, M) \to \mathbb{1}_{{}_RMod}(M)$ for all $M \in Ob({}_RMod)$ such that $\varphi_M(\alpha) \mapsto \alpha(1_R)$, which we know is equivalent to some $m \in M$. Then for any $f \in Mor_{{}_RMod}(M, N)$, the following diagram commutes:

$$\begin{array}{ccc} Mor_R(R, M) & \xrightarrow{\varphi_M} & M \\ {\scriptstyle h_M(f)}\downarrow & & \downarrow{\scriptstyle f} \\ Mor_R(R, N) & \xrightarrow[\varphi_N]{} & N \end{array}$$

which implies that for some $\alpha \in Mor_R(R, M)$,

$$\begin{array}{ccc} \alpha & \xrightarrow{\varphi_M} & \alpha(1_R) \\ {\scriptstyle h_R(f)}\downarrow & & \downarrow{\scriptstyle f} \\ h_R(f)(\alpha) & \xrightarrow[\varphi_N]{} & \varphi_N(h_R(f)(\alpha)) = f(\alpha(1_R)) \end{array}$$

We know that $h_R(f)(\alpha) = \beta \in Mor_R(R, N)$ where $f(\alpha(r)) = \beta(r)$ for all $r \in R$, so $\varphi_N(\beta) = \beta(1_R) = f(\alpha(1_R))$; therefore this diagram indeed commutes.

Having defined natural transformations, we digress to revisit the relation between the forgetful functor Φ_{Grp} and the free group functor (Examples 2.5 and 2.6).

Definition 3.6. Let $F : C \to D$, $G : D \to C$ be functors of the same variance. The pair (F, G) is called an *adjoint pair*, where F is the *left adjoint* functor and G is the *right adjoint* functor, if for all $A \in Ob(C)$, $B \in Ob(D)$, there exists a bijection $\varphi : Mor_D(FA, B) \to Mor_C(A, GB)$ such that the following diagrams commute:

$$\begin{array}{ccc} Mor_D(FA,B) & \xrightarrow{\varphi_{A,B}} & Mor_C(A,GB) \\ {\scriptstyle h_{FA}(g)}\downarrow & & \downarrow{\scriptstyle h_A(Gg)} \\ Mor_D(FA,B') & \xrightarrow[\varphi_{A,B'}]{} & Mor_C(A,GB') \end{array}$$

for a fixed $A \in Ob(C)$ and $g \in Mor_D(B, B')$; and

$$\begin{array}{ccc} Mor_D(FA,B) & \xrightarrow{\varphi_{A,B}} & Mor_C(A,GB) \\ {\scriptstyle h^B(Ff)}\uparrow & & \uparrow{\scriptstyle h^{GB}(f)} \\ Mor_D(FA',B) & \xrightarrow[\varphi_{A',B}]{} & Mor_C(A',GB) \end{array}$$

for a fixed $B \in Ob(D)$ and $f \in Mor_C(A, A')$.

Example 3.7. The free group functor F is the left adjoint of the forgetful functor Φ_{Grp}. Let S be a set and G be a free group; we want to show a natural bijection

$$Mor_{Grp}(F(S), G) \cong Mor_{Set}(S, \Phi_{Grp}(G)).$$

We know that $\Phi_{Grp}(G)$ is simply the underlying set of G, so any function $f' : S \to \Phi_{Grp}(G)$ can be passed very easily to a morphism $f : S \to G$, since all the objects and morphisms in G are still in place. Thus to prove this bijection we simply need to find a unique homomorphism $g : F(S) \to G$ for every function $f : S \to G$. We can define $g : F(S) \to G$ such that for any word $w_1^{\epsilon_1} \dots w_n^{\epsilon_n} \in F(S)$,

$$g(w_1^{\epsilon_1} \dots w_n^{\epsilon_n}) = f(w_1^{\epsilon_1}) \dots f(w_n^{\epsilon_n}) \in G.$$

Thus for every $g \in Mor_{Grp}(F(S), G)$, we can assign the function

$$f : S \to G, \text{ where } f(s) = g([s]),$$

and for every $f : S \to G$, which is equivalent to a $f' \in Mor_{Set}(S, \Phi_{Grp}(G))$, we can assign the group homomorphism

$$g : F(S) \to G, \text{ where } g(w_1^{\epsilon_1} \dots w_n^{\epsilon_n}) = f(w_1^{\epsilon_1}) \dots f(w_n^{\epsilon_n}).$$

Therefore, there is indeed a bijection

$$Mor_{Grp}(F(S), G) \cong Mor_{Set}(S, \Phi(G))$$

giving us the adjoint pair (F, Φ).

With the new layer of abstraction offered by natural transformations, we can also think of functors as objects and natural transformations as their morphisms.

Definition 3.8. Let C, D be categories. A *functor category* $[C, D]$ is a category whose objects are functors $F : C \to D$ and morphisms are natural transformations between these functors.

An important example of a functor category is the category of presheaves.

Example 3.9. Let C be a category. A *presheaf* on C is a functor

$$F : C^{op} \to Set.$$

The *category of presheaves* on C, denoted $[C^{op}, Set]$, is a functor category with functors $F : C^{op} \to Set$ as objects and natural transformations between presheaves as morphisms.

Presheaves have many uses in category theory, many of which reach beyond the scope of this paper. One important use of presheaves is in the Yoneda Lemma and its corollaries.

4. The Yoneda Lemma

Unlike traditional mathematics, category theory emphasizes understanding objects by their relations to other objects rather than defining them on their own—the Yoneda Lemma motivates this perspective.

Definition 4.1. A functor $F : C \to Set$ is *representable* if there exists some object $X \in C$ such that $F \cong h_X$ if F is covariant and $F \cong h^X$ if F is contravariant. We say that the functor F is *represented by* the object X.

Lemma 4.2 (Yoneda Lemma)**.** *For any covariant functor $F : C \to Set$ and any $A \in Ob(C)$, there is a natural bijection*

$$Nat(h_A, F) \cong F(A)$$

such that $\alpha \in Nat(h_A, F) \leftrightarrow \alpha_A(\mathbb{1}_A) \in F(A)$.

There are a few variations to the proof of this lemma; the following proof, which avoids using the opposite category, originates from [1].

Proof. Let $\varphi : h_A \to F$ be a natural transformation where for any $X \in Ob(C)$, $\varphi_X : h_A(X) \to F(X)$. Then for all $X, Y \in Ob(C)$, $f \in Mor_C(X, Y)$, the following diagram must commute:

$$\begin{CD} h_A(X) @>{\varphi_X}>> F(X) \\ @V{h_A(f)}VV @VV{F(f)}V \\ h_A(Y) @>>{\varphi_Y}> F(Y) \end{CD}$$

where, as defined in Example 2.6, $h_A(f) : g \mapsto f \circ g$ for each $g \in Mor_C(A, X)$. In particular, for the fixed $A \in Ob(C)$, we have $\varphi_A : h_A(A) \to F(A)$. Note that $h_A(A) = Mor_C(A, A)$, which contains the identity functor for A, $\mathbb{1}_A$. Now consider the map $Nat(h_A, F) \to F(A)$; we know $\varphi \in Nat(h_A, F)$, so we define this map $\mu_\varphi : \varphi \mapsto \varphi_A(\mathbb{1}_A)$. Thus, for any $B \in Ob(C)$ and $f \in Mor_C(A, B)$, the following diagram commutes:

$$\begin{CD} h_A(A) @>{\varphi_A}>> F(A) \\ @V{h_A(f)}VV @VV{F(f)}V \\ h_A(B) @>>{\varphi_B}> F(B) \end{CD}$$

which means that

$$\begin{array}{ccc} \mathbb{1}_A & \longrightarrow & \mu_\varphi \\ \downarrow & & \downarrow \\ f & \longrightarrow & \varphi_B(f). \end{array}$$

This implies that $\varphi_B(f) = F(f)(\mu_\varphi)$, which shows that φ_B is determined by μ_φ for all $B \in Ob(C)$, giving us injectivity for the map $Nat(h_A, F) \to F(A)$ (where $\varphi \mapsto \mu_\varphi$).

Conversely, take some $u \in F(A)$. For every $B \in Ob(C)$ and $f : A \to B$, we define $\psi : h_A \to F$ such that

$$\psi_B(f) = F(f)(u).$$

To prove surjectivity of the map $Nat(h_A, F) \to F(A)$, we must prove that ψ is a natural transformation, which means proving the commutativity of the following diagram for some arbitrary $g : B \to C$:

$$\begin{array}{ccc} h_A(B) & \xrightarrow{\psi_B} & F(B) \\ {\scriptstyle h_A(g)}\downarrow & & \downarrow{\scriptstyle F(g)} \\ h_A(C) & \xrightarrow[\psi_C]{} & F(C). \end{array}$$

Let $f : A \to B$. Then

$$F(g) \circ (\psi_B(f)) = F(g) \circ F(f)(u) = F(g \circ f)(u),$$

and

$$\psi_C \circ (h_A(g)(f)) = \psi_C(g \circ f) = F(g \circ f)(u).$$

Thus, $F(g) \circ \psi_B = \psi_C \circ h_A(g)$, so the diagram is commutative, which implies that $Nat(h_A, F) \to F(A)$ is also surjective. Therefore, $Nat(h_A, F) \cong F(A)$. □

Remark 4.3. The dual statement of the Yoneda lemma also holds [4], which uses a contravariant $F : C \to Set$ and the contravariant hom functor instead: $Nat(h^A, F) \cong F(A)$. Some category theorists regard this dual statement as the actual Yoneda Lemma, and its covariant counterpart as the dual, because of the emphasis on "looking at" A from other objects' perspectives rather than "looking at" other objects from A's perspective.

5. Corollaries and Applications

A direct result of the Yoneda Lemma is the Yoneda embedding functor [5]:

Definition 5.1. The *Yoneda embedding* for a category C is a covariant functor

$$Y : C \to [C^{op}, Set]$$

which maps an object $A \in C$ to the functor $h^A \in [C^{op}, Set]$ and a morphism $f \in Mor_C(A, B)$ to the natural transformation $\varphi : h^A \to h^B$.

There is also a contravariant Yoneda embedding functor, $C^{op} \to [C, Set]$; however, the contravariant Yoneda embedding is simply the covariant functor applied to C^{op}, so we'll simply use the covariant Yoneda embedding. The Yoneda embedding functor allows us to consider some category C in terms of its representables in the category of presheaves on C.

Corollary 5.2. *Yoneda embedding is a fully faithful functor.*

Proof. Let $A, B \in Ob(C)$ and $F = h^B$; by the dual statement of the Yoneda Lemma (Remark 4.3), we know that there exists a natural bijection

$$Nat(h^A, h^B) \cong h^B(A) = Mor(A, B).$$

Let $\varphi : h^A \to h^B$ be a natural transformation; from the Yoneda Lemma, we know that the bijection gives rise to $\varphi_A(\mathbb{1}_A) : A \to B$ from φ; thus every natural transformation φ has exactly one corresponding $f = \varphi_A(\mathbb{1}_A) \in Mor(A, B)$. Since there exists a natural bijection mapping $Mor(A, B)$ to $Nat(h^A, h^B)$ for every $A, B \in Ob(C)$, then the Yoneda embedding, which maps from one to the other, must be fully faithful. □

Remark 5.3. The contravariant Yoneda embedding functor is also fully faithful, using a similar proof with the Yoneda Lemma as we've proven it (Lemma 4.2, not the dual statement).

This corollary shows that the Yoneda embedding is truly an "embedding": every map $A \to B$ in C has an equivalent map $h^A \to h^B$ in $[C^{op}, Set]$ and vice versa. What this tells us is that there is a direct relationship between an object and all its relations to other objects; analyzing the relations that an object has with other objects is equivalent to analyzing the object itself. This leads us to another corollary: the uniqueness of representing objects.

Corollary 5.4. *Let C be a category; for all $A, B \in Ob(C)$, $h_A \cong h_B$ if and only if $A \cong B$.*

The isomorphism established between a category C and its category of presheaves $[C^{op}, Set]$, gives us this corollary rather trivially. However, the following is a different proof, taken from [1], which demonstrates the uniqueness of representing objects directly from the Yoneda Lemma, without Corollary 5.2.

Proof. The Yoneda Lemma gave us the bijection

$$Nat(h_A, h_B) \cong Mor(B, A)$$

which associated a natural transformation $\varphi^f : h_A \to h_B$ to $f : B \to A$, where

$$\varphi^f_X : h_A(X) \to h_B(X), g \mapsto g \circ f \text{ for every } g : A \to X.$$

Consider some $f_1 : B \to A$ and $f_2 : A \to B$, and let $\varphi^{f_1} : h_A \to h_B$ and $\varphi^{f_2} : h_B \to h_A$ be natural transformations such that for any $X \in Ob(C)$,

$$\varphi^{f_1}_X : h_A(X) \to h_B(X) \text{ where } g \in Mor(A, X) \mapsto g \circ f_1 \in Mor(B, X)$$

and

$$\varphi^{f_2}_X : h_B(X) \to h_A(X) \text{ where } h \in Mor(B, X) \mapsto h \circ f_2 \in Mor(A, X).$$

Then

$$\varphi^{f_2} \circ \varphi^{f_1} : h_A(X) \to h_A(X) \text{ where } g \mapsto g \circ (f_1 \circ f_2),$$

which shows that $\varphi^{f_2} \circ \varphi^{f_1} = \varphi^{f_1 \circ f_2}$ (similarly $\varphi^{f_1} \circ \varphi^{f_2} = \varphi^{f_2 \circ f_1}$). If $f_1 \circ f_2 = \mathbb{1}_A$, then $\varphi^{f_2} \circ \varphi^{f_1} = \varphi^{f_1 \circ f_2}$ is the identity transformation $h_A \to h_A$. The parallel for $f_2 \circ f_1 = \mathbb{1}_B$ is implied, so $\varphi^{f_2} \circ \varphi^{f_1} = \varphi^{f_2 \circ f_1}$ is the identity transformation $h_B \to h_B$. Thus, if $A \cong B$ then $h_A \cong h_B$.

Proving the other direction, let $\varphi : h_A \to h_B$ be a natural equivalence and consider $\varphi_A : h_A(A) \to h_B(A)$. We know that $\mathbb{1}_A \in Mor_C(A, A)$, so $\varphi_A(\mathbb{1}_A) \in$

$Mor_C(B,A)$. Let $\psi : h_B \to h_A$ be a natural equivalence; by a similar argument $\psi_B(\mathbb{1}_B) \in Mor_C(A,B)$. If we compose ψ and φ we get the natural transformation $\psi \circ \varphi : h_A \to h_A$ which produces $(\psi \circ \varphi)_A(\mathbb{1}_A) \in Mor_C(A,A)$. Let $f = \varphi_A(\mathbb{1}_A)$ and $g = \psi_B(\mathbb{1}_B)$, and consider the following commutative diagram:

$$\begin{array}{ccccc} h_A(A) & \xrightarrow{\varphi_A} & h_B(A) & \xrightarrow{\psi_A} & h_A(A) \\ & & {\scriptstyle h_B(f)}\uparrow & & \uparrow{\scriptstyle h_A(f)} \\ & & h_B(B) & \xrightarrow[\psi_B]{} & h_A(B). \end{array}$$

This gives us

$$\begin{aligned} (\psi \circ \varphi)_A(\mathbb{1}_A) &= \psi_A(\varphi_A(\mathbb{1}_A)) \\ &= \psi_A(f) \\ &= \psi_A(h_B(f)(\mathbb{1}_B)) \\ &= h_A(f)(\psi_B(\mathbb{1}_B)) \\ &= h_A(f)(g) \\ &= f \circ g. \end{aligned}$$

Thus, if $\psi \circ \varphi$ is the identity natural transformation $h_A \to h_A$, then $f \circ g = (\psi \circ \varphi)_A(\mathbb{1}_A) = \mathbb{1}_A$. If we rearrange the commutative diagram, then by a similar argument we can find that $(\varphi \circ \psi)_B(\mathbb{1}_B) = g \circ f$, so if $\varphi \circ \psi$ is the identity natural transformation $h_B \to h_B$, then $g \circ f = \mathbb{1}_B$. If $\psi \circ \varphi$ and $\varphi \circ \psi$ are both identity natural transformations, then it follows that $h_A \cong h_B$; thus, if $h_A \cong h_B$ then $A \cong B$.

Therefore, for all $A, B \in Ob(C)$, $h_A \cong h_B$ if and only if $A \cong B$. □

From the perspective of the Yoneda Lemma and Yoneda embedding, Corollary 5.3 states a simple, yet powerful, concept: if the relations between other objects and A are the same as the relations between other objects and B, then A must be "equivalent" (isomorphic) to B, and vice versa.

Together, the Yoneda Lemma and these two corollaries show us that every object is determined up to isomorphism by its relationships with other objects. This empowers category theory to be more than simply an organizational structure for mathematics; with the Yoneda Lemma, category theory can actively analyze various mathematical objects (with little to no regard to which branch they are from) despite its emphasis on relations instead of objects. The Yoneda Lemma leads to many more complex results in category theory, such as Tannaka reconstruction theorems, Isbell conjugation, and many more concepts which reach far beyond the scope of this introductory paper [3].

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