HOPF ALGEBRAS AND EXOTIC CHARACTERISTIC CLASSES

ANDREW BURKE

Abstract. In this paper, we introduce the basic notions of Hopf algebras and Hopf rings, after which we construct the Steenrod algebra and Dyer-Lashof algebra as concrete examples. We then apply our algebraic framework to define certain exotic characteristic classes for spherical fibrations and to prove their nontriviality.

Contents

1. Introduction

A spherical fibration is a fibration for which each fiber is homotopy equivalent to a sphere. There exists a structure group SG and a classifying space BSG for oriented spherical fibrations. It can be shown that BSG has the homotopy type of $\lim_{n\to\infty}(\Omega^n S^n)_1$ where $(\Omega^n S^n)_1$ denotes the space of degree one basepoint-preserving maps $S^n \to S^n$. The above colimit is naturally a subspace of $QS^0 = \lim_{n\to\infty} \Omega^n S^n$, a space whose homology has the algebraic structure of a Hopf ring and is acted on by the Steenrod algebra A and the Dyer-Lashof algebra R, two examples of Hopf algebras.

Our goal in this paper is to define certain exotic classes $e_k \in H^{p^k r-1}(BSG;\mathbb{Z}_p)$ modulo indeterminacy introduced by ordinary characteristic classes. We will, with some effort, prove that each of these classes is nonzero.

Before defining exotic classes however, we must introduce the algebraic structures mentioned above. In section 2, we define Hopf algebras. Briefly stated, a Hopf algebra A is a module together with a product operation $\phi : A \otimes A \to A$ and a coproduct operation $\psi : A \to A \otimes A$ satisfying certain compatibility conditions. In section 3, we construct the Steenrod algebra and show that it is a Hopf algebra which acts on the cohomology of any topological pair. In section 4, we construct the Dyer-Lashof algebra. It is analogous to the Steenrod algebra, except that it acts on the homology of infinite loop spaces. One may view a Hopf algebra as an abelian group over the category of coalgebras. With this interpretation in mind, in section 5 we define Hopf rings to be a commutative ring over the category of coalgebras. Having developed the above algebraic machinery, we introduce exotic characteristic classes of spherical fibrations in section 6 and prove that these classes are nontrivial in section 7.

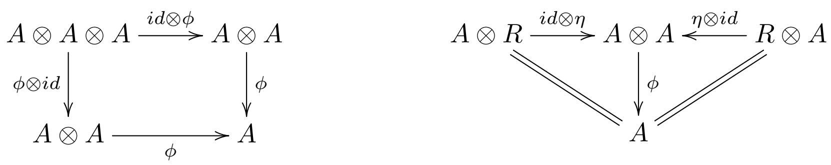
We assume only a basic knowledge of homology and cohomology theory throughout this paper.

2. Hopf Algebras

We now introduce the notion of a Hopf algebra and provide several examples. Throughout, let R be a fixed commutative ring with unit.

A Hopf algebra is, roughly speaking, a set A that is both an algebra and a coalgebra. Furthermore, it is equipped with a map $\xi : A \to A$ called its antipode; this may be thought of as a sort of conjugation. Hence, in order to formally define a Hopf algebra, we must start with the simpler concepts of an algebra and a coalgebra. A coalgebra is, in some sense, the dual of an algebra. Our definitions contain mirror commutative diagrams to illustrate this duality.

Definition 2.1. An R-algebra $A = (A, \phi, \eta)$ is a graded R-module A together with R-module homomorphisms $\phi : A \otimes A \to A$ and $\eta : R \to A$ such that the following diagrams commute.



The map ϕ is called the product map. The first diagram shows that multiplication is associative. The map η is called the unit; it is determined uniquely by the image $\eta(1) \in A$. By the second diagram, this element is the multiplicative unit in A.

If A and B are graded R-modules, then $A \otimes B$ is given the grading

$$(A \otimes B)_n = \bigoplus_{i+j=n} (A_i \otimes B_j)$$

where $(-)_n$ denotes the direct summand of a graded module consisting of its homogeneous elements of degree n. If A and B are, additionally, R-algebras, then $A \otimes B$ is an R-algebra with unit $\eta_A \otimes \eta_B : R \cong R \otimes R \to A \otimes B$ and product

$$(A \otimes B) \otimes (A \otimes B) \xrightarrow{id \otimes \gamma \otimes id} A \otimes A \otimes B \otimes B \xrightarrow{\phi_A \otimes \phi_B} A \otimes B$$

where $\gamma(b \otimes a) = (-1)^{\deg(a)\deg(b)} a \otimes b$. More explicitly,

$$(a \otimes b) \cdot (a' \otimes b') = (-1)^{\deg(a')\deg(b)} aa' \otimes bb'$$

We say that the R-algebra A is (graded) commutative if ϕ is an R-algebra homomorphism. This condition holds if and only if $x \cdot y = (-1)^{\deg(x)\deg(y)} y \cdot x$ for all $x, y \in A$.

Definition 2.2. A map $f : A \to B$ between R-algebras (A, ϕ_A, η_A) and (B, ϕ_B, η_B) is an R-module homomorphism $f : A \to B$ such that $f \circ \eta_A = \eta_B$ and $f \circ \phi_A = \phi_B \circ (f \otimes f)$.

Example 2.3. The cohomology $H^*(X;R)$ of any topological space X is an R-algebra whose product is given by the cup product.

Definition 2.4. An R-coalgebra $A = (A, \psi, \epsilon)$ is a graded R-module A together with R-module homomorphisms $\psi : A \to A \otimes A$ and $\epsilon : A \to R$ such that the following diagrams commute.

$$\begin{array}{ccc} A & \xrightarrow{\psi} & A \otimes A \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle id\otimes\psi} \\ A \otimes A & \xrightarrow[\psi\otimes id]{} & A \otimes A \otimes A \end{array} \qquad \begin{array}{ccccc} & & A & & \\ & \swarrow\!\!= & \downarrow{\scriptstyle\psi} & =\!\!\searrow & \\ A \otimes R & \xleftarrow[id\otimes\epsilon]{} & A \otimes A & \xrightarrow[\epsilon\otimes id]{} & R \otimes A \end{array}$$

Note that these diagrams are dual to the corresponding diagrams in the definition of an algebra. The map ψ is called the coproduct and ϵ is the counit. The first diagram shows that the coproduct is coassociative and the second displays the behavior of the counit.

If A, B are R-coalgebras, so is $A \otimes B$ with counit $\epsilon_A \otimes \epsilon_B : A \otimes B \to R \otimes R \cong R$ and coproduct

$$A \otimes B \xrightarrow{\psi_A \otimes \psi_B} A \otimes A \otimes B \otimes B \xrightarrow{id \otimes \gamma \otimes id} A \otimes B \otimes A \otimes B$$

We say that A is (graded) cocommutative if ψ is an R-coalgebra homomorphism. This condition holds if and only if $\psi \circ \gamma = \psi$.

Definition 2.5. A map $f : A \to B$ between R-coalgebras (A, ψ_A, ϵ_A) and (B, ψ_B, ϵ_B) is an R-module homomorphism $f : A \to B$ such that $\epsilon_B \circ f = \epsilon_A$ and $\psi_B \otimes f = (f \otimes f) \circ \psi_A$.

Example 2.6. The homology $H_*(X;F)$ of any topological space X is an F-coalgebra if F is a field. Indeed, the diagonal map $X \to X \times X$ induces the coproduct $H_*(X;F) \to H_*(X \times X;F) \cong H_*(X;F) \otimes H_*(X;F)$ where the isomorphism comes from the Kunneth Theorem.

We now state the conditions under which algebra and coalgebra structures are compatible.

Definition 2.7. An R-bialgebra $(A, \phi, \psi, \eta, \epsilon)$ is an algebra (A, ϕ, η) and a coalgebra (A, ψ, ϵ) such that ϕ and η are R-coalgebra homomorphisms or, equivalently, such that ψ and ϵ are R-algebra homomorphisms. This condition is expressed by the commutativity of the following diagrams.

$$\begin{array}{ccc} R & = & R \\ & {\scriptstyle\eta}\searrow \quad \nearrow{\scriptstyle\epsilon} & \\ & A & \end{array} \qquad \begin{array}{ccc} A \otimes A & \xrightarrow{\phi} & A \\ & {\scriptstyle\epsilon\otimes\epsilon}\searrow \quad \nearrow{\scriptstyle\epsilon} & \\ & R \otimes R \cong R & \end{array}$$

$$\begin{array}{ccc} & R \otimes R \cong R & \\ & {\scriptstyle\eta\otimes\eta}\swarrow \quad \searrow{\scriptstyle\eta} & \\ A \otimes A & \xleftarrow[\psi]{} & A \end{array} \qquad \begin{array}{ccccc} A \otimes A & \xrightarrow{\phi} & A & \xrightarrow{\psi} & A \otimes A \\ {\scriptstyle\psi\otimes\psi}\downarrow & & & & \uparrow{\scriptstyle\phi\otimes\phi} \\ A \otimes A \otimes A \otimes A & & \xrightarrow[id\otimes\gamma\otimes id]{} & & A \otimes A \otimes A \otimes A \end{array}$$

The assertion that a coalgbera is, essentially, the dual of an algebra is expressed rigorously as follows.

Proposition 2.8. *Let A be a projective R-module of finite type. Then*

(1) (A, ϕ, η) is an algebra if and only if (A^*, ϕ^*, η^*) is a coalgbra where A^* denotes the dual R-module to A and ϕ^*, η^* denote the dual maps.
(2) $(A, \phi, \psi, \eta, \epsilon)$ is a bialgebra if and only if $(A^*, \psi^*, \phi^*, \epsilon^*, \eta^*)$ is a bialgebra.

We may now finally define Hopf algebras.

Definition 2.9. A Hopf Algebra $(A, \phi, \psi, \eta, \epsilon, \xi)$ is a bialgebra $(A, \phi, \psi, \eta, \epsilon)$ together with an antipode ξ, an R-module homomorphism $\xi : A \to A$ such that the following diagrams commute.

We think of the antipode as being a sort of conjugation on A. Though the term antipode suggests that $\xi^2 = id$, this need not hold. Indeed, if we let n be the minimum number for which $\xi^n = id$, then n can be any even number or it can even be infinite. However, $n = 2$ for each example in this paper. Given a bialgebra, the existence and uniqueness of an antipode does not hold in general, but can be shown in a special case.

Proposition 2.10. *Let A be a connected R-bialgebra, one for which $A_i = 0$ for $i < 0$ and $A_0 = R$. Then A admits a unique antipode ξ. Furthermore, $\xi^2 = id_A$ if A is either commutative or cocommutative.*

Proof. See [6], Proposition 21.3.3. □

We now consider a few examples.

Example 2.11. Let G be a group and let R be a ring. Then the group ring

$$R[G] = \left\{ \sum_{i=0}^{N} r_i g_i : r_i \in R, g_i \in G \right\}$$

is a Hopf Algebra over R. Indeed, define the unit by the relation $\eta(1) = 1$, the product by $\phi(g, g') = gg'$, the coproduct by $\psi(g) = g \otimes g$, the counit by $\epsilon(g) = 1$, and the antipode by $\xi(g) = g^{-1}$ for all $g \in G$. Each of these maps is extended linearly over $R[G]$. More explicitly, $\eta(r) = r \cdot 1$,

$$\phi\left(\sum_{i=0}^{N} r_i g_i, \sum_{j=0}^{M} r_i' g_i' \right) = \sum_{i=0}^{N} \sum_{j=0}^{M} r_i r_j' g_i g_j'$$

and $\psi(\sum r_i g_i) = \sum r_i (g_i \otimes g_i)$, $\epsilon(\sum r_i g_i) = \sum r_i$, $\xi(\sum r_i g_i) = \sum r_i g_i^{-1}$. One can show that, with these definitions, $(R[G], \phi, \psi, \eta, \epsilon, \xi)$ is a Hopf algebra.

Example 2.12. Let V be a vector space over a field k. The tensor algebra $T(V)$ is well-known as a k-algebra. It can, additionally, be given the structure of a Hopf algebra. Define the coproduct by $\psi(1) = 1 \otimes 1$ and $\psi(x) = x \otimes 1 + 1 \otimes x$ for $x \in T^1(V)$ and extend it as a k-algebra homomorphism over all of $T(V)$. Thus

$$\psi(x_1 \cdots x_n) = \psi(x_1) \cdots \psi(x_n)$$

for $x_1 \dots x_n \in T^1(V)$. For instance,

$$\begin{aligned}\psi(xy) =& \psi(x)\psi(y) = (x \otimes 1 + 1 \otimes x)(y \otimes 1 + 1 \otimes y) \\ =& (xy \otimes 1) + (x \otimes y) + (y \otimes x) + (1 \otimes xy)\end{aligned}$$

The counit is the identity on the summand $k \subset T(V)$ and is zero elsewhere. Finally, the antipode ξ is given by $\xi(x_1 \cdot \dots x_n) = (-1)^n x_n \cdots x_1$, so that, in particular, $\xi(1) = 1$ and $\xi(x) = -x$.

Example 2.13. A lie algebra is a vector space $\mathfrak{g}$ over a field k together with an operation $[-,-] : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ that is bilinear, antisymmetric, and satisfies the Jacobi identity

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]].$$

The universal enveloping algebra $U(\mathfrak{g})$ is defined as $T(\mathfrak{g})/I$ where $I \subset T(\mathfrak{g})$ is the two-sided ideal generated by elements of the form $xy - yx - [x, y]$ for $x, y \in \mathfrak{g}$. The Hopf algebra structure on $T(\mathfrak{g})$ descends to give $U(\mathfrak{g})$ the structure of a Hopf algebra. To see that the coproduct and counit are well-defined, note that

$$\psi(xy - yx - [x, y]) = (xy - yx - [x, y]) \otimes 1 + 1 \otimes (xy - yx - [x, y]) \in I$$

$$\epsilon(xy - yx - [x, y]) = -xy + yx + [x, y] \in I.$$

Hence $U(\mathfrak{g})$ is a Hopf algebra.

Example 2.14. In particular, let $\mathfrak{g}$ be a k-vector space with basis $x_1 \dots x_n$ and set $[v, w] = 0$ for all $v, w \in \mathfrak{g}$. Then $U(\mathfrak{g}) \cong k[x_1 \dots x_n]$, the polynomial algebra generated over k by algebraically independent elements $x_1 \dots x_n$.

Example 2.15. The homology of an H-space can be shown to be a Hopf algebra as long as the product is associative up to homotopy.

3. Steenrod Algebra

The Steenrod algebra A is an important Hopf algebra. Throughout, let p be a fixed odd prime and let $H^*(X)$ denote $H^*(X; \mathbb{Z}_p)$ unless otherwise specified. Everything that follows can be adjusted to the case $p = 2$ with minor changes. Here we require p to be an odd prime for convenience. Since the following quantity will appear often in what follows, set $r = 2p - 2$.

The Steenrod operation P^i is a homomorphism $P^i : H^j(X, V) \to H^{j+ri}(X, V)$ defined for all pairs of spaces (X, V) and all integers $i, j \geq 0$. It is uniquely characterized by the following properties.

(1) $P^i(x) = x$ if $i = 0$, $P^i(x) = x^p$ if $i = \deg(x)/2$, and $P^i(x) = 0$ if $i > \deg(x)/2$
(2) (Naturality) $f^* P^i = P^i f^*$ for any map $f : (X, V) \to (Y, W)$
(3) (Cartan Formula) $P^n(x \cup y) = \sum_{i+j=n} P^i(x) \cup P^j(y)$

Additionally, we will need the coboundary operation $\delta : H^j(X, V) \to H^{j+1}(X, V)$ associated with the coefficient sequence $0 \to \mathbb{Z}_p \to \mathbb{Z}_{p^2} \to \mathbb{Z}_p \to 0$. It satisfies naturality as well as the properties $\delta\delta = 0$ and $\delta(x \cup y) = (\delta x) \cup y + (-1)^{\deg x} x \cup (\delta y)$.

Let F be the free graded associative (but not commutative) algebra over $\mathbb{Z}_p$ generated by δ and P^i for $i \geq 0$ where $\dim(\delta) = 1$ and $\dim(P^i) = ir$. Then F clearly acts on $H^*(X,V)$ for pairs (X,V) by composition of functions. Let I be the two-sided ideal of F consisting of all elements $f \in F$ for which $fx = 0$ for all pairs (X,V) and all elements $x \in H^*(X,V)$. Then A is defined to be the quotient $\mathbb{Z}_p$-algebra F/I. It acts on cohomology rings $H^*(X,V)$ as before.

Proposition 3.1. *(Ádem relations)*

$$P^a P^b = \sum_i (-1)^{a+i} \binom{(p-1)(b-i)-1}{a-pi} P^{a+b-i} P^i$$

for $a < pb$ and

$$\begin{aligned} P^a \delta P^b = & \sum_i (-1)^{a+i} \binom{(p-1)(b-i)}{a-pi} \beta P^{a+b-i} P^i \\ & + \sum_i (-1)^{a+i+1} \binom{(p-1)(b-i)-1}{a-pi-1} P^{a+b-i} \beta P^i \end{aligned}$$

for $a \leq pb$.

It can further be shown that the elements

$$\delta^{\epsilon_0} P^{s_1} \delta^{\epsilon_1} \dots P^{s_k} \delta^{\epsilon_k}$$

where each ϵ_i is either zero or one and

$$s_1 \geq ps_2 + \epsilon_1, s_2 \geq ps_3 + \epsilon_2, \dots, s_{k-1} \geq ps_k + \epsilon_{k-1}, s_k \geq 1$$

form an additive basis for A ([3], page 154).

As has already been stated, A is a $\mathbb{Z}_p$-algebra. In order to show that it is, additionally, a Hopf algebra, we must define the coproduct. The coproduct ψ is defined as in the following lemma.

Lemma 3.2. *Given an element $a \in A$, there exists a unique element*

$$\psi(a) = \sum a_i' \otimes a_i'' \in A \otimes A$$

such that

$$a(x \cup y) = \sum (-1)^{\deg(a_i'')\deg(x)} a_i' x \cup a_i'' y$$

for all pairs (X,V) and elements $x, y \in H^*(X,V)$.

Proof. We prove the existence of such elements; uniqueness is shown in [3], lemma 3.1.

For convenience, let $A \otimes A$ act on $H^*(X,V) \otimes H^*(X,V)$ by the rule

$$(a \otimes b)(x \otimes y) = (-1)^{\deg(x)\deg(b)} ax \otimes by$$

Let $c : H^*(X,V) \otimes H^*(X,V) \to H^*(X,V)$ be the cup product. The required identity for $\psi(a) \in A \otimes A$ can now be written as $ac(x \otimes y) = c\psi(a)(x \otimes y)$.

The relations

$$\delta(x \cup y) = (\delta x) \cup y + (-1)^{\deg x} x \cup (\delta y)$$
$$P^n(x \cup y) = \sum_{i+j=n} P^i(x) \cup P^j(y)$$

show that $\psi(\delta) = \delta \otimes 1 + 1 \otimes \delta$ and $\psi(P^n) = \sum_{i+j=n} P^i \otimes P^j$ exist.

Let J denote the set of all $a \in A$ for which such an element $\psi(a) \in A \otimes A$ exists. For $a, b \in J$, we see

$$abc(x \otimes y) = ac\psi(b)(x \otimes y) = c\psi(a)\psi(b)(x \otimes y)$$

so $ab \in J$ with $\psi(ab) = \psi(a)\psi(b)$. Similarly, J is closed under addition; hence J is an ideal. Since we have already shown that the generators $\delta, P^n \in J$, it follows that $J = A$.

□

Theorem 3.3. *The Steenrod algebra A has the structure of a Hopf algebra.*

Proof. Let the counit be the map that is identically zero on elements of positive degree and the identity on elements of degree zero.

In the course of the proof of the above lemma, we showed that ψ is an algebra homomorphism. Thus every condition for A to be a bialgebra follows easily except for coassociativity. For coassociativity, we need only consider the elements $P^n, \delta \in A$ since ψ is an algebra homomorphism. We compute

$$(\psi \otimes id)\psi(P^n) = \sum_{i+j+k=n} P^i \otimes P^j \otimes P^k = (id \otimes \psi)\psi(P^n)$$

$$(\psi \otimes id)\psi(\delta) = \delta \otimes 1 \otimes 1 + 1 \otimes \delta \otimes 1 + 1 \otimes 1 \otimes \delta = (id \otimes \psi)\psi(\delta)$$

as desired. Finally, A is connected, hence it admits a unique antipode ξ making it a Hopf algebra.

□

4. Dyer-Lashof Algebra

Analogous to the Steenrod algbera, there exists an algebra of homology operations on infinite loop spaces. In this section, we define the Dyer-Lashof algebra R and state its basic properties.

Definition 4.1. An infinite loop space is a space X together with a collection of spaces X_i for $i \geq 0$ such that $X_0 = X$ and $X_i \simeq \Omega X_{i+1}$ for all i. A map $f : X \to Y$ of infinite loop spaces is a sequence of maps $f_i : X_i \to Y_i$ for which $f_i = \Omega f_{i+1}$.

Let $\delta : H_i(X) \to H_{i-1}(X)$ be the homology Bockstein homomorphism associated with the sequence $\mathbb{Z}_p \to \mathbb{Z}_{p^2} \to \mathbb{Z}_p$ It is the dual of the cohomology Bockstein homomorphism introduced earlier. Similarly, let $P^a : H_i(X) \to H_{i-ra}(X)$ be the dual of the Steenrod operation P^a. By abuse of notation, we use the same variables as before. Each of δ and P^a acts on the homology of an arbitrary space.

Theorem 4.2. *There exist natural homomorphisms $Q^s : H_i(X) \to H_{i+rs}(X)$ defined for all infinite loop spaces X and all integers $i \geq 0$ that satisfy the following properties.*

(1) $Q^s(x) = x^p$ if $2s = \deg(x)$ and $Q^s(x) = 0$ if $2s < \deg(x)$
(2) $Q^s(\phi) = 0$ if $s > 0$ and $\phi \in H_0(X)$ is the identity element
(3) Cartan Formula:

$$Q^s(xy) = \sum_{i=0}^{s} Q^i(x)Q^{s-i}(y)$$

(4) *Adem Relations:*

$$Q^sQ^t = \sum_i (-1)^{s+i}\binom{(pi-s}{s-(p-1)t-i-1}Q^{s+t-i}Q^i$$

if $r > ps$ *and*

$$\begin{aligned} Q^s\delta Q^t =& \sum_i (-1)^{s+i}\binom{pi-s}{s-(p-1)t-i}\delta Q^{s+t-i}Q^i \\ &- \sum_i (-1)^{s+i}\binom{pi-s-1}{s-(p-1)t-i}Q^{s+t-i}\delta Q^i \end{aligned}$$

if $r \geq ps$

(5) *Nishida Relations:*

$$P^aQ^s = \sum_i (-1)^{a+i}\binom{a-pi}{s(p-1)-ap+pi}Q^{s-a+i}P^i$$

$$\begin{aligned} P^a\beta Q^s =& \sum_i (-1)^{a+i}\binom{a-pi}{s(p-1)-ap+pi-1}\delta Q^{s-a+i}P^i \\ &+ \sum_i (-1)^{a+i}\binom{a-pi-1}{s(p-1)-ap+pi}Q^{s-a+i}P^i\delta \end{aligned}$$

Proof. See [5], Theorem 1.1. □

The Dyer-Lashof algebra is defined in a similar way to the Steenrod algebra. Let F be the free associative (but not commutative) algebra over $\mathbb{Z}_p$ generated by Q^s for $s \geq 0$ and βQ^s for $s > 0$ where $\dim(Q^s) = rs$ and $\dim(\beta Q^s) = rs - 1$. Then F acts on $H_*(X)$ for infinite loop spaces X by composition of maps. Let I be the two-sided ideal of F consisting of all elements $f \in F$ for which $fx = 0$ for all spaces X and all elements $x \in H_*(X)$. Then the Dyer-Lashof algebra R as defined to be the quotient algebra F/I. It acts on homology groups $H_*(X)$ as before and is itself acted on by A.

Theorem 4.3. *R is a Hopf algebra with coproduct defined on generators by*

$$\psi(Q^s) = \sum_{i+j=n} Q^i \otimes Q^s \quad \textit{and} \quad \psi(\beta Q^{s+1}) = \sum_{i+j=n} \beta Q^{i+1} \otimes Q^j + Q^i \otimes \beta Q^{j+1}.$$

Furthermore, R admits a $\mathbb{Z}_p$-basis consisting of elements

$$\beta^{\epsilon_1}Q^{s_1}\cdots\beta^{\epsilon_k}Q^{s_k}$$

where $k \geq 1$, ϵ_j is zero or one, $s_j \geq \epsilon_j$, $s_j - \epsilon_j \geq s_{j-1}$ for each j, and

$$2s_1 - \epsilon_1 - \sum_{j=2}^{k}(2s_j(p-1) - \epsilon_j \geq 0.$$

Proof. See [5], Theorem 2.3. □

5. Hopf Rings

In this section, we introduce the notion of a Hopf ring. A Hopf ring is a Hopf algebra that has two separate multiplication operations, $*$ and $\circ$. The $*$ operation will be called the additive product and $\circ$ the multiplicative product. These operations must satisfy a variant of the distributive law. Formally, a Hopf ring is a graded commutative ring object over the category of coalgebras. We proceed first to define this category.

Let R be a commutative ring. Let $\mathscr{D} = CoAlg_R$ be the category of graded cocommutative coalgebras. An object in $\mathscr{D}$ is a graded cocommutatative coalgebra and a map in $\mathscr{D}$ is a map of coalgebras.

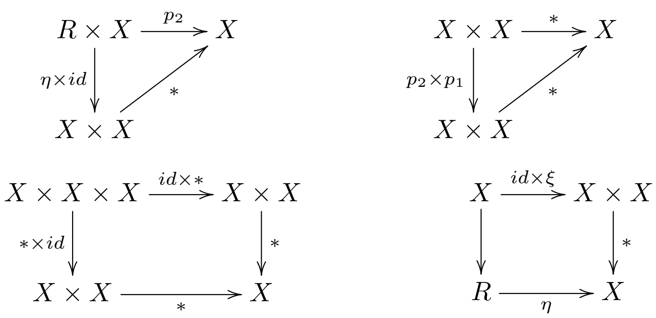
It can be shown that the tensor product $A \otimes B$ of two coalgebras A, B is the categorical product in $\mathscr{D}$. Indeed, the projections are the maps $id_A \otimes \epsilon_B : A \otimes B \to A$ and $\epsilon_A \otimes id_B : A \otimes B \to B$ and maps $f : C \to A$ and $g : C \to B$ induce a map $(f \otimes g) \circ \phi_C : C \to A \otimes B$.

In addition, the element $R \in \mathscr{D}$ is a terminal object. The unique map from an object $A \in \mathscr{D}$ is given by $\epsilon_A : A \to R$.

We now define what we mean by an abelian object over a given category $\mathscr{C}$. With this definition, if $\mathscr{C}$ is the category of sets, an abelian object over $\mathscr{C}$ is the same as an ordinary abelian group. If $\mathscr{C} = CoAlg_R$, an abelian group object over $\mathscr{C}$ is the same as a commutative and cocommutative Hopf algebra. Thus, with this terminology we should really rename Hopf algebras to Hopf groups. This terminology, however, is nonstandard so, at the risk of confusion, we proceed with the name Hopf algebra.

Throughout, let $\mathscr{C}$ be an arbitrary category with finite products and a terminal object R.

Definition 5.1. An abelian group object of $\mathscr{C}$ is an object $X \in \mathscr{C}$ together with maps $\eta : R \to X$ (zero), $* : X \times X \to X$ (addition) and $\xi : X \to X$ (additive inverse) such that the following diagrams commute:



Note that these diagrams give the standard properties of an abelian group. The first diagram is the additive identity, the second is commutativity, the third associativity, and the fourth additive inverses.

The category $G\mathscr{C}$ of graded objects of $\mathscr{C}$ has as its objects sequences $X_* = \{X_n\}_{n \in \mathbb{Z}}$ of objects $X_n \in \mathscr{C}$ and has maps $f_* : X_* \to Y_*$ sequences of maps $f_n : X_n \to Y_n$ where each f_n is a morphism in $\mathscr{C}$.

Definition 5.2. A graded commutative ring object with unit over $\mathscr{C}$ is an abelian group object $X_* \in G\mathscr{C}$ (meaning each X_n is an abelian group object) with zero $\eta = (\eta_n)$, addition $* = (*_n)$, and additive inverse $\xi = (\xi_n)$. It is required to also have maps $e : R \to X_0$ (unit) and $\circ_{ij} : X_i \times X_j \to X_{i+j}$ (multiplication) such that the following diagrams commute:

$$\begin{CD}
X_i \times X_j \times X_k @>{id\times\circ}>> X_i \times X_{j+k} \\
@V{\circ\times id}VV @VV{\circ}V \\
X_{i+j} \times X @>>{\circ}> X
\end{CD}
\qquad
\begin{CD}
X_i \times X_j @>{\circ}>> X_{i+j} \\
@V{p_2\times p_1}VV @VV{(\xi)^{ij}}V \\
X_j \times X_i @>>{\circ}> X_{i+j}
\end{CD}$$

$$\begin{array}{ccc}
R \times X_i & \xrightarrow{p_2} & X_i \\
\downarrow{\scriptstyle e\times id} & \nearrow{\scriptstyle \circ} & \\
X_0 \times X_i & &
\end{array}
\qquad
\begin{CD}
R \times X_i @>{p_1}>> R \\
@V{\eta\times id}VV @VV{\eta}V \\
X_i \times X_j @>>{\circ}> X_{i+j}
\end{CD}$$

$$\begin{CD}
X_i \times X_j \times X_j @>>> X_i \times X_i \times X_j \times X_j @>>> X_i \times X_j \times X_i \times X_j \\
@V{id\times *}VV @. @VV{\circ\times\circ}V \\
X_i \times X_j @>>{\circ}> X_{i+j} @<<{*}< X_{i+j} \times X_{i+j}
\end{CD}$$

Again, these diagrams give the standard properties of a graded commutative ring. The first is associativity, the second graded commutativity, the third the multiplicative unit, the fourth multiplication by zero, and the fifth the distributive property.

In the case where $\mathscr{C}$ is the category of sets, a graded commutative ring object with unit over $\mathscr{C}$ is just a graded commutative ring.

Definition 5.3. A Hopf ring is a graded commutative ring object with unit over the category of $\mathscr{D} = CoAlg_R$ of coalgebras.

This definition is rather abstract, so we now summarize its contents. Though we leave out some detail, the following should convey the main idea behind a Hopf ring more clearly. A Hopf ring consists of:

- A sequence of Hopf algebras X_n. The product within each X_n is denoted $*$; it is called the additive product.
- A multiplicative product $\circ : X_i \times X_j \to X_{i+j}$
- A distributive law
$$x \circ (y * z) = \sum (-1)^{\deg(y)\deg(x'')}(x' \circ y) * (x'' \circ z)$$
when $x \in X_i$, $y, z \in X_j$ and $\psi(x) = \sum x' \otimes x''$

Example 5.4. We have already shown that a group ring $R[G]$ is a Hopf algebra, a group object over the category $\mathscr{D}$. We show that if we replace G by a graded commutative ring S, the group ring, or "ring ring", $R[S]$ is a Hopf ring. Since S_n is, in particular, an abelian group we may define a Hopf algebra structure on $R[S_i]$ is usual and denote the additive product $*$. Then the multiplication structure $S_i \times S_j \to S_{i+j}$ on S gives us a multiplicative product $\circ : R[S_i] \times R[S_j] \to R[S_{i+j}]$. Since $\psi(s) = s \otimes s$ by definition for all $s \in S$, the distributive law

$$x \circ (y * z) = \sum (-1)^{\deg(y)\deg(x'')}(x' \circ y) * (x'' \circ z)$$

is equivalent to the distributive law

$$x(y+z) = xy + xz$$

when $x, y, z \in S$ are homogeneous. The general distributive law holds for the same reason when we expand by R-linearity. It can be checked that $R[S]$ satisfies all additional axioms for a Hopf ring.

Example 5.5. The homology of an E_∞ ring space (in particular, $QS^0 = \lim_{n\to\infty} \Omega^n S^n$) is a Hopf ring. Moreover, the homology of such an E_∞ ring space is acted on by the two Hopf algebras A and R.

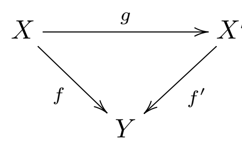
6. Exotic Characteristic Classes of Spherical Fibrations

Our goal in this section is to define characteristic classes of spherical fibrations. We will use much of the algebraic machinery developed up to this point. In the next section, it will be shown, with some effort, that these exotic classes are all nonzero. All unproven results may be found in [1].

As before, a spherical fibration is a fibration in which each fiber is homotopy equivalent to a sphere. Denote by G the structure group and BG the classifying space for spherical fibrations and by SG the structure group and BSG the classifying space for oriented spherical fibrations. Then SG has the homotopy type of $\lim_{n\to\infty}(\Omega^n S^n)_1$ where $(\Omega^n S^n)_1$ is the space of degree one basepoint-preserving maps from S^n to itself. Our exotic classes will be defined as elements $e_k \in H^{p^k r-1}(BSG)$ modulo some indeterminacy.

6.1. The Algebra A(Y).

Recall that the Steenrod algebra A acts on the cohomology $H^*(X,V)$ of any pair (X,V). To construct the classes e_k, we will require a similar algebra to act on the category of pairs over a fixed space Y. An object (X,V,f) in this category is a topological pair (X,V) together with a map $f : X \to Y$. A morphism between objects (X,V,f) and (X',V',f') is a map $g : X \to X'$ with $g(V) \subset V'$ that fits into a commutative diagram as shown.



To define the algebra acting on pairs over Y, we introduce a general construction. Let R be an algebra over a Hopf algebra A; this means that R is an A-module such that $a(xy) = \sum(-1)^{deg(a_i'')deg(x)}(a_i'x)(a_i''y)$ for all $x, y \in R$ where we write $\psi(a) = \sum a_i' \otimes a_i''$. Then the semitensor product $A(R)$ is $R \otimes A$ given the following multipication:

$$(x \otimes a)(y \otimes b) = \sum(-1)^{deg(a_i')deg(y)} x(a_i'y) \otimes a_i''b$$

This product gives $A(R)$ the structure of a ring.

In particular, if Y is a space, then $H^*(Y)$ is an algebra over the Steenrod algebra A (indeed, this is how we defined the coproduct on A). We shall denote the semitensor product $A(H^*(Y))$ by the simpler notation $A(Y)$.

It follows that $A(Y)$ acts on pairs over Y, for if $f : X \to Y$ is a continuous map and $V \subset X$, then $H^*(X,Y)$ has an $A(Y)$-module structure defined by $(y \otimes a)x = f^*(y) \cup ax$ for $y \in H^*(Y)$, $a \in A$, and $x \in H^*(X,V)$. Moreover, if $g : (X,V,f) \to$

(X', V', f') is a morphism of pairs over Y, the map $g^* : H^*(X', V') \to H^*(X, V)$ is an $A(Y)$-module homomorphism.

6.2. **Twisted Secondary Cohomology Operations (TSCOs).** We now define twisted secondary cohomology operations (TSCOs). Much as secondary operations are derived from relations in A, twisted secondary operations are derived from relations in $A(Y)$. Let

$$C : C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0$$

be a chain complex of free $A(Y)$-modules. Each C_i is required to be graded and the maps d_i must respect the grading. Let

$$\alpha : Hom^k_{A(Y)}(C_1, H^*(X, V)) \longrightarrow Hom^k_{A(Y)}(C_2, H^*(X, V))$$

$$\beta : Hom^k_{A(Y)}(C_0, H^*(X, V)) \longrightarrow Hom^k_{A(Y)}(C_1, H^*(X, V))$$

be the maps induced by d_2 and d_1 respectively. To clarify notation, note that an element $f \in Hom^k_{A(Y)}(C_1, H^*(X, V))$, for instance, is an $A(Y)$-module homomorphism $f : C_1 \to H^*(X, V)$ that raises degree by k, that is such that $f((C_1)_i) \subset H^{i+k}(X, V)$ for all i.

Now if M and N are R-modules, an additive relation $r : M \rightharpoonup N$ is by definition a submodule of $M \oplus N$. Denote submodules

$$\text{Domain}(r) = \{m \in M : (m, n) \in r \text{ for some } n \in N\} \subset M$$

$$\text{Ind}(r) = \{n \in N : (0, n) \in r\} \subset N$$

We think of r as a partially-defined function $M \to N$; in this interpretation, Domain(r) is its domain of definition and Ind(r) is its indeterminacy. Thus if $m \in$ Domain(r), its image $r(m) = \{n \in N : (m, n) \in r\}$ is a subset of N in which any two elements differ by an element of Ind(r).

Definition 6.1. A twisted secondary cohomology operation (TSCO) is an additive relation

$$\phi : Hom^k_{A(Y)}(C_0, H^*(X, V)) \rightharpoonup Hom^{k-1}_{A(Y)}(C_2, H^*(X, V))$$

defined for all $k \in \mathbb{Z}$ and for all pairs (X, V, f) over Y. It is required to satisfy Domain$(\phi) = \ker(\beta)$ and Ind$(\phi) = \text{Im}(\alpha)$, as well as several other conditions.

These additional conditions are not relevant here, except that they can be used to deduce existence and uniqueness in the following way.

Proposition 6.2. *(Existence and Uniqueness of TSCOs)*

For any chain complex $C : C_2 \to C_1 \to C_0$ of free $A(Y)$-modules, there exists an associated TSCO ϕ.

Moreover, if ϕ_0, ϕ_1 are two TSCO's associated with C, then they differ by a twisted primary operation. Explicitly, this means that there exists a map $d : C_2 \to C_0$ of degree -1 such that $Hom(d, 1)(\epsilon) \in \phi_0(\epsilon) - \phi_1(\epsilon)$ for each $\epsilon \in Domain(\phi_0) = Domain(\phi_1)$.

In addition, the construction of TSCOs is natural.

Proposition 6.3. *(Naturality of TSCOs)*

Let $f : X \to Y$ be a continuous map, C^Y a chain complex of free $A(Y)$-modules and β a corresponding TSCO. Then, letting $C^X = A(X) \otimes_{A(Y)} C^Y$ be the corresponding chain complex of free $A(X)$-modules, there is a natural isomorphism

$$\gamma : Hom_{A(Y)}(C^Y_i, H^*(X, V)) \to Hom_{A(X)}(C^X_i, H^*(X, V))$$

such that $\alpha = \gamma\beta\gamma^{-1}$ is a TSCO associated with C^X. Moreover, Ind(α) $\supset$ Ind(β) and Domain(α) $\subset$ Domain(β).

6.3. **Peterson's Operation.** We develop an example of a TSCO. Let ESG be the total space of the universal bundle over BSG and let MSG be the corresponding Thom space. We may think of MSG as the pair (BSG, ESG) over BSG. Let $q_i \in H^{ir}(BSG)$ be the ith Wu class of the universal bundle; it is defined by the equation $q_i u = P^i u \in H^*(MSG)$ where $P^i \in A$ is a Steenrod operation and $u \in H^n(MSG)$ is the Thom class.

Let $\theta : A \to A(BSG)$ be the ring homomorphism defined by the relations $\theta(P^n) = \sum_{i+j=n} q_i \otimes P^j$ and $\theta(\delta) = 1 \otimes \delta$. It can be shown that this map is a well-defined injection of Hopf algebras. Therefore, $A(BSG)$ can be made into an A-module with the multiplication

$$A \otimes A(BSG) \xrightarrow{\theta\otimes id} A(BSG) \otimes A(BSG) \xrightarrow{\phi} A(BSG)$$

Consider the chain complex

$$C^A : C_2^A \xrightarrow{d_2} C_1^A \xrightarrow{d_1} C_0^A$$

of free A-modules where $C_0^A = A$, C_1^A has as A-basis the set of elements p_i for $i > 0$ where $\dim p_i = ir$, and C_2^A has as a basis the elements e_k for $k > 0$ where $\dim e_k = p^k r$. In each of these modules, multiplication by P^i raises degree by ir and multiplication by δ raises degree by 1 as usual. Define the map d_1 by the relation $d_1(p_i) = P^i$ and the map d_2 by

$$d_2(e_k) = \sum_{i=1}^{p^{k-1}} a_{k,i} P^{p^k - i} p_i$$

where $\sum_{i=1}^{p^{k-1}} a_{k,i} P^{p^k-i} P^i = 0$ is the Adem relation for $P^{(p-1)p^{k-1}} P^{p^{k-1}}$.

Taking the tensor product of this complex with $A(BSG)$ over A, we obtain a free $A(BSG)$ complex $C^{A(BSG)}$. By our above results, this yields a TSCO of the form

$$\phi^{BSG} : Hom^n_{A(BSG)}(C_0^{BSG}, H^*(X,V)) \rightharpoonup Hom^{n-1}_{A(BSG)}(C_2^{BSG}, H^*(X,V))$$

defined on pairs (X, V) over BSG. We will consider, in particular the pair $MSG = (BSG, ESG)$ of spaces over BSG. Note that

$$Hom^n_{A(BSG)}(C_0^{BSG}, H^*(MSG)) \cong H^n(MSG)$$

$$Hom^{n-1}_{A(BSG)}(C_2^{BSG}, H^*(MSG)) \cong \prod_{k>0} H^{p^k r+n-1}(MSG)$$

The first equality follows because $C_0^{BSG} = A(BSG) \otimes_A A \cong A(BSG)$ and the second because $Hom(\bigoplus -, -) \cong \prod Hom(-,-)$. Thus ϕ^{BSG} can be seen as providing an additive relation

$$\phi^{BSG} : H^n(MSG) \to \prod_{k>0} H^{p^k r+n-1}(MSG)$$

We hope to apply ϕ^{BSG} to the Thom class $u \in H^n(MSG)$ to obtain well-defined elements of $H^{p^k r+n-1}(MSG)$ for $k > 0$.

Proposition 6.4. *$\phi^{BSG}(u)$ is defined and has zero indeterminacy.*

Proof. Consider the diagram

$$C_2^{BSG} \xrightarrow{d_2} C_1^{BSG} \xrightarrow{d_1} A(BSG) \xrightarrow{u} H^*(MSG)$$

Now $\phi^{BSG}u$ is defined exactly if $u \in \ker(\beta)$ or equivalently, if $u \circ d_1 = 0$. This occurs if and only if $(u \circ d_1)(p_m) = 0$ for all $m > 0$. We compute

$$(u \circ d_1)(p_m) = u(P^m) = \theta(P^m)u = (\sum_{i+j=m} q_i \otimes P^j)u$$
$$= \sum_{i+j=m} q_i' \cup P^j u = \sum_{i+j=m} q_i' \cup q_j u = (\sum_{i+j=m} q_i' \cup q_j) \cup u = 0$$

where q_i' is the ith Wu class of the Whitney inverse of the universal bundle. This holds because the map $H^*(BSG) \to H^*(MSG) = H^*(BSG, ESG) \subset H^*(BSG)$ maps q_i to q_i' and the final expression is zero by the Whitney product theorem. Hence $\phi^{BSG}u$ is defined.

It can be shown by a similar computation that the indeterminacy is zero. □

At this point, we would like to use ϕ^{BSG} to define the exotic classes. We would let e be the unique total class satisfying $eu = \phi^{BSG}u$; its components $e_k \in H^{p^k r-1}(BSG)$ would be defined modulo a certain indeterminacy from the choice of the TSCO ϕ^{BSG}. However, this is too imprecise. Two choices of ϕ^{BSG} may differ by a twisted primary operation, by multiplication by any element of $H^*(BSG)$. We will require a modification to resolve this difficulty.

6.4. **The Definition of Exotic Characteristic Classes.** To make our choice of TSCO more precise, we replace ϕ^{BSG} by a new complex ϕ^Z. Let $Z = \Pi_{i>0} K(\mathbb{Z}_p, ir)$ and let $q : BSG \to Z$ be a map corresponding to the total Wu class in $H^*(BSG)$.

We make use of the following lemmas.

Lemma 6.5. *There exists a chain complex C^Z of free $A(Z)$-modules such that $C^{BSG} = A(BSG) \otimes_{A(Z)} C^Z$*

Lemma 6.6. *$Im(q^*) = \mathbb{Z}_p[q_i : i > 0] \otimes E[\delta q_i : i > 0]$ where $E[-]$ denotes the exterior algebra on the indicated generators*

We may finally define the exotic classes.

Definition 6.7. Let ϕ^Z be a TSCO associated with C^Z. Define the total exotic class $e \in H^*(BSG)/Im(q^*)$ by $eu = \phi^Z u$ where $u \in H^*(MSG)$ is the Thom class. Let e_k denote the $(p^k r - 1)$-dimensional component of e.

This definition makes sense, for $\phi^Z u = \phi^{BSG}u$ for a suitable choice of ϕ^{BSG} by the naturality of TSCOs, and hence is defined with zero indeterminacy. Any two choices of ϕ^{BSG} differ by a twisted primary operation over Z. Such an operation applied to u yields an element in $(Im(q^*))u$ and hence $e \in H^*(BSG)$ is well-defined modulo elements of $Im(q^*)$, the ordinary characteristic classes.

We will use the following result in the next section to show that these characteristic classes are nonzero. It again follows from the naturality of TSCOs.

Proposition 6.8. *$\phi^A u = \phi^Z u$ if u is any Thom class on which ϕ^A is defined.*

Here, ϕ^A is obtained from the chain complex C^A of free A-modules as an ordinary secondary operation. Equivalently, C^A is a chain complex of free $A(Y)$-modules where Y is a point so that ϕ^A is an associated TSCO.

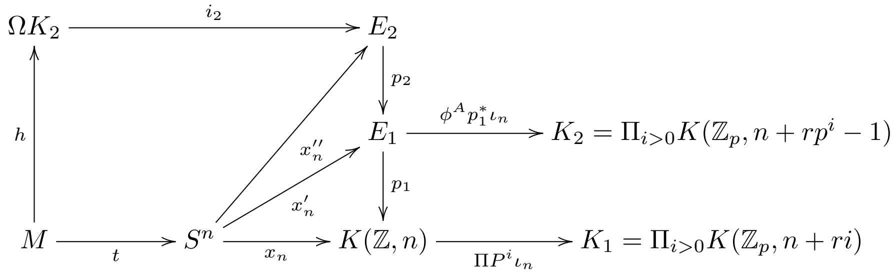
7. Nontriviality of Exotic Classes

In this section, we prove that the exotic classes defined above are nontrivial.

Theorem 7.1. *There exists a spherical fibration ξ over a space X such that the operation ϕ^A is defined on the Thom class of ξ and such that $e_k(\xi) \neq 0$ for all $k > 0$.*

Corollary 7.2. *The exotic classes $e_k \in H^{p^k r-1}BSG$ are nonzero modulo $Im q^*$ for all $k > 0$.*

In order to define ξ, consider the following diagram for large n.



Here $x_n \in H^n(S^n, \mathbb{Z})$ is a generator of the cohomology group and $P^i \iota_n \in H^{n+ir}(K(\mathbb{Z}, n))$ is the Steenrod operation $P^i \in A$ applied to the fundamental class $\iota_n \in H^n(K(\mathbb{Z}, n))$. Then $E_1 = F(\Pi P^i \iota_n)$ is the homotopy fiber of the indicated map. The function $(\Pi P^i \iota_n) \circ x_n$ is nulhomotopic because K_1 is n-connected, hence we obtain a map $x_n' : S^n \to E_1$ from the fiber sequence such that $p_1 \circ x_n' \simeq x_n$.

To show that $\phi^A(p_1^* \iota_n)$ is well-defined, we must show that $p_1^* \iota_n \in \ker \beta$ or, equivalently, that the below composition is zero.

$$C_1^A \xrightarrow{d_1} A \xrightarrow{p_1^* \iota_n} H^*(E_1)$$

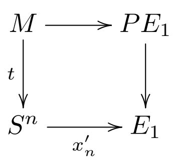
This is true if and only if $P^i(p_1^* \iota_n) = p_1^*(P^i \iota_n) = 0$ for $i > 0$. But this result holds by the fiber sequence.

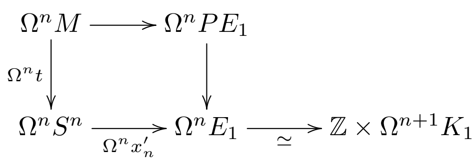
Next, $E_2 = F(\phi^A p_1^* \iota_n)$ is the homotopy fiber. Again K_2 is n-connected, so $\phi^A p_1^* \iota_n \circ x_n'$ is nulhomotopic, hence we get a map x_n'' as shown. Then ΩK_2 and i_2 are a continuation of the fiber sequence

$$\cdots \longrightarrow \Omega K_2 \xrightarrow{i_2} E_2 \xrightarrow{p_2} E_1 \xrightarrow{\phi^A p_1^* \iota_n} K_2$$

and, finally, M is the pullback of i_2 and x_n''.

Now ΩE_1 is the fiber of t and we have two fiber squares





where PE_1 is the path space of E_1.

We note that M is n-connected so that $\Omega^n M$ is connected and its image under $\Omega^n t$ lies in the degree zero component of $\Omega^n S^n$, which in turn is canonically homotopy equivalent to SG. This yields a map $\Omega^n M \to SG \simeq \Omega BSG$ and, by adjunction, a map $\Sigma\Omega^n M \to BSG$. Therefore we obtain an orientable spherical fibration ξ on the space $X = \Sigma\Omega^n M$. This is the fibration for which we will prove the theorem.

First, we state two lemmas which will be used later on.

Lemma 7.3. *The class* $[-p^2]\beta Q^{(p-1)p^i}\beta Q^{p^i}([1]) \in H_*(\Omega^n S^n)$ *is in the image of the map* $(\Omega^n t)_* : H_*(\Omega^n M) \to H_*(\Omega^n S^n)$

The proof uses the Eilenberg-Moore spectral sequence of the fiber sequence $\Omega^n M \to (\Omega^n S^n)_0 \to \Omega^{n+1} K_1$ from the diagram above.

Lemma 7.4. *The fundamental class* $b_i \in H_{p^{i+1}r-2}(\Omega^n E_2)$ *obtained from the Serre spectral sequence is of the form*

$$(\Omega^n x_n'')_*([-p^2]\beta Q^{(p-1)p^i}\beta Q^{p^i}([1]))$$

The map $\Omega^n t : \Omega^n M \to \Omega^n S^n$ has an adjoint $t' : \Sigma^n\Omega^n M \to S^n$ and it is straightforward to see that the Thom space $T\xi$ is homotopy equivalent to $\Sigma Ct'$ where Ct' is the mapping cone of t'. Consider the following commutative diagram

$$\begin{array}{ccccccc}
\Omega K_2 & \xrightarrow{i_2} & E_2 & \xrightarrow{p_2} & E_1 & \xrightarrow{\phi^A p_1^*\iota_n} & K_2 \\
\uparrow{\scriptstyle h} & & \downarrow{\scriptstyle x_n''} & & \| & & \uparrow{\scriptstyle \Sigma h} \\
M & \xrightarrow{t} & S^n & \xrightarrow{x_n'} & E_1 & \xrightarrow{v} & \Sigma M \\
\uparrow{\scriptstyle \alpha} & & \| & & \uparrow{\scriptstyle \alpha'} & & \uparrow{\scriptstyle \Sigma\alpha} \\
\Sigma^n\Omega^n M & \xrightarrow{t'} & S^n & \xrightarrow{\tilde{x}_n'} & Ct' & \xrightarrow{v'} & \Sigma^{n+1}\Omega^n M
\end{array}$$

Each row is a fiber sequence in the stable range.

Our main theorem is equivalent to the component of $\phi^A(u) = \phi^Z(u)$ being nonzero in each dimension $n + p^i r - 1$. We know that the map $(\phi^A p_1^*\iota_n \circ \alpha')^*$ sends each fundamental cohomology class of K_2 to the corresponding class in $\phi^A(u)$ where u is the Thom class. To prove the theorem, it therefore suffices to show that every fundamental homology class in K_2 is in the image of $(\phi^A p_1^*\iota_n \circ \alpha')_*$. For if a fundamental class $v \in H_*(K_2)$ is of the form $v = f_*(x)$ where $f = \phi^A p_1^*\iota_n$ and $x \in H_*(Ct')$, then

$$\langle f^*(v^*), x\rangle = \langle v^*, f_*(x)\rangle = \langle v^*, v\rangle \neq 0$$

where $v^* \in H^*(K_2)$ is the dual of v, hence $f^*(v^*) \neq 0$.

Using the commutative diagram, we compute

$$Im((\phi^A p_1^*\iota_n \circ \alpha')_*) = Im((\Sigma h \circ \Sigma\alpha \circ v')_*) = Im((\Sigma(h \circ \alpha)_*)$$

where the last equality holds because v'_* is an isomorphism for large n. Considering the map $\Sigma(h \circ \alpha) : \Sigma^n\Omega^n M \to \Omega K_2$, we see that $\Omega^n h : \Omega^n M \to \Omega^{n+1} K_2$ is an adjunction. It hence suffices to prove that the fundamental homology classes of $\Omega^{n+1} K_2$ are in the image of $(\Omega^n h)_*$. Consider the commutative square

$$\begin{array}{ccc}
\Omega^{n+1} K_2 & \xrightarrow{\Omega^n i_2} & \Omega^n E_2 \\
\uparrow{\scriptstyle \Omega^n h} & & \uparrow{\scriptstyle \Omega^n x_n''} \\
\Omega^n M & \xrightarrow[\Omega^n t]{} & \Omega^n S^n
\end{array}$$

We may now finally apply our two lemmas. The classes $b_i \in H_{p^{i+1}r-2}(\Omega^n E_2)$ are in the image of

$$(\Omega^n x_n'' \circ \Omega^n t)_* = (\Omega^n i_2 \circ \Omega^n h)_*$$

and $\Omega^n i_2$ maps the fundamental classes of $H_*(\Omega^{n+1}K_2)$ to the b_i, hence these classes must be in the image of $(\Omega^n h)_*$ as was asserted. This proves our theorem and consequently shows that the exotic classes $e_k \in H^{p^k r-1}BSG$ are nonzero modulo $\mathrm{Im}(q^*)$ for all $k > 0$.

Acknowledgments

I would like to thank my mentor, Weinan Lin, for his tremendous help, guidance, and patience throughout the summer. I am also grateful to Professor Peter May for organizing this virtual University of Chicago REU.

References

[1] D.C. Ravenel, A definition of exotic characteristic classes of spherical fibrations, Comment. Math. Helv. 47 (1972) 421-436.
[2] D. C. Ravenel and W. S. Wilson, The Hopf ring for complex cobordism, J. Pure and Applied Algebra, 9 (1977), 241-280.
[3] J. Milnor. The Steenrod algebra and its dual. Annals of Math. 67(1958), 150–171.
[4] J. Milnor and James D. Stasheff. Characteristic Classes. Princeton University Press and University of Tokyo Press. 1974.
[5] J. P. May. The Homology of Iterated Loop Spaces. Springer Lecture Notes in Mathematics Vol. 533. Springer, 1976.
[6] J.P. May and K. Ponto. More Concise Algebraic Topology. Chicago Lectures in Mathematics. The University of Chicago Press. Chicago. 2012.
[7] W. Lin. Homology Operations on E_∞ Ring Spaces. To appear.