MORLEY'S THEOREM

ALEX BURKA

Abstract. We expose the basic notions of stability theory in the context of the Morley categoricity theorem, following "Categoricity in Power" [3].

Contents

Introduction

A theory is ***categorical*** if it has exactly one model up to isomorphism. Such a theory is, from a logical point of view, ideal: it encodes all structural information of its model within the apparatus of its ambient logic.

Of course, no theory of first-order logic with infinite models is categorical: the Löwenheim–Skolem theorem furnishes structures in every cardinality at least the size of the language.[1] In this case, the most we can hope for is ***κ-categoricity*** in a particular cardinal κ: T is ***κ-categorical*** if it has one model of cardinality κ. Examples from the following infinite categoricity types were well-known to early model theorists: T is

(i) ***totally categorical*** if κ-categorical for every infinite κ. Evidently, pure identity theory is totally categorical. So is the theory of infinite abelian groups in which every element has order 2.

(ii) ***uncountably categorical*** if κ-categorical for every uncountable κ. The prototypical example is the theory of algebraically closed fields of characteristic p,

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[1]If $\mathcal{L}$ is a signature and not the first-order language it generates, we mean $|\mathcal{L}| + \aleph_0$ by $|\mathcal{L}|$.

ACF$_p$. The theory of torsion-free divisible abelian groups is also uncountably categorical. Neither is categorical in $\aleph_0$.

(iii) ***countably categorical*** if $\kappa = \aleph_0$. The theory of dense linear orders without endpoints DLO is countably categorical, though it has the maximum number 2^κ of non-isomorphic models in every uncountable κ.

In 1954, J. Los announced that he could only produce countable complete theories of type (i)-(iii), leading to

Conjecture 0.1. *If κ-categorical for some uncountable κ, T is κ-categorical for every uncountable κ.*

M. Morley confirmed Los' conjecture in his thesis "Categoricity in Power" [3].[2] The so-called ***Morley categoricity theorem*** is the apparent aim of the present paper. In fact, this is secondary to another interest—namely, understanding the model-theoretic devices Morley discovered on his way to proving Los' conjecture. We first develop Morley analysis, a method of quantifying the complexity of a theory's types using ideas from classical descriptive set theory. This leads us to a combinatorial characterization known as stability, which justifies Morley analysis in the context of Los' conjecture: a theory categorical in some uncountable power is necessarily stable. "Dual" to the notion of a stable theory is a saturated model, one that realizes as many types as possible. In the third section, we enlist the machinery of the first two sections to show if T is categorical in some uncountable power, then every uncountable model of T is saturated. Morley's theorem is a simple corollary of this fact.

We assume the reader has familiarity with basic model theory. A first course in mathematical logic (e.g. Math 277 at Chicago) along with an understanding of elementary embeddings and types will suffice.

1. Types, Topology, and Totally Transcendental Theories

1.1. **Provisos.** In what follows, T denotes a countable, complete theory. The latter assumption is harmless in view of Los' conjecture: A complete theory has fewer models than any of its incomplete subtheories. Furthermore, any theory with exclusively infinite models is already complete if categorical in some infinite cardinal.

The countability proviso is more subtle. Most of our proofs dispense with it, but the assumption is crucial for (1.4), (2.1), and (2.6). S. Shelah managed to eliminate the assumption $|\mathcal{L}| = \aleph_0$ a decade after [3] by developing new set-theoretic techniques, which we do not discuss in the current exposition.

$\mathcal{K}(T)$ and $\mathcal{B}(T)$ denote the class of models and of substructures of models of T, respectively.[3] Without loss, we assume $\mathcal{L}$ is a relational language.[4] Then, if $\mathfrak{M} \in \mathcal{K}(T)$ and $A \subseteq M$, the domain of the substructure $\mathfrak{A} \in \mathcal{B}(T)$ generated by A in $\mathfrak{M}$ is just A.

We require that T eliminates quantifiers. This assumption appears quite strong and opaque, but greatly facilitates our development of Morley analysis and in fact

[2]Advised by S. Maclane at Chicago, but owing much to R. Vaught at Berkeley.

[3]"Model" means "model of T," and "substructure" means "substructure of a model of T."

[4]It is straightforward to convert a theory to one conveying the same information but using predicates in place of function and constant symbols.

imposes no loss of generality. Recall the ***diagram*** $\mathcal{D}(A)$ is the set of all atomic sentences and negations of atomic sentences true in $\mathfrak{A}$. Then:

Theorem 1.1. *The following are equivalent:*
(a) T eliminates quantifiers.
(b) T is **substructure complete***: for every $\mathfrak{A} \in \mathcal{B}(T)$, $T(A) := T \cup \mathcal{D}(A)$ is complete.*
(c) T has the **amalgamation property***: for every $\mathfrak{A} \in \mathcal{B}(T)$, $\mathfrak{B}_0, \mathfrak{B}_1 \in \mathcal{K}(T)$, and embeddings f_0, f_1 of A into $\mathfrak{B}_0$, $\mathfrak{B}_1$, there is $\mathfrak{C} \in \mathcal{K}(T)$ and embeddings g_0, g_1 such that $g_0 f_0 = g_1 f_1$ and g_0 is elementary.*

Proof. See [4]. □

It is straightforward to expand T to a theory that eliminates quantifiers: for each $n < \omega$ and each formula $\phi \in \mathcal{L}$ whose free variables are among $v_0 \ldots v_{n-1}$, append a new n-ary predicate R_ϕ to the language of T. Then, expand T to a theory T^* via the universal closure of $\phi(\mathbf{v}) \leftrightarrow R_\phi(\mathbf{v})$. T^* is called a ***Morleyization*** of T. Furthermore, T^* evidently eliminates quantifiers, whence it exhibits substructure completeness and the amalgamation property, too. In addition, elementary embeddings among substructures in $\mathcal{B}(T)$ biject with embeddings among substructures in $\mathcal{B}(T^*)$. Therefore, we not only require that T eliminates quantifiers, but that T is the Morleyization of some theory T'. It is routine to check T is κ-categorical iff T^* is κ-categorical, which justifies proviso in view of considerations about categoricity.

1.2. **The Stone Functor.** We want to identify conditions under which T has few or many models in a given cardinality. More broadly, how can we probe the models of T using devices of first-order logic? A reasonable approach is to find isomorphism-invariant logical specifications, and construct models of T that either realize or omit these specifications. Of course, we cannot use finitely many first-order formulas to distinguish between models of a complete theory. Instead, we consider infinite families of formulas:[5]

Fix a substructure $\mathfrak{A}$, and let p be a set of formulas in $\mathcal{L}(A)$ in n free variables. p is an ***n-type*** if consistent with $T(A)$—equivalently, the conjunction of any finite number of formulas in p is satisfied in some model of T containing $\mathfrak{A}$. An n-type is ***complete*** if maximal consistent. $S_n(A)$ denotes the set of complete n-types over the substructure $\mathfrak{A}$.

Later on, we will be interested in bounds on $|S_n(A)|$. It's enough to establish the desired bound for $|S_1(A)|$ in order to establish that for $|S_n(A)|$ for every $n < \omega$.[6] Hence we restrict attention to $S(A) = S_1(A)$. Unless otherwise noted, "type" always means "complete 1-type."

Let $p \in S(A)$ and $\mathfrak{M} \in \mathcal{K}(T)$ such that $\mathfrak{A} \subseteq \mathfrak{M}$. We say $\mathfrak{M}$ ***realizes*** p if there exists $a \in M$ that satisfies every formula in p. By compactness, $\mathfrak{M}$ has an elementary extension that realizes p. Well-ordering $S(A) = \langle p_\alpha : \alpha < \gamma \rangle$, we can produce an elementary chain $\langle \mathfrak{M}_\alpha : \alpha < \gamma \rangle$ such that $\mathfrak{M}_\alpha$ realizes p_β for $\beta < \alpha$. The union of this chain realizes every type over A.

Every element of a structure realizes a type: if $a \in M$, let $\mathrm{tp}(a/A)$ be the set of $\mathcal{L}(A)$-formulas satisfied by a in $\mathfrak{M}$. Clearly $|A| \leq |S(A)|$, since $a \neq b$ for $a, b \in A$ implies $\mathrm{tp}(a/A) \neq \mathrm{tp}(b/A)$.

[5]For details in this subsection, refer to Ch. 4 of [2].
[6]See [5], I.2.

If no $a \in M$ realizes p, we say $\mathfrak{M}$ ***omits*** p. Omitting types requires much more care, as we shall see.

At this point, it is important to note that realization/omission of a type is an isomorphism invariant.

Let $B(A)$ be the set of $\mathcal{L}(A)$-formulas in one free variable equivalent modulo $T(A)$, so that $B(A)$ forms a boolean algebra under union, intersection, and complementation. Then, p is a type iff the corresponding subset of $B(A)$ is an ultrafilter.[7] The connection here suggests a natural topology for $S(A)$: if $\phi \in \mathcal{L}(A)$, put $[\phi] = \{p \in S(A) : \phi \in p\}$. Then $U(A) = \{[\phi] : \phi \in \mathcal{L}(A)\}$ amounts to a basis for a compact, totally disconnected, Hausdorff topology on $S(A)$, the ***Stone topology***.[8]

An embedding $f : \mathfrak{A} \to \mathfrak{B}$[9] induces an injective map $\mathcal{L}f : \mathcal{L}(A) \to \mathcal{L}(B)$ as follows: Given $\phi \in \mathcal{L}(A)$, let $(\mathcal{L}f)(\phi)$ be the formula obtained by replacing instances of A-parameters in ϕ by their respective images under f. Then $\mathcal{L}f$ induces a map $Sf : S(B) \to S(A)$ via

$$(Sf)(p) = \{(\mathcal{L}f)^{-1}(\phi) : \phi \in p\}.$$

Notice that since p is complete and consistent with $T(B)$, $(Sf)(p)$ is complete and consistent with $T(A)$. Also, if q is a type over A and a realizes q in some $\mathfrak{N}$ containing $\mathfrak{B}$, then $(Sf)(\mathrm{tp}(a/B)) = q$. And, for any $\phi \in \mathcal{L}(A)$, the preimage of $[\phi]$ in $U(A)$ under Sf is $[\phi]$ in $U(B)$. Thus Sf is well-defined, surjective, and continuous with respect to the Stone topology.

A type p is ***isolated*** if there is some $\phi \in \mathcal{L}(A)$ such that $\psi \in p$ iff

$$T(A) \models \forall v(\phi(v) \to \psi(v)).$$

The terminology is not accidental, for p is isolated iff $\{p\} = [\phi]$ for some ϕ in $S(A)$. This topological viewpoint carries us quite far, ultimately leading us to the notion of a stable theory.

1.3. **Morley Analysis.** Morley's analysis of $S(A)$ pays homage to the Cantor–Bendixson theorem, an early result in descriptive set theory. The Cantor–Bendixson derivatives $X^{(\alpha)}$ of a topological space X are defined by removing isolated points at successor ordinals and taking intersections at limits. Morley defines an analogous, but stronger notion of a derivative for $S(A)$:

Definition 1.2. The ***Morley derivatives*** $S^\alpha(A)$ and the ***transcendence ranks*** $\mathrm{Tr}^\alpha(A)$ are defined by simultaneous induction on α:

$$\begin{aligned} S^\alpha(A) &= S(A) - \bigcup\{\mathrm{Tr}^\beta(A) : \beta < \alpha\} \\ \mathrm{Tr}^\alpha(A) &= \{p \in S^\alpha(A) : \\ &\qquad (Sf)^{-1}(p) \cap S^\alpha(B) \text{ is isolated in } S^\alpha(B) \text{ for all } f : \mathfrak{A} \to \mathfrak{B}\}. \end{aligned}$$

If $p \in \mathrm{Tr}^\alpha(A)$, p has ***Morley rank*** $\mathrm{RM}(p) = \alpha$. If $p \notin \mathrm{Tr}^\alpha(A)$ for any α, put $\mathrm{RM}(p) = \infty$.[10]

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[7] i.e. a maximal, upward-closed, confluent, nonempty subset.

[8] Compactness follows by compactness of first-order logic, while totally disconnected and Hausdorff hold because $S(A)$ consists of *complete* types.

[9] Every map of the form $\mathfrak{A} \to \mathfrak{B}$ is assumed to be an embedding.

[10] For the Francophiles, "RM" may be read "rang de Morley."

Notice $p \in S^\alpha(A)$ has rank α not just if it's isolated in $S^\alpha(A)$, but if for every extension $\mathfrak{B} \supseteq \mathfrak{A}$, every completion of p to a complete 1-type over B with rank at least α is isolated in $S^\alpha(B)$. Among other things (1.4), this condition ensures that maps of the form (Sf) preserve Morley derivative. The following proposition lists this and other basic consequences of Definition 1.2:

Proposition 1.3. *Suppose $f : \mathfrak{A} \to \mathfrak{B}$ is an embedding among substructures $\mathfrak{A}, \mathfrak{B}$. Then for all α:*
(a) $S^\alpha(A)$ is a closed (hence compact) subspace of $S(A)$.
(b) (Sf) preserves Morley derivative: $(Sf)(S^\alpha(B)) = S^\alpha(A)$.
(c) If $p \in S^\alpha(A)$, then $p \in Tr^\alpha(A)$ iff $(Sf)^{-1}(p) \cap S^\alpha(B) \subseteq Tr^\alpha(B)$.
(d) If p has rank α, there exists n such that for all embeddings $f : \mathfrak{A} \to \mathfrak{B}$,

$$\left|(Sf)^{-1}(p) \cap S^\alpha(B)\right| \leq n.$$

The least such n is called the **Morley degree** *of p, denoted $\deg(p)$. We have*

$$\deg(p) = \sum \left\{\deg(q) : q \in (Sf)^{-1}(p) \cap Tr^\alpha(B)\right\}.$$

(e) If $(Sf)(p) = q$, then $RM(p) \leq RM(q)$ and $\deg p \leq \deg q$.
(f) If p has rank α, there is a finite $F \subseteq A$ such that $(Si_{FA})(p)$ has rank α.

Proof. See [3]. □

The Cantor–Bendixson theorem states that every closed, uncountable subset $X \subseteq \mathbb{R}$ has a nonempty perfect subset $X^{(\gamma)}$, where γ is the least ordinal at which the Cantor–Bendixson derivatives of X stabilize. An analogous, but stronger result holds for a theory's Stone spaces: (1.2) ensures that the Morley derivatives $S^\alpha(A)$ stabilize uniformly over all substructures $\mathfrak{A}$.

Theorem 1.4. *There exists α such that for all $\mathfrak{A} \in \mathcal{B}(T)$ and all $\beta > \alpha$, we have $S^\alpha(A) = S^\beta(A)$. The least such α is called the* **Morley rank** *of T, denoted α_T. $S^{\alpha_T}(A)$ vanishes for all substructures $\mathfrak{A}$ if it vanishes for some $\mathfrak{A}$.*

Proof. If $\mathrm{Tr}^\alpha(A)$ is nonempty, then $\mathrm{Tr}^\alpha(F)$ is nonempty for some finite $F \subseteq A$ by (1.3f). Recall that elementary equivalence determines isomorphism type among finite structures. Since $\mathcal{L}$ is countable, we have at most $2^{\aleph_0}$ isomorphism types of finite structures in $\mathcal{B}(T)$.[11] Notice $|S(F)| \leq 2^{\aleph_0}$ if $\mathfrak{F}$ is finite: For each $p \in S(F)$ is a subset of $\mathcal{L}(F)$ and $|\mathcal{L}(F)| = |\mathcal{L}|$. Thus, there is a least α_F at which $\mathrm{Tr}^{\alpha_F}(F) = \emptyset$. Define $\alpha_T = \sup\{\alpha_F : \text{finite } \mathfrak{F}\}$, so that $\mathrm{Tr}^{\alpha_T}(F) = \emptyset$ for all finite $\mathfrak{F}$. This is enough to get $\mathrm{Tr}^{\alpha_T}(A) = \emptyset$ for all substructures $\mathfrak{A}$—otherwise, if $\mathfrak{F}$ is a finite subset of $\mathfrak{A}$ witnessing transcendence rank α_T, then $\mathrm{Tr}^{\alpha_T}(F) = \emptyset$.

Let $\mathfrak{A}, \mathfrak{B} \in \mathcal{B}(T)$ and $\beta \geq \alpha_T$. Suppose $S^\beta(A)$ vanishes. Let $\mathfrak{C}$ amalgamate $\mathfrak{A}$ and $\mathfrak{B}$ into a common superstructure along embeddings f, g. Since (Sf) preserves Morley derivative, we have $S^\beta(C) = \emptyset$ whence $S^\beta(B)$ also vanishes. □

Note $\alpha_T < (2^{\aleph_0})^+$, by regularity of $(2^{\aleph_0})^+$ and that $\alpha_F < (2^{\aleph_0})^+$ for all finite $\mathfrak{F}$. Also, in case $S^{\alpha_T}(A)$ vanishes for some (all) $\mathfrak{A}$, α_T is not a limit ordinal: if $S^\beta(A)$ is nonempty for all $\beta < \alpha$, then $S^\alpha(A) = \bigcap\{S^\beta(A) : \beta < \alpha\}$ is the intersection of a nested sequence of closed, nonempty sets in a compact space.

[11] Fix an enumeration $\phi_0, \phi_1 \dots$ of $\mathcal{L}$-sentences and put $[\sigma] = \{\phi_{\sigma(n)} : \sigma(n) = 1\} \cup \{\neg\phi_{\sigma(n)} : \sigma(n) = 0\}$ for $\sigma \in 2^\omega$. If $\mathfrak{F}$ is a finite substructure, there exists some $\sigma \in 2^\omega$ such that F satisfies exactly the members of $[\sigma]$. Distinct isomorphism types yield distinct infinite paths in 2^ω, so this witnesses desired bound follows.

Definition 1.5. T is ***totally transcendental*** if $S^{\alpha_T}(A) = \emptyset$ for all $\mathfrak{A} \in \mathcal{B}(T)$.

That is, T is totally transcendental if all its types are ranked.

1.4. **Prime Extensions.** If T is totally transcendental, we can obtain information about T and its types by Morley analysis. This technique allows us to construct prime extensions of continuous chains in $\mathcal{B}(T)$, for instance.

Definition 1.6. Let $\mathfrak{M} \in \mathcal{K}(T)$ and $\mathfrak{A} \subseteq \mathfrak{M}$. $\mathfrak{M}$ is a ***prime extension*** of $\mathfrak{A}$ if every embedding of $\mathfrak{A}$ into a model of T factors through an elementary embedding of $\mathfrak{M}$ into that model.

Assuming T is a Morleyization of some theory allows the following, equivalent definition: every embedding of $\mathfrak{A}$ into a model of T factors through an embedding from the prime extension.

Totally transcendental theories necessarily exhibit Stone spaces whose isolated types form a dense subset:

Lemma 1.7. *If T is totally transcendental, $S(A) \setminus S'(A)$ is dense in $S(A)$ for $\mathfrak{A} \in \mathcal{B}(T)$.*

Proof. If T is totally transcendental and $[\phi]$ is any basic open set in $S(A)$, then every member of $[\phi]$ is ranked. The element of minimal rank is isolated: Let U be open in $S(A)$ and $p \in U$ be of minimal transcendence rank α among the points of U (if some of U is ranked). By definition, there is some open $V \subseteq S(A)$ such that $\{p\} = V \cap S^{\alpha}(A)$. But $U \cap S^{\alpha}(A) = U$ because α is minimal. Hence $V \cap U = \{p\}$, so p is isolated in $S(A)$. □

Lemma 1.8. *Let $\mathfrak{A}$ be a substructure of a model of T. Then $\mathfrak{A}$ models T iff $\mathfrak{A}$ realizes a dense subset of $S(A)$.*

Proof. See [3]. □

Theorem 1.9. *Suppose the isolated points of $S(A)$ are dense in $S(A)$ for every substructure $\mathfrak{A}$. Then every $\mathfrak{A}$ has a prime extension. If, in fact, T is totally transcendental, then any continuous chain in $\mathcal{B}(T)$ admits a chain of prime extensions in $\mathcal{K}(T)$.*

Proof. Suppose the isolated points of $S(A)$ are dense in $S(A)$ for all $\mathfrak{A} \in \mathcal{B}(T)$, and let $\kappa = |A| + \aleph_0$. If $p \in S(A)$ is isolated, there is a formula such that $\{p\} = [\phi]$. Thus, we can enumerate the isolated points of $S(A)$ by a κ-termed sequence $\langle p_\alpha : \alpha < \kappa \rangle$. We construct an increasing, continuous chain $\langle \mathfrak{A}_\alpha : \alpha < \kappa \rangle$ starting at $\mathfrak{A}$ in $\mathcal{B}(T)$ as follows: If $\mathfrak{A}_\alpha$ omits p_α, let $q \in S(A_\alpha)$ be an isolated extension of p_α.[12] Then, if $\mathfrak{M}$ is some elementary extension of $\mathfrak{A}_\alpha$ that realizes q via a_α, put $A_{\alpha+1} = A_\alpha \cup \{a_\alpha\}$. If instead $\mathfrak{A}_\alpha$ realizes p_α, put $A_{\alpha+1} = A_\alpha$.

Now, let $\mathfrak{A}_\kappa = \bigcup\{\mathfrak{A}_\alpha : \alpha < \kappa\}$. Then $\mathfrak{A}_\kappa$ is bounded by κ in power and realizes every isolated point of $S(A)$, but not necessarily of $S(A_\kappa)$. Enumerating the isolated points of $S(A_\kappa)$ by a κ-termed sequence and iterating the above procedure a countable number of times, we obtain a sequence $\langle \mathfrak{A}_{\kappa \cdot n} : n < \omega \rangle$ such that $\mathfrak{A}_{\kappa \cdot (n+1)}$ realizes the isolated types of $S(A_{\kappa \cdot n})$. Let $\mathfrak{A}_{\kappa \cdot \omega} = \bigcup\{\mathfrak{A}_{\kappa \cdot n} : n < \omega\}$. Then $\mathfrak{A}_{\kappa \cdot \omega}$ realizes the isolated types of $S(A_{\kappa \cdot \omega})$ due to the finitary nature of formulas. Indeed, any $\phi \in \mathcal{L}(A_{\kappa \cdot \omega})$ lives in $\mathcal{L}(A_{\kappa \cdot n})$ for some sufficiently large n. If $U = [\phi]$ in $S(A_{\kappa \cdot \omega})$

[12]Note $(Sf)^{-1}(p)$ is open in $S(B)$ and the isolated types are dense in $S(B)$.

and $V = [\phi]$ in $S(A_{\kappa\cdot n})$, it is easy to see $(Si_{A_{\kappa\cdot n}A_{\kappa\cdot\omega}})^{-1}(V) = U$. Thus, any type isolated in $S(A_{\kappa\cdot\omega})$ must be isolated in some $S(A_{\kappa\cdot n})$ for sufficiently large n. By (1.8) and hypothesis, we conclude $\mathfrak{A}_{\kappa\cdot\omega}$ is a model of T.

In fact, $\mathfrak{A}_{\kappa\cdot\omega}$ is a prime extension of A. Let $\mathfrak{B}$ be a model of T and f_0 be an embedding $\mathfrak{A} \to \mathfrak{B}$. We construct an increasing, continuous chain

$$\langle (f_\alpha : \mathfrak{A}_\alpha \to \mathfrak{C}) : \alpha < \kappa \rangle$$

starting at f_0: The only nontrivial case occurs when $A_{\alpha+1} \setminus A_\alpha = \{a_\alpha\}$. In this case, $(Sf_\alpha)^{-1}(q)$ is open in $S(C)$. But the isolated points are dense in $S(C)$, whence there is some isolated $q' \in (Sf_\alpha)^{-1}(q)$. By (1.8), $\mathfrak{C}$ realizes q' via some $c \in C$. $f_{\alpha+1}(a_\alpha) = c$ thus extends f_α to an embedding $\mathfrak{A}_{\alpha+1} \to \mathfrak{C}$. Let $f_\kappa = \bigcup\{f_\alpha : \alpha < \kappa\}$, and iterate this process $\kappa \cdot \omega$ times to obtain a morphism $f_{\kappa\cdot\omega} : \mathfrak{A}_{\kappa\cdot\omega} \to \mathfrak{C}$ that extends f_0.

Now suppose T is totally transcendental and let $\langle \mathfrak{A}_\delta : \delta < \alpha \rangle$ be a continuous chain in $\mathcal{B}(T)$. Without loss, suppose α is a limit ordinal. We define a sequence of continuous chains $\langle \mathfrak{A}_{\delta,\beta} : \delta \leq \alpha\}$ by induction on β. First, put $\mathfrak{A}_{\delta,0} = \mathfrak{A}_\delta$ and $\mathfrak{A}_{\delta,\beta+1} = \mathfrak{A}_{\delta,\beta}$ in case $\mathfrak{A}_{\delta,\beta}$ is a model of T. If $\mathfrak{A}_{\gamma,\beta}$ models T for all $\gamma < \delta$ but $\mathfrak{A}_{\delta,\beta}$ does not, we define a sequence $\langle p_{\gamma,\beta} : \delta \leq \gamma \leq \alpha \rangle$ of isolated types by induction on γ. Since $\mathfrak{A}_{\delta,\beta}$ is not a model of T, $\mathfrak{A}_{\delta,\beta}$ omits some isolated type $p_{\beta,\delta} \in S(A_{\delta,\beta})$. If $p_{\gamma,\beta}$ is isolated in $S(A_{\gamma+1,\beta})$, then $p_{\gamma,\beta}$ pulls back to an open set in $S(A_{\gamma+1,\beta})$. The element $p_{\gamma+1,\beta}$ of minimal rank and of minimal degree in that rank is isolated in $S(A_{\gamma+1,\beta})$. Finally, suppose λ is a limit ordinal; further suppose $p_{\gamma,\beta}$ pulls back to $p_{\gamma+1,\beta}$ of minimal rank and minimal degree in that rank, $p_{\gamma,\beta}$ is isolated in $S(A_{\gamma,\beta})$ whenever $\gamma < \lambda$, and $p_{\gamma,\beta}$ is a preimage of $p_{\gamma',\beta}$ whenever $\gamma' < \gamma < \lambda$. Let

$$p_{\lambda,\beta} = \bigcup\{p_{\gamma,\beta} : \gamma < \lambda\}.$$

The rank of $p_{\gamma,\beta}$ can only decrease finitely many times, since there can be no infinite descending chain of ordinals. Likewise for degree after this point. Hence there exists $\mu < \lambda$ where the ranks and degrees of the $p_{\gamma,\beta}$ stabilize for $\mu \leq \gamma < \lambda$. In fact, when $\mu \leq \gamma < \lambda$, $p_{\gamma,\beta}$ is the unique preimage of $p_{\mu,\beta}$. It follows, $p_{\lambda,\beta}$ is isolated in $S(A_{\lambda,\beta})$.

With δ as above, we proceed as follows: If $\mathfrak{A}_{\gamma,\beta}$ realizes $p_{\gamma,\beta}$ via a_γ for some $\gamma > \delta$, and γ_0 is the least such γ, let $\mathfrak{A}_{\gamma,\beta+1}$ be the substructure determined by $A_{\lambda,\beta} \cup \{a_{\gamma_0}\}$ whenever $\lambda \geq \delta$ and $\mathfrak{A}_{\gamma,\beta+1} = \mathfrak{A}_{\beta,\gamma}$ whenever $\gamma < \delta$. If $\mathfrak{A}_{\gamma,\beta}$ omits $p_{\gamma,\beta}$ whenever $\gamma > \delta$, let $\mathfrak{A}_{\gamma,\beta+1}$ expand $\mathfrak{A}_{\beta,\gamma}$ by a single element that realizes $p_{\alpha,\beta}$ whenever $\gamma \geq \delta$, and put $\mathfrak{A}_{\gamma,\beta+1} = \mathfrak{A}_{\gamma,\beta}$ when $\gamma < \delta$. The argument for (a) shows $\bigcup_\beta \mathfrak{A}_{\gamma,\beta}$ is a prime model over $\mathfrak{A}_\gamma$ when $\gamma < \delta$.

□

2. Stability and Indiscernibles

2.1. **Stable Theories.** Morley analysis and its concomitant topological perspective are a mixed blessing. Once we know a theory is totally transcendental, we gain access useful constructions made possible by Morley analysis, e.g. (1.9). But in most cases, it is cumbersome or downright unfeasible to decide whether a given theory is totally transcendental by computing its Morley rank. A combinatorial characterization of totally transcendental theories, on the other hand, would greatly simplify this task. It is not unreasonable to anticipate a bound on $|S(A)|$ for any substructure $\mathfrak{A}$: for a totally transcendental theory, we have $S^\beta(A) = \emptyset$ for all

$\beta \geq \alpha_T$, which suggests the types do not become too complicated. As it turns out, this holds in a strong sense: if T is totally transcendental, it admits the smallest possible number of types over infinite substructures of its models. Establishing the minimal bound for $|S(A)|$ over countable $\mathfrak{A}$ turns out to be enough for total transcendence, too.

Theorem 2.1. *Suppose T has a countable language. If $S(A)$ is countable for every countable $\mathfrak{A}$, then T is totally transcendental. Conversely, if T is totally transcendental, then $|S(A)| + \aleph_0 = |A| + \aleph_0$ for all substructures $\mathfrak{A}$.*

Proof. Suppose T is not totally transcendental. We claim there is a countable sequence $\mathfrak{A}_1 \subseteq \mathfrak{A}_2 \subseteq \dots$ of finite substructures in $\mathcal{B}(T)$ and a family $\{U_s : s \in 2^{<\omega}\}$ such that for all n and $s \in 2^n$,

(1) U_s is open in $S^{\alpha_T}(A_n)$ and nonempty,
(2) $S^{\alpha_T}(A_n) = \bigcup\{U_s : s \in 2^n\}$,
(3) U_s and U_t are disjoint if $s \neq t$, and
(4) $(Si_{A_n A_{n+1}})^{-1}(U_s) = U_{s\frown 0} \cup U_{s\frown 1}$.

Assuming we have found such $\mathfrak{A}_n$ and U_s, let $\mathfrak{A} = \bigcup_n \mathfrak{A}_n$. For $\sigma \in 2^\omega$, we put

$$U_\sigma = \bigcap_{n<\omega} (Si_{A_n A})^{-1}(U_{\sigma\restriction_n}) \cap S^{\alpha_T}(A).$$

If $\sigma \neq \tau$, then $U_\sigma \neq U_\tau$ since $U_{\sigma\restriction_n} \neq U_{\tau\restriction_n}$ for some n.

Also, U_σ is nonempty: Some p_s inhabits U_s for every $s \in 2^n$. By (1.3b), we have

$$S^{\alpha_T}(A_n) = (Si_{A_n A})(S^{\alpha_T}(A)).$$

So $(Si_{A_n A})^{-1}(U_s) \cap S^{\alpha_T}(A)$ is nonempty for all n and $s \in 2^n$. $U_\sigma \neq \emptyset$ follows from the fact that $S(A_n)$ is compact and U_s is closed in $S^{\alpha_T}(A_n)$, hence in $S(A_n)$.

Therefore $|S(A)| = 2^{\aleph_0}$ though A is countable. It remains to establish (1)–(4). First, observe there is some substructure $\mathfrak{A}_1$ with $|S^{\alpha_T}(A_1)| > 1$: otherwise, if p_A denotes the unique member of $S^{\alpha_T}(A)$ for $\mathfrak{A} \in \mathcal{B}(T)$, then we have

$$(Sf)^{-1}(p_A) \cap S^{\alpha_T}(A) \subseteq \{p_B\}$$

whenever $f : \mathfrak{A} \to \mathfrak{B}$ is an embedding among substructures in $\mathcal{B}(T)$. So $S^{\alpha_T+1}(A)$ is empty for all $\mathfrak{A}$, since each p_A trivially has rank α_T. But this contradicts the definition of α_T.

Let p, q be distinct members of $S^{\alpha_T}(A_1)$ and $\phi \in \mathcal{L}(A_1)$ be such that $\phi \in p$ and $\neg\phi \in q$. Form a clopen partition of $S^{\alpha_T}(A_1)$ via $U_0 = S^{\alpha_T}(A_1) \cap [\neg\phi]$ and $U_1 = S^{\alpha_T}(A_1) \cap [\phi]$. Now, if F_1 denotes the set of A-parameters mentioned in ϕ, then a clopen partition of $S^{\alpha_T}(F_1)$ is given by $(Si_{F_1 A_1})(U_0)$ and $(Si_{F_1 A_1})(U_1)$. So we can assume A_1 is finite without loss.

By a similar argument, there is a substructure $\mathfrak{B}$ containing $\mathfrak{A}_1$ such that

$$|(Si_{A_1 B})^{-1}(U_0) \cap S^{\alpha_T}(B)| > 1.$$

(Otherwise, every member of U_0 has rank α_T.) Similarly for some substructure $\mathfrak{C}$ with U_1. Let $\mathfrak{A}_2$ amalgamate $\mathfrak{B}$ and $\mathfrak{C}$ along $\mathfrak{A}_1$. If p, q are distinct members of $(Si_{A_1 B})^{-1}(U_0) \cap S^{\alpha_T}(B)$, there are $r, s \in S^{\alpha_T}(A_2)$ whose images under Si_{BA_2} are, respectively, p and q (1.3b). By our choice of p and q, r and s live in

$$(Si_{A_1 A_2})^{-1}(U_0) \cap S^{\alpha_T}(A_2).$$

Since r and s are distinct, we can form a clopen partition U_{00}, U_{01} of this set. Similarly, we can partition $(Si_{A_1A_2})^{-1}(U_1) \cap S^{\alpha_T}(A_2)$ via some U_{10}, U_{11}. As before, we assume A_2 is finite without loss. Iterating this procedure thus establishes the desired properties.

For the converse, recall we already established $|A| \leq |S(A)|$ for all substructures $\mathfrak{A}$. If T is totally transcendental, every $p \in S(A)$ has ordinal-valued rank. By definition, there is a formula ϕ_p such that $[\phi_p] \cap S^{(\mathrm{RM}(p))}(A) = \{p\}$. If q exceeds p in rank, then $[\phi_p] \cap S^{(\mathrm{RM}(p))}(A)$ is empty, whence $\phi_p \neq \phi_q$. The same holds when p, q have the same rank but $p \neq q$. This establishes an injection $S(A) \to U(A)$. We conclude $|A|$ and $|S(A)|$ are equivalent modulo $\aleph_0$, since $\mathcal{L}$ is countable. □

Definition 2.2. Let κ be infinite. T is ***κ-stable*** if $|S(A)| = \kappa$ for all substructures $\mathfrak{A}$ of cardinality κ. If $\kappa = \aleph_0$, T is ***ω-stable***. T is ***stable*** if κ-stable for some κ. Otherwise, T is ***unstable***.

Rephrasing (2.1), a countable theory is totally transcendental iff ω-stable iff stable in every κ.

2.2. **Indiscernible Sequences.** We use (2.1) to prove that T is totally transcendental if categorical in some uncountable power. The argument is simple: If T is unstable in $\aleph_0$, then $|S(A)|$ is uncountable for some countable substructure $\mathfrak{A}$. A routine Löwenheim–Skolem argument delivers for each $\kappa > \aleph_0$ a model $\mathfrak{M}_\kappa$ such that $\mathfrak{A} \subseteq \mathfrak{M}_\kappa$, $|M_\kappa| = \kappa$, and $\mathfrak{M}_\kappa$ realizes at least $\aleph_1$ types over A. To show T is not κ-categorical for any uncountable κ, it thus suffices to construct models $\mathfrak{N}_\kappa$ in κ that realize $\aleph_0$ types over countable substructures. In fact, we produce models that realize at most $|B| + \aleph_0$ types over any substructure $\mathfrak{B}$.

Models generated from so-called ***indiscernibles*** meet this specification. Indiscernibles arise naturally in this context: If a model $\mathfrak{A}$ contains a substructure $\mathfrak{B}$ with $|S(B)| < |A|$, the pigeonhole principle implies that multiple elements of A must realize the same type over B. And, if $\mathrm{tp}(a/B) = \mathrm{tp}(b/B)$, then a and b are "indiscernible" from the point of view of first-order logic. The notion of an indiscernible sequence is a particularly strong form of this idea:

Definition 2.3. Let $\mathfrak{A}$ be a model and X be a subset of A that carries a linear order $<$[13]. For $B \subseteq A$, $(X, <)$ is an ***indiscernible sequence*** over B in $\mathfrak{A}$ if whenever $n < \omega$, any two increasing n-tuples $x_1 < \cdots < x_n$, $y_1 < \cdots < y_n$ in X realize the same type over B in $\mathfrak{A}$, i.e. $\mathrm{tp}(\mathbf{x}/B) = \mathrm{tp}(\mathbf{y}/B)$.[14]

For instance, $\mathbb{N}$ is indiscernible in $(\mathbb{Q}, <)$: increasing n-length sequences in $\mathbb{N}$ satisfy the same quantifier-free formulas in n-free variables, which is enough since DLO eliminates quantifiers. Another example comes from field theory: in $(\mathbb{C}, +, \cdot, -, 0, 1)$, any enumeration of a transcendence basis for $\mathbb{C}$ is indiscernible.

Towards showing categoricity in an uncountable power suffices for ω-stability, we prove any theory with infinite models exhibits a model in which a given linear order is an indiscernible sequence. This result—a construction due to A. Ehrenfeucht and A. Mostowski—depends on infinite Ramsey's theorem, a combinatorial fact about finitely colored infinite hypergraphs. [1] details a nonconstructive proof using ultrafilters.

[13] $<$ may or may not be definable in $\mathfrak{A}$.

[14] Order matters: the notation $\mathbf{x}$ is short for $(x_1 \ldots x_n)$, not any permutation thereof.

Lemma 2.4. *(Ramsey) Let A be an infinite set. Put $A^{(m)} = \{X \subseteq A : |X| = m\}$. Suppose $P_1 \dots P_n$ partition $A^{(m)}$. Then, there is an infinite $B \subseteq A$ such that $B^{(m)} \subseteq P_j$ for some j.*

Theorem 2.5. *(Ehrenfeucht–Mostowski) Suppose T has infinite models. If $(X, <)$ is an infinite linear order, then T has a model in which $(X, <)$ is an indiscernible sequence.*

Proof. In $\mathcal{L}(X)$, let Σ contain T along with the assertions $c_x \neq c_y$ for distinct $x, y \in X$ and $\phi(c_{x_1} \dots c_{x_n}) \leftrightarrow \phi(c_{y_1} \dots c_{y_n})$ whenever $n < \omega$, ϕ is an $\mathcal{L}$-formula, and $x_1 < \cdots < x_n$, $y_1 < \cdots < y_n$ are increasing n-tuples in X. The reduct of a model of Σ to $\mathcal{L}$ will satisfy the theorem. Let $S \subseteq \Sigma$ be finite, denote by $\psi_1 \dots \psi_k$ the finitely many sentences in S of the form $\phi_i(c_{\mathbf{x}}) \leftrightarrow \phi_i(c_{\mathbf{y}})$, and let m be the least m such that the free variables of ϕ_i are among $v_1, \dots, v_m$ for $1 \leq i \leq k$. By hypothesis, there exists an infinite model $\mathfrak{A}$ of T. Equip A with an arbitrary linear order, so that $A^{(m)}$ bijects with the set of increasing m-tuples in A. If $I \subseteq \{1 \dots k\}$ and

$$P_I = \{a_1 < \cdots < a_m \in A : \mathfrak{A} \models \phi_i[\mathbf{a}] \text{ iff } i \in I\},$$

then $P_I : I \subseteq \{1 \dots k\}$ amounts to a finite partition of $A^{(m)}$. Ramsey's theorem delivers an infinite $D \subseteq A$ such that $D^{(m)} \subseteq P_{I_0}$ for some I_0. If X_S is the set of parameters from X used in S, then there exists an order-preserving map $F : X_S \hookrightarrow D$. Expand $\mathfrak{A}$ to an $\mathcal{L}(X)$-structure $\mathfrak{A}_X$ as follows: If $x \in X_S$, put $c_x^{\mathfrak{A}_X} = F(x)$; otherwise interpret $c_x^{\mathfrak{A}_X}$ arbitrarily. $\mathfrak{A}_X$ evidently satisfies S, whence the theorem follows by compactness. □

Recall a theory has ***definable Skolem terms*** if its language supplies enough operation symbols to witness existentials—that is, for every $\phi \in \mathcal{L}$ there exists a term $t_\phi \in \mathcal{L}$ such that

$$T \models \forall \mathbf{w}(\exists v \phi(v, \mathbf{w}) \to \phi(t_\phi(\mathbf{w}), \mathbf{w})).$$

A theory with definable Skolem terms is ***model complete***, i.e. every substructure of a model of T is elementary. Indeed, observe $\mathfrak{B}$ is closed under Skolem witnesses $t_\phi(\mathbf{b})$ to any $\mathcal{L}(B)$-formula $\phi(v, \mathbf{b})$ so that $\phi(t_\phi(\mathbf{b}), \mathbf{b})$ holds in $\mathfrak{B}$ iff it holds in $\mathfrak{A}$.

Every T admits an expansion $\overline{T}$ with definable Skolem terms such that its language has power $|\mathcal{L}|$. To do this, one produces a countable chain $\langle T_n \rangle$ such that $T_0 = T$ and T_{n+1} claims the terms $t_\phi : \phi \in \mathcal{L}_n$ in $\mathcal{L}_{n+1}$ bear witness to the corresponding existentials in the language of T_n. The union $\overline{T} = \bigcup_n T_n$ witnesses all existentials in its own language, since formulas are finitary and $\overline{\mathcal{L}} = \bigcup_n \mathcal{L}_n$.

If $\mathfrak{A}$ models T and $X \subseteq A$, the ***Skolem hull*** $\overline{\mathfrak{H}}(X)$ is the substructure of $\overline{\mathfrak{A}}$ generated by X, where $\overline{\mathfrak{A}}$ models $\overline{T}$. Observe $H(X) = \operatorname{dom} \overline{\mathfrak{H}}(X)$ is between $|X|$ and $|X| + |\mathcal{L}|$ in power. In particular, if indiscernible in $\overline{\mathfrak{A}}$, $(X, <)$ is also indiscernible in $\overline{\mathfrak{H}}(X)$ by model completeness.

A model generated by indiscernibles in this way realizes the smallest possible bound on the number of types it realizes over infinite substructures:

Theorem 2.6. *Any countable theory T has a model in power κ that realizes at most $|A| + \aleph_0$ types over any substructure $\mathfrak{A}$.*

Proof. Let $(X, <)$ have order type κ. Let $\overline{T}$ be an expansion of T with definable Skolem terms and a countable language. (2.5) delivers a model of $\overline{T}$ in which X is indiscernible. By the remarks above, X is also indiscernible in $\overline{\mathfrak{H}}(X)$. We show that the reduct $\mathfrak{H}(X)$ of $\overline{\mathfrak{H}}(X)$ to $\mathcal{L}$ satisfies the theorem. First, observe $|H(X)| = \kappa$ since $|X| = \kappa$ and $|H(X)| \leq |X| + |\overline{\mathcal{L}}|$.

Let $\mathfrak{A} \subseteq \mathfrak{H}(X)$. For each $a \in A$, there is a Skolem term t and $x_1 < \cdots < x_n$ in X such that $t(\mathbf{x}) = a$ in $\overline{\mathfrak{H}}(X)$. Choose one such representation for each $a \in A$, identify A with this set, and let Y be the set of X-parameters used to generate these representations of elements of A. Then Y is the union of finite sets indexed by A, which establishes the bound $|Y| \leq |A| + \aleph_0$.

For each $x \in X$, let $\mu(x) = \inf\{z \in Y : x \leq z\}$ or ∞ in case $x > Y$. Since X is well-ordered, $\mu(x)$ is well-defined for all $x \in X$. For $n < \omega$ and increasing n-length sequences $\mathbf{x}, \mathbf{y}$ in X, we write $\mathbf{x} \sim_n \mathbf{y}$ if $\mu(x_k) = \mu(y_k)$ for each k. That is, $\mathbf{x} \sim_n \mathbf{y}$ iff they interpolate among the members of Y in the same way.

By indiscernibility and rearranging free variables, $\mathbf{x} \sim_n \mathbf{y}$ implies $\phi(\mathbf{x}) \leftrightarrow \phi(\mathbf{y})$ in $\overline{\mathfrak{H}}(X)$ whenever $\phi \in \overline{\mathcal{L}}(Y)$ has n free variables. Each $\overline{\mathcal{L}}(A)$-formula is equivalent to $\overline{\mathcal{L}}(Y)$-formula in $\overline{\mathfrak{H}}(X)$: just expand A-parameters by their corresponding representation as a Skolem term applied to some increasing sequence in X. It follows, $\mathbf{x} \sim_n \mathbf{y}$ implies $\phi(\mathbf{x}) \leftrightarrow \phi(\mathbf{y})$ in $\overline{\mathfrak{H}}(X)$ whenever $\phi \in \overline{L}(A)$ has n free variables. So if t is any n-ary Skolem term and $\mathbf{x} \sim_n \mathbf{y}$, then $\mathrm{tp}(t(\mathbf{x})/A) = \mathrm{tp}(t(\mathbf{y})/A)$ in $\overline{\mathfrak{H}}(X)$, hence in $\mathfrak{H}(X)$.

Contraposing, if $s(\mathbf{x})$ and $t(\mathbf{y})$ are members of $H(X)$ that realize distinct types over A in $\mathfrak{H}(X)$, then either $s \neq t$ or $\mathbf{x}, \mathbf{y}$ have the same length n and $\mathbf{x} \sim_n \mathbf{y}$. For each n, $|X/\sim_n|$ is precisely $|Y|$, the number of increasing n-length sequences in Y. It follows,

$$\left|\bigcup_n X_n/\sim_n\right| \leq |Y| + \aleph_0.$$

Since $\overline{\mathcal{L}}$ is countable and $|Y| \leq |A| + \aleph_0$, we conclude $\mathfrak{H}(X)$ realizes at most $|A| + \aleph_0$ types over A. □

Corollary 2.7. *T is ω-stable if κ-categorical for some uncountable κ.*

2.3. **Indiscernible Sets.** An indiscernible set is an indiscernible sequence in which the order of elements is irrelevant:

Definition 2.8. X is an ***indiscernible set*** over B in $\mathfrak{A}$ if for all n and n-tuples $x_1 \dots x_n$, $y_1 \dots y_n$ in X, $\mathbf{x}$ and $\mathbf{y}$ have the same type over B. An indiscernible sequence that is an indiscernible set is called ***totally***[15] ***indiscernible***.

Is every indiscernible sequence totally indiscernible? In general, no: $(\mathbb{N}, <)$ is not totally indiscernible in $(\mathbb{Q}, <)$. On the other hand, an enumeration of a transcendence basis for $\mathbb{C}$ is. This is because ACF_0 is uncountably categorical and thus ω-stable.

Theorem 2.9. *If T is ω-stable, then every infinite indiscernible sequence in a model of T is totally indiscernible.*

Proof. Suppose T has a model $\mathfrak{A}$ in which $(X, <)$ is indiscernible but not totally indiscernible. That is, suppose there is some formula $\phi \in \mathcal{L}$, increasing sequence $\mathbf{x}$

[15]Following the standard model-theoretic convention of overloading a small set of terms in the metalanguage.

in X, and permutation $\rho \in S_n$ such that $\phi(\mathbf{x})$ and $\neg\phi(\rho\mathbf{x})$ hold in $\mathfrak{A}$. The following defines a partition of S_n:

$$S_n^+ = \{\rho \in S_n : \mathfrak{A} \models \phi(\rho\mathbf{x})\},$$
$$S_n^- = \{\rho \in S_n : \mathfrak{A} \models \neg\phi(\rho\mathbf{x})\}.$$

By assumption, S_n^+ and S_n^- are nonempty. Hence there are $\theta = (k, k+1)$, $\sigma \in S_n^+$, and $\tau \in S_n^-$ such that

$$\tau = \theta\sigma.^{16}$$

Define $\psi(\mathbf{v}) = \phi(\sigma\mathbf{v})$, so that $\psi(\mathbf{x})$ and $\neg\psi(\theta\mathbf{x})$ hold in $\mathfrak{A}$ by construction. In fact, by indiscernibility, these hold for any increasing n-length sequence in X.

In $\mathcal{L}(\mathbb{R})$, consider the theory

$$\Sigma = T \cup \{\psi(c_{r_1} \dots c_{r_n}) \wedge \neg\psi(\theta(c_{r_1} \dots c_{r_n})) : r_1 < \cdots < r_n \in \mathbb{R}\}.$$

Because X is infinite, and $\psi(\mathbf{x})$ and $\neg\psi(\theta\mathbf{x})$ hold in $\mathfrak{A}$ whenever $x_1 < \cdots < x_n$ in X, compactness guarantees Σ is satisfiable. Let $\mathfrak{B}$ be the reduct of a model of Σ to $\mathcal{L}$ that contains $\mathbb{R}$. If $r < s$ in $\mathbb{R}$, we can find $n-1$ rationals such that

$$q_1 < \cdots < q_{k-1} < r < q_{k+1} < s < \cdots < q_n.$$

But the following holds in $\mathfrak{B}$ by construction:

$$\mathfrak{B} \models \phi(q_1 \dots q_{k-1}, r, q_{k+1} \dots q_n) \wedge \neg\phi(q_1 \dots q_{k-1}, s, q_{k+1} \dots q_n).$$

We've shown distinct reals realize distinct types over the rationals in $\mathfrak{B}$. So T is unstable in $\aleph_0$. □

A model with a large set of indiscernibles realizes a small number of types, as we show in the following section. In the interest of building indiscernible sets, (2.9) greatly simplifies this task when the ambient theory is totally transcendental: it suffices to produce indiscernible sequences.

We examined how to construct prime extensions (of continuous chains) for substructures of models of totally transcendental theories via Morley analysis. The machinery presented there also helps in the way of constructing models with large sets of indiscernibles. Note that up to this point in the present section, we can omit the assumption that T is a Morleyization of some theory. In what follows, we resume this proviso in order to freely utilize Morley analysis as developed in the preceding section.

Theorem 2.10. *Suppose T is totally transcendental. Let $\mathfrak{A} \in \mathcal{B}(T)$ be uncountable and $\mathfrak{B}$ be a substructure of strictly smaller cardinality. For each $\kappa < |A|$, there is an indiscernible set X over B of cardinality $> \kappa$. In fact, if $|A|$ is regular, such an X exists with $|X| = |A|$.*

Proof. We only need to prove the last sentence in the statement of the theorem. Indeed, κ^+ is regular for any κ.

Fix a substructure $\mathfrak{C}$ such that $B \subseteq C \subseteq A$ and $|C| < |A|$. By stability, we have $|S(C)| < |A|$. If p has less than $|A|$ realizations in A for each $p \in S(C)$, then

[16] Suppose S_n^+ were such that $(k, k+1)\sigma$ never landed in S_n^-. Let $\tau \in S_n$ be arbitrary. Recall transpositions of the form $(k, k+1)$ generate S_n. Thus $\theta = \tau\sigma^{-1}$ can be written as a product

$$\prod_{i<n_\tau} (k_i, k_i + 1).$$

By induction, $\tau = \theta\sigma$ must be in S_n^+ i.e. $S_n^+ = S_n$.

by regularity, $|A|$ bounds the set of realizations of all $p \in S(C)$ in power. But A realizes at least $|A|$ types over C via $\mathrm{tp}(a/C)$ for $a \in A$. We conclude there must be some $p \in S(C)$ that has $|A|$ realizations in A. Of the set of pairs $(\mathfrak{C}, p)$ satisfying this criterion, let $(\mathfrak{C}_0, p_0)$ be such that p_0 has minimal rank ν and minimal degree n among those of rank ν. Similarly, if $\mathfrak{C}' \in \mathcal{B}(T)$ is such that $C_0 \subseteq C' \subseteq A$ and $|C'| < |A|$, p_0 pulls back to some $p' \in (Si_{C_0C'})^{-1}(p_0)$ that has $|A|$ realizations in A: for $|S(C')|$, hence $|(Si_{C_0C'})^{-1}(p_0)|$, is bounded by $|A|$ in power. We have $\mathrm{RM}(p') \geq \nu$ by minimality, whence $\mathrm{RM}(p') = \nu$: rank is monotonically decreasing with respect to preimages (1.3b). Likewise for $\deg(p')$. By (1.3d), p_0 pulls back to a unique type over C' of rank ν and degree n, and it has $|A|$ realizations in A.

By induction, there exists a set $\{x_\alpha : \alpha < \kappa\} \subseteq A \setminus C_0$ such that p_α is the unique member of C_α of rank ν and degree n in $(Si_{C_0C_\alpha})^{-1}(p_0)$, where

$$C_\alpha = C_0 \cup \{x_\beta : \beta < \alpha\}$$

and $p_\alpha = \mathrm{tp}(x_\alpha/C_\alpha)$: if $\{x_\beta : \beta < \alpha\}$ is defined, the argument above guarantees $|A|$ members of B that realize p_α. From these, we can choose our x_α arbitrarily. In particular, notice $(Si_{C_\beta C_\alpha})(p_\alpha) = p_\beta$ whenever $\beta < \alpha$, whence x_α realizes p_β for all $\beta < \alpha$.

It suffices to produce an isomorphism

$$\mathfrak{C}_0(x_{\beta_1} \dots x_{\beta_m}) \cong \mathfrak{C}_0(x_{\beta'_1} \dots x_{\beta'_m})$$

whenever $m < \omega$ and $\beta_i, \beta'_i < \alpha$. By (2.9), we can assume $\beta_1^{(')} < \cdots < \beta_m^{(')}$.

For induction, suppose we have an isomorphism f_{m-1} such that $f_{m-1} \restriction_{C_0}$ is the identity and $f_{m-1}(x_{\beta_i}) = x_{\beta'_i}$ for each i. Let q be the type realized by x_{β_m} over $C_0 \cup \{x_{\beta_1} \dots x_{\beta_{m-1}}\}$, and likewise for q'. Then, extend f_{m-1} to the map f_m via $f_m(x_{\beta_m}) = x_{\beta'_m}$. It suffices to show $(Sf_{m-1})(q') = q$.

Recall x_{β_m} realizes $p_0 \in S(C_0)$ and $p_{\beta_m} \in S(C_{\beta_m})$, both of rank ν and degree n. Since $C_0 \subseteq C_0 \cup \{x_{\beta_1} \dots x_{\beta_m}\} \subseteq C_{\beta_m}$, q must also have rank ν and degree n. In fact, by the argument above, q is the unique preimage of p_0 of rank ν and degree n. The analogous statements hold for q'.

By hypothesis, (Sf_{m-1}) is a homeomorphism. Therefore, $(Sf_{m-1})(q')$ has RM ν and degree n. But $(Sj) = (Si)(Sf_{m-1})$ implies

$$(Sf_{m-1})(q') \in (Si)^{-1}(p_0),$$

and q is the unique preimage of p_0 of rank ν and degree n. The desired result thus follows: $(Sf_{m-1})(q') = q$. □

The proof of (2.10) suggests a more general construction: Let T be a complete, Morleyized theory, and let $Y, \{a_\alpha : \alpha < \gamma\}$ be subsets of a substructure $\mathfrak{A}$. For $\alpha < \gamma$, put $X_\alpha = Y \cup \{a_\beta : \beta < \alpha\}$ and $p_\alpha = \mathrm{tp}(a_\alpha/X_\alpha)$. Suppose p_0 has ordinal-valued rank, and p_α pulls back to p_β under $Si_{X_\alpha X_\beta}$ for all $\alpha < \beta < \gamma$. Then, p_α has ordinal-valued rank for all $\alpha < \gamma$. $\{a_\alpha : \alpha < \gamma\}$ is called a ***Morley sequence*** over Y in $\mathfrak{A}$ if every p_α has the same rank and degree as p_0. The proof of (2.10) is easily adapted to prove a general fact: any infinite Morley sequence is totally indiscernible over Y in $\mathfrak{A}$.

3. Saturation

ω-stable theories have as few types as possible; saturated models realize as many types as reasonably expected.

Definition 3.1. An infinite model $\mathfrak{M}$ is ***κ-saturated*** if it realizes every type over every substructure of cardinality $< \kappa$. $\mathfrak{M}$ is ***saturated*** if $\kappa = |M|$.[17]

Saturated models are unique up to isomorphism in a given cardinality. This follows by a routine back-and-forth argument: Suppose $\mathfrak{A}, \mathfrak{B}$ are saturated models with the same complete theory and $|A| = |B| =: \kappa$. We show $\mathfrak{A} \cong \mathfrak{B}$ by constructing a continuous chain of elementary maps $f_\alpha : \mathfrak{A}_\alpha \to \mathfrak{B}$ by induction on $\alpha < \kappa$. Enumerate $A = \{a_\alpha : \alpha < \kappa\}$ and $B = \{b_\alpha : \alpha < \kappa\}$. Suppose f_β is defined, $a_\gamma \in A_\beta$ and $b_\gamma \in f_\beta(A_\beta)$ for $\gamma < \beta$, and $|A_\beta| < \kappa$. Let

$$p \in (Sf_\beta)^{-1}(\mathrm{tp}(a_\beta/A_\beta))$$

so that p is a type over $f_\beta(A_\beta)$. Because $\mathfrak{B}$ is saturated and $|A_\beta| < \kappa$, some $b \in \mathfrak{B}$ realizes p. Put $f'_\beta = f_\beta \cup \{(a_\beta, b)\}$ to extend f_β to an elementary embedding $\mathfrak{A}_\beta(a_\beta) \to \mathfrak{B}$. Similarly, we can find some a that realizes $(Sf'_\beta)^{-1}(\mathrm{tp}(b_\beta/A_\beta \cup \{a_\beta\}))$ and extend f'_β to $f_{\beta+1}$ via (a, b_β). The union of this chain is a bijective elementary embedding $\mathfrak{A} \cong \mathfrak{B}$, as desired.

Furthermore, a saturated substructure of a model of T must be a model of T. If $\mathfrak{A} \in \mathcal{B}(T) \setminus \mathcal{K}(T)$, there is some $\phi \in \mathcal{L}(A)$ such that $\mathfrak{A}$ omits every $p \in [\phi]$ (1.7). Let F be the set of A-parameters mentioned in ϕ, and let $p \in [\phi]$ be arbitary. If $\mathfrak{A}$ is saturated, then some $a \in A$ realizes $(Si_{FA})(p)$. But then $\mathrm{tp}(a/A)$ contains ϕ, a contradiction. The same argument works for $\aleph_1$-saturated substructures as well.

Stability in every infinite κ allows for the constructuction of $\aleph_1$-saturated models in every uncountable cardinality. Let $\kappa > \aleph_0$. Fix an arbitrary model $\mathfrak{B}_0 \in \mathcal{K}(T)$ in κ, so that $|S(B_0)| = \kappa$ by stability. By induction, there is a continuous chain $\langle \mathfrak{B}_\alpha : \alpha < \omega_1 \rangle$ in $\mathcal{K}(T)$ such that $\mathfrak{B}_{\alpha+1}$ realizes every type over B_α and each $|B_\alpha| = \kappa$. Let $\mathfrak{B}$ be the union of this chain. Then $|B| = \kappa$ because each $|B_\alpha| = \kappa$ and $\kappa > \aleph_0$. By regularity of ω_1, there exists $\alpha < \omega_1$ such that $A \subseteq B_\alpha$ whenever $A \subseteq B$ is countable. Then $\mathfrak{B}_{\alpha+1}$, hence $\mathfrak{B}$, realizes every type over B_α, hence A.

We summarize these basic properties in

Proposition 3.2. *Let $\mathfrak{A}, \mathfrak{B} \in \mathcal{B}(T)$.*

(a) If $\mathfrak{A}$ and $\mathfrak{B}$ are saturated and $|A| = |B|$, then $\mathfrak{A} \cong \mathfrak{B}$.
(b) If ($\aleph_1$-)saturated, $\mathfrak{A}$ is a model of T.
(c) If ω-stable, T has an $\aleph_1$-saturated model in every uncountable cardinal.

We can finally prove a stronger form of Morley's theorem: if T is κ-categorical for some uncountable κ, then every uncountable model of T is saturated. The argument goes as follows: If ω-stable, T has an $\aleph_1$-saturated model in every uncountable power. So it suffices to produce a model of T in every uncountable cardinal that is not $\aleph_1$-saturated, assuming T has an uncountable, unsaturated model. (2.10) and the Löwenheim–Skolem theorem deliver a countable model of T that omits a type over a countable substructure and contains a countable set of indiscernibles. By compactness, we obtain a continuous chain in $\mathcal{B}(T)$ whose members omit this type and contain large sets of indiscernibles. The union of a prime extension of this chain gives a model that is not $\aleph_1$-saturated, as we show:

Theorem 3.3. *Let T be totally transcendental. Suppose $\mathfrak{B} \in \mathcal{K}(T)$ is uncountable and unsaturated.*

[17] We can't realize all types over every substructure of cardinality $|M|$: the set $\{v \neq m : m \in M\}$ extends to a type over M that $\mathfrak{M}$ cannot realize.

(a) $\mathfrak{B}$ *has a countable substructure* $\mathfrak{A}$*, a model of* T*, that contains subsets* A', Y *such that* Y *is infinite and indiscernible over* A' *and* $\mathfrak{A}$ *omits some* $q \in S(A')$.
(b) T *has a model in every uncountable power that is not* $\aleph_1$*-saturated.*

Proof. For (a): By hypothesis, $\mathfrak{B}$ omits a type p over some substructure $\mathfrak{C}$ of strictly smaller cardinality. By (2.10), there is a countable set Y of indiscernibles over C. Löwenheim–Skolem furnishes a countable elementary substructure $\mathfrak{A}_0 \preceq \mathfrak{B}$ that contains Y. Since $\mathfrak{B}$ omits p, we have for each $a \in A_0$ an $\mathcal{L}(C)$-formula ϕ_a such that ϕ_a inhabits $\mathrm{tp}(a/C)$ while $\neg\phi_a$ inhabits p.

Let C_a be the C-parameter set of ϕ_a, and let A'_1 be the union of C_a for $a \in A_0$. Evidently, no element of A_0 realizes $(Si_{A'_1C})(p)$ in $S(A'_1)$: if $a \in A_0$ realizes this type, then $\neg\phi_a \in p$ implies $\neg\phi_a \in (Si_{A'_1C})(p)$. But this contradicts $\phi_a \in \mathrm{tp}(a/C)$.

Again, Löwenheim–Skolem furnishes a countable elementary substructure $\mathfrak{A}_1 \preceq \mathfrak{B}$ that extends $A_0 \cup A'_1$. Then, induction on n gives a sequence of countable models $\langle \mathfrak{A}_n \rangle$ and subsets $A'_1 \subseteq A'_2 \subseteq \dots$ such that A'_n is contained in $A_n \cap C$ and $\mathfrak{A}_n$ omits the type $(Si_{A'_{n+1}C})(p)$.

Let $\mathfrak{A}$ and A' denote the unions of these two chains. Then $\mathfrak{A}$ omits $(Si_{A'C})(p)$. Since $A' \subseteq C$, $Y \subseteq A$, and $\mathfrak{A} \succeq \mathfrak{A}_n \preceq \mathfrak{B}$ for all n, Y is indiscernible over A' in $\mathfrak{A}$.

For (b): Part (a) shows there exists a countable substructure $\mathfrak{A} \subseteq \mathfrak{B}$ satisfying the following properties: $\mathfrak{A}$ is a model of T that contains subsets Y and A' such that Y is indiscernible over A' and $\mathfrak{A}$ omits some $q \in S(A')$. By compactness, there exists a model $\mathfrak{A}_\kappa$ of cardinality κ containing $A' \cup Y$ such that $A_\kappa \setminus A'$ is an indiscernible set over A' of cardinality κ. Let $\{y_\alpha : \alpha < \kappa\}$ be a well-ordering of $A_\kappa \setminus A'$, and let $A_\alpha = A' \cup \{y_\beta : \beta < \alpha\}$. (1.9b) furnishes a prime extension $\langle \mathfrak{B}_\alpha : \alpha < \kappa \rangle$ of $\langle \mathfrak{A}_\alpha : \alpha < \kappa \rangle$.

It suffices to show $\mathfrak{B}_\alpha$ omits q for each $\alpha < \kappa$: For then the union $\mathfrak{B}_\kappa$ is a model of T that omits q. The result is trivial when α is a limit ordinal: if $\mathfrak{B}_\alpha$ realizes q, so would $\mathfrak{B}_\beta$ for some $\beta < \alpha$ by continuity.

Assume $\alpha = \beta + 1$ is a successor ordinal. Suppose first that α is finite. Recall $\mathfrak{A}$ is a model of T that extends $A' \cup Y$ and omits q. Since $\mathfrak{B}_\alpha$ is prime over $A' \cup \{y_0 \dots y_\beta\}$, there is an embedding of $\mathfrak{B}_\alpha$ into $\mathfrak{A}$. Thus $\mathfrak{B}_\alpha$ also omits q. Now suppose α is infinite. By indiscernibility of $A_\kappa \setminus A'$ over A', there exists an isomorphism $\mathfrak{A}_\alpha \cong \mathfrak{A}_\beta$ which is the identity on A'. This isomorphism extends to an elementary embedding $\mathfrak{B}_\alpha \to \mathfrak{B}_\beta$ among the given prime extensions of $\mathfrak{A}_\alpha$ and $\mathfrak{A}_\beta$. By induction, $\mathfrak{B}_\beta$ omits q. So $\mathfrak{B}_\alpha$ omits q, given $\mathfrak{B}_\alpha \preceq \mathfrak{B}_\beta$. □

Theorem 3.4. *If* T *is* κ*-categorical for some uncountable* κ*, then every uncountable model of* T *is saturated.*

Proof. T is ω-stable by (2.7). By (3.2), it has an $\aleph_1$-saturated model $\mathfrak{A}_\kappa$ in every uncountable κ. If T has an uncountable, unsaturated model, it has a model $\mathfrak{B}_\kappa$ in every uncountable κ that is not $\aleph_1$-saturated. So T is not κ-categorical for any uncountable κ, a contradiction. □

Morley's theorem follows by (3.4) and (3.2a).

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