CLASSIFYING THE FINITE SUBGROUPS OF SO_3

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Abstract. In this paper, we classify the finite subgroups of SO_3, the group of rotations of $\mathbb{R}^3$. We prove that all finite subgroups of SO_3 are isomorphic to either a cyclic group, a dihedral group, the tetrahedral group, the octahedral group or the icosahedral group.

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1. Introduction

The aim of this paper is to prove the following classification theorem:
A finite subgroup of SO_3 is one of the following groups:

- C_k: the cyclic group of rotation by multiples of $2\pi/k$ about a line, with k arbitrary;
- D_k: the dihedral group of symmetries of a regular k-gon, with k arbitrary;
- T: the tetrahedral group of 12 rotational symmetries of a tetrahedron;
- O: the octahedral group of 24 rotational symmetries of a cube or an octahedron;
- I: the icosahedral group of 60 rotational symmetries of a dodecahedron or an icosahedron.

More than just itself being an interesting classificatory result, this theorem sheds light upon the importance of the Platonic solids from a group-theoretic perspective. Oftentimes we define the Platonic solids as regular convex polyhedra that are constructed by congruent regular polygonal faces with the same number of faces meeting at each vertex. We could show that there are exactly five Platonic solids by counting the number of ways that we can build one by bringing together congruent

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regular polygons at a vertex. However, as opposed to the geometric understanding, our classification theorem emphasizes the group-theoretic one. It tells us that the Platonic solids can be thought of as expressing some kind of symmetry, or to be more accurate, the symmetries of the Platonic solids are fundamentally important in group theory because they are essentially the finite subgroups of SO_3.

This paper is a self-contained proof of the classification of the finite subgroups of SO_3 based on arguments that can be found in [1]. The images are either drawn or from [1] or [2].

2. Basic Group Theory: Groups, Subgroups and Cosets

We must first acquaint ourselves with some preliminary definitions in group theory.

Definition 2.1. A *group* is a set G together with a law of composition that has the following properties:

- The law of composition is associative: $(ab)c = a(bc)$ for all a, b, c in G.
- G contains an identity element 1, such that $1a = a = a1$ for all a in G.
- Every element a of G has an inverse, an element b such that $ab = 1 = ba$.

Definition 2.2. The *order* of a group G is the number of elements that it contains. We denote the order by $|G|$. If the order is finite, G is said to be *finite*. If not, G is *infinite*.

Definition 2.3. A subset H of a group G is a *subgroup* of G if it has the following properties:

- Closure: If a and b are in H, then ab is in H.
- Identity: 1 is in H.
- Inverses: If a is in H, then a^{-1} is in H.

The first condition tells us that the law of composition on the group G defines a law of composition on H, called the *induced law*. The second and third conditions say that H is a group with respect to this induced law.

Definition 2.4. The subgroup *generated by a subset X of G* is the smallest subgroup of G that contains X.

Definition 2.5. If H is a subgroup of G and if a is an element of G, the subset $aH = \{ah \mid h \in H\}$ is called a *left coset*.

Corollary 2.6. *The left cosets of a subgroup H of a group G partition the group.*

Proof. The left cosets of H in G are the equivalence classes for the congruence relation

$$a \equiv b \text{ if } b = ah \text{ for some } h \text{ in } H.$$

We can verify this congruence is an equivalence relation:

Transitivity: Suppose that $a \equiv b$ and $b \equiv c$. This means that $b = ah$ and $c = bh'$ for some elements h and h' of H. Therefore, $c = ahh'$. Since H is a subgroup, hh' is in H. Thus, $a \equiv c$.

Symmetry: Suppose $a \equiv b$, so that $b = ah$. Then $a = bh^{-1}$ and h^{-1} is in H. So $b \equiv a$.

Reflexivity: $a = a1$ and 1 is in H, so $a \equiv a$. □

Lemma 2.7. *All left cosets aH of a subgroup H of a group G have the same order.*

Proof. Left multiplication by a defines a map $H \to aH$ that sends h to ah. This map is bijective because its inverse is the left multiplication by a^{-1}. □

Since the cosets all have the same order, and since they partition the group, we obtain the important Counting Formula for a finite group G:

$$|G| = |H|[G : H]. \tag{2.8}$$

$$(\text{order of } G) = (\text{order of } H)(\text{number of cosets}).$$

3. Homomorphism, Isomorphism, and Conjugation

Now that we know the basics of groups, it is important that we understand what it means for groups to have the same structure, and to "classify groups".

Definition 3.1. Let G and G' be groups. A *homomorphism* $\varphi\colon G \to G'$ is a map from G to G' such that for all a and b in G,

$$\varphi(ab) = \varphi(a)\varphi(b).$$

Definition 3.2. A *isomorphism* $\varphi\colon G \to G'$ from a group G to a group G' is a bijective group homomorphism – a bijective map such that $\varphi(ab) = \varphi(a)\varphi(b)$ for all a and b in G. Two groups G and G' are said to be *isomorphic* if there exists an isomorphism φ from G to G'. An isomorphism $\varphi\colon G \to G$ from a group G to itself is called an *automorphism.*

Lemma 3.3. *If $\varphi\colon G \to G'$ is an isomorphism, the inverse map $\varphi^{-1}\colon G' \to G$ is an isomorphism.*

Proof. The inverse of a bijective map is bijective. We will show that for all x and y in G', $\varphi^{-1}(x)\varphi^{-1}(y) = \varphi^{-1}(xy)$. We let $a = \varphi^{-1}(x)$, $b = \varphi^{-1}(y)$, and $c = \varphi^{-1}(xy)$. Since φ is a homomorphism, $\varphi(ab) = \varphi(a)\varphi(b) = xy = \varphi(c)$. Since φ is bijective, $ab = c$. □

Definition 3.4. The groups isomorphic to a given group G form the *isomorphism class* of G.

We consider groups that are isomorphic to be the same. This is because since the law of composition on the group is preserved, we can think of an isomorphism as a relabelling of the group's elements. Therefore, when we speak of classifying groups, what we mean is to describe these isomorphism classes.

Definition 3.5. Two elements x and x' of a group G are *conjugate* if $x' = gxg^{-1}$ for some g in G. The element gxg^{-1} is the *conjugate of x by g. Conjugation by g* is the map φ_g from G to itself defined by $\varphi_g(x) = gxg^{-1}$.

Lemma 3.6. *Conjugation is an automorphism.*

Proof. Let φ_g be a conjugation by g. This is a homomorphism because $\varphi_g(xy) = gxyg^{-1} = gxg^{-1}gyg^{-1} = \varphi_g(x)\varphi_g(y)$. It is bijective because $\varphi_{g^{-1}}$ is the inverse function. □

These lemmas tell us that the conjugate gxg^{-1} behaves in much the same way as the element x itself.

4. Orthogonal Matrices and The Rotation Group SO_3

In this section, we define the rotation group SO_3 and its elements, the orthogonal matrices.

Definition 4.1. A vector X is *orthogonal* to another vector Y if and only if $X^tY = 0$.

Definition 4.2. An *orthonormal basis* $B = (v_1, ..., v_n)$ of $\mathbb{R}^n$ is a basis of orthogonal unit vectors. Another way to say this is that B is an orthonormal basis if

$$(v_i \cdot v_j) = \delta_{ij},$$

where δ_{ij}, the *Kronecker delta*, is the (i, j)-entry of the identity matrix, which is equal to 1 if $i = j$ and to 0 if $i \neq j$.

Definition 4.3. A real $n \times n$ matrix A is *orthogonal* if $A^tA = I$, which is to say, A is invertible and its inverse is A^t.

Lemma 4.4. *An $n \times n$ matrix A is orthogonal if and only if its columns form an orthonormal basis of $\mathbb{R}^n$.*

Proof. Let A_i denote the i-th column of A. Then A_i^t is the i-th row of A^t. The (i, j)-entry of A^tA is $A_i^tA_j$, so $A^tA = I$ if and only if $A_i^tA_j = \delta_{ij}$ for all i and j. □

Proposition 4.5. ***(a)*** *The product of orthogonal matrices is orthogonal, and the inverse of an orthogonal matrix, its transpose, is orthogonal.*
(b) *The determinant of an orthogonal matrix is ± 1.*

Proof. **(a)** Let A and B be orthogonal matrices. Since $A^tA = I$ and $B^tB = I$, we get $(AB)^t(AB) = (B^tA^t)AB = B^t(A^tA)B = B^tB = I$. Therefore, the product of orthogonal matrices is orthogonal. We also have $(A^{-1})^{-1} = A = (A^t)^t = (A^{-1})^t$, so the inverse of an orthogonal matrix is orthogonal.
(b) We know that $\det(A) = \det(A^t)$ for any A, and the determinant of the product is the product of the determinants. Thus, for A orthogonal: $1 = \det(I) = \det(A^tA) = \det(A^t)\det(A) = (\det A)^2$. □

It follows from part ***(a)*** of the proposition that the orthogonal matrices form a subgroup O_n of GL_n, called the *orthogonal group*. The orthogonal matrices with determinant 1 form a subgroup SO_n of O_n, called the *special orthogonal group*.

Definition 4.6. An *orthogonal operator* T on $\mathbb{R}^n$ is a linear operator that preserves the dot product: For every pair X, Y of vectors,

$$(TX \cdot TY) = (X \cdot Y).$$

Proposition 4.7. *A linear operator T on $\mathbb{R}^n$ is orthogonal if and only if its matrix A with respect to the standard basis is an orthogonal matrix.*

Proof. If A is the matrix of T, then $(TX \cdot TY) = (AX)^t(AY) = X^t(A^tA)Y$. The operation is orthogonal if and only if the right side is equal to X^tY for all X and Y. We can write this condition as $X^t(A^tA - I)Y = 0$. The next lemma shows that this is true if and only if $A^tA - I = 0$, and therefore A is orthogonal. □

Lemma 4.8. *Let M be an $n \times n$ matrix. If $X^tMY = 0$ for all column vectors X and Y, then $M = 0$.*

Proof. The product $e_i^t M e_j$ evaluates to the (i,j)-entry of M. For instance,

$$\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = m_{21}.$$

If $e_i^t M e_j = 0$ for all i and j, then $M = 0$. □

Definition 4.9. An orthogonal operator is *orientation-preserving* if its determinant is 1 and *orientation-reversing* if its determinant is -1.

Now we describe the orthogonal 2×2 matrices.

A linear operator T on $\mathbb{R}^2$ is a *reflection* if it has orthogonal eigenvectors v_1 and v_2 with eigenvalues 1 and -1, respectively.

Because it fixes v_1 and changes the sign of the orthogonal vector v_2, such an operator reflects the plane about the one-dimensional subspace spanned by v_1. Reflection about the e_1-axis is given by the matrix

$$S_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{4.10}$$

Theorem 4.11. ***(a)*** *The orthogonal 2×2 matrices with determinant 1 are the matrices*

$$R = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}, \tag{4.12}$$

with $c = \cos\theta$ and $s = \sin\theta$, for some angle θ. The matrix R represents counter-clockwise rotation of the plane $\mathbb{R}^2$ about the origin and through the angle θ.

(b) *The orthogonal 2×2 matrices A with determinant -1 are the matrices*

$$S = \begin{bmatrix} c & s \\ s & -c \end{bmatrix} = RS_0, \tag{4.13}$$

with c and s as above. The matrix S reflects the plane about the one-dimensional subspace of $\mathbb{R}^2$ that makes an angle $\frac{1}{2}$ with the e_1-axis.

Proof. Say that

$$A = \begin{bmatrix} c & * \\ s & * \end{bmatrix}$$

is orthogonal. Then by Lemma 4.4, its columns are unit vectors, so the point $(c,s)^t$ lies on the unit circle, and $c = \cos\theta$ and $s = \sin\theta$, for some angle θ. We inspect the product $P = R^t A$, where R is the matrix (4.12):

$$P = R^t A = \begin{bmatrix} 1 & * \\ 0 & * \end{bmatrix}.$$

Since R^t and A are orthogonal, so is P. By Lemma 4.4, the second column is a unit vector orthogonal to the first one. So

$$P = \begin{bmatrix} 1 & 0 \\ 0 & \pm 1 \end{bmatrix}.$$

Working back, $A = RP$, so $A = R$ if $\det A = 1$ and $A = S = RS_0$ if $\det A = -1$.

We know that R represents a rotation. We will identify the operator defined by the matrix S. The characteristic polynomial of S is $t^2 - 1$, so its eigenvalues are 1 and -1. Let X_1 and X_2 be unit-length eigenvectors with these eigenvalues. Because S is orthogonal,

$$(x_1 \cdot X_2) = (SX_1 \cdot SX_2) = (X_1 \cdot -X_2) = -(X_1 \cdot X_2).$$

It follows that $(X_1 \cdot X_2) = 0$. The eigenvectors are orthogonal. The span of X_1 will be the line of reflection. To determine this line, we write a unit vector X as (c', s'), with $c' = \cos\alpha$ and $s' = \sin\alpha$. Then

$$SX = \begin{bmatrix} cc' + ss' \\ sc' - cs' \end{bmatrix} = \begin{bmatrix} \cos(\theta - \alpha) \\ \sin(\theta - \alpha) \end{bmatrix}.$$

When $\alpha = \frac{1}{2}\theta$, X is an eigenvector with eigenvalue 1, a fixed vector. □

Now we describe the 3×3 rotation matrices.

Definition 4.14. A *rotation* of $\mathbb{R}^3$ about the origin is a linear operator with these properties:

- ρ fixes a unit vector u, called a *pole* of ρ, and
- ρ rotates the two-dimensional subspace W orthogonal to u.

The *axis of rotation* is the line l spanned by u. We also call the identity operator a rotation, though its axis is indeterminate.

If multiplication by a 3×3 matrix R is a rotation of $\mathbb{R}^3$, R is called a *rotation matrix.*

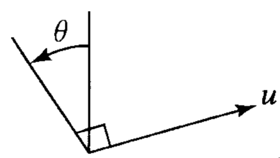


FIGURE 1. A rotation of $\mathbb{R}^3$. [1]

When u is the vector e_1, the set (e_2, e_3) will be a basis for W, and the matrix of ρ will have the form

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix}, \tag{4.15}$$

with $c = \cos\theta$, $s = \sin\theta$ for some angle θ. The matrix M represents a counterclockwise rotation of $\mathbb{R}^3$ about the axis of rotation and through some angle θ.

A rotation that is not the identity is described by the pair (u, θ), called a *spin*, that consists of a pole u and a nonzero angle of rotation θ.

Every rotation ρ different from the identity has two poles, the intersections of the axis of rotation l with the unit sphere in $\mathbb{R}^3$.

Lemma 4.16. *A 3×3 orthogonal matrix M with determinant 1 has an eigenvalue equal to 1.*

Proof. To show that 1 is an eigenvalue, we show that the determinant of the matrix $M - I$ is 0. If B is an $n \times n$ matrix, $\det(-B) = (-1)^n \det B$. We are dealing with 3×3 matrices, so

$$\det(M - I) = -\det(M - I)^t = \det M \det(M - I)^t = \det(M(M^t - I)) = \det(I - M).$$

The relation $\det(M - I) = \det(I - M)$ shows that $\det(M - I) = 0$. □

Theorem 4.17. ***Euler's Theorem*** *The 3×3 rotation matrices are the orthogonal 3×3 matrices with determinant 1, the elements of the special orthogonal group SO_3.*

Proof. Suppose that M represents a rotation ρ with spin (u, α). We form an orthonormal basis B of V by appending to u an orthonormal basis of its orthogonal space W. The matrix M' of ρ with respect to this basis will have the form (4.15), which is orthogonal and has determinant 1. Moreover, $M = PM'P^{-1}$, where the matrix P is equal to $[B]$, the basechange matrix from the standard basis to B. Since its columns are orthonormal, $[B]$ is orthogonal. Therefore, M is also orthogonal, and its determinant is equal to 1.

Conversely, let M be an orthogonal matrix with determinant 1, and let T denote left multiplication by M. Let u be a unit-length eigenvector with eigenvalue 1, which exists by Lemma 4.16, and let W be the two-dimensional space orthogonal to u. Since T is an orthogonal operator that fixes u, it sends W to itself. So W is a T-invariant subspace, and we can restrict the operator to W.

Since T is orthogonal, it preserves lengths, so its restriction to W is orthogonal too. Now, W has dimension 2, and we know by Theorem 4.11 that the orthogonal operators in dimension 2 are the rotations and the reflections. The reflections are operators with determinant -1. If an operator T acts on W as a reflection and fixes the orthogonal vector u, its determinant will be -1 too. Since this is not the case, the restriction of T to W is a rotation. This verifies the second condition of Definition 4.14, and shows that T is a rotation. □

A natural question to ask at this point, after we have discussed SO_3, would be: How do we classify the finite subgroups of SO_3?

This question is answered by our classification theorem, whose complete statement is in the Introduction. A finite subgroup of SO_3 is either a cyclic group, a dihedral group, the tetrahedral group, the octahedral group or the icosahedral group. The proof of this theorem will be presented in the last section of this paper.

5. Isometries and Groups and Subgroups of Isometries of $\mathbb{R}^n$

Now that we have introduced the rotation group SO_3 and the classification of its finite subgroups, we want to know what these subgroups are. In the next section, we will formally define the cyclic group, the dihedral group, the tetrahedral group, the octahedral group, and the icosaheral group as orientation-preserving *isometries.* Therefore, in this section, we define isometries and discuss relevant theorems and propositions.

Definition 5.1. An *isometry* of n-dimensional space $\mathbb{R}^n$ is a distance-preserving map f from $\mathbb{R}^n$ to itself, a map such that, for all u and v in $\mathbb{R}^n$,

$$|f(u) - f(v)| = |u - v|.$$

An isometry will map a figure to a congruent figure.

Proposition 5.2. *Orthogonal operators are isometries.*

Proof. Let $\varphi\colon \mathbb{R}^n \to \mathbb{R}^n$ be an orthogonal operator. Since φ is linear, $\varphi(u)-\varphi(v) = \varphi(u-v)$, so $|\varphi(u) - \varphi(v)| = |\varphi(u-v)|$. Since φ is orthogonal, $|\varphi(u-v)| = \sqrt{(\varphi(u-v)\cdot\varphi(u-v))} = \sqrt{((u-v)\cdot(u-v))} = |u-v|$. □

Theorem 5.3. *The following conditions on a map $\varphi\colon \mathbb{R}^n \to \mathbb{R}^n$ are equivalent:*

(a) *φ is an isometry that fixes the origin: $\varphi(0) = 0$.*

(b) *φ preserves the dot products: $(\varphi(v)\cdot\varphi(w)) = (v\cdot w)$, for all v and w.*

(c) *φ is an orthogonal linear operator.*

Proof. ***(c ⇒ a)*** follows from Proposition 5.2.

(b ⇒ c) Let T be a map that preserves the dot product. Then by Definition 4.6, it will be orthogonal if it is a linear operator. We will use $T(x)$ and Tx interchangeably. To show that T is a linear operator, we must show that $T(u+v) = T(u) + T(v)$ and that $T(cv) = cT(v)$, for all u and v and all scalars c.

We let $w = u + v$. We will show $Tw = Tu + Tv$.

The lemma following this proof will prove that to show $Tw = Tu+Tv$, it suffices to show that

$$(Tw \cdot Tw) = (Tw \cdot (Tu + Tv)) = ((Tu + Tv) \cdot (Tu + Tv)).$$

Expanding the second and the third dot products, we get

$$(Tw \cdot Tw) = (Tw \cdot Tu) + (Tw \cdot Tv) = (Tu \cdot Tu) + 2(Tu \cdot Tv) + (Tv \cdot Tv).$$

By hypothesis, T preserves the dot products. So it suffices to show that

$$(w \cdot w) = (w \cdot u) + (w \cdot v) = (u \cdot u) + 2(u \cdot v) + (v \cdot v). \tag{5.4}$$

Now, whereas $Tw = Tu + Tv$ is to be shown, $w = u + v$ is true by definition. So we may substitute $u + v$ for w. Then (5.4) becomes true.

To show that $T(cv) = cT(v)$, we let $u = cv$. Then we want to show that $Tu = cTv$. The proof is analogous to the one we have just given.

(a ⇒ b) Let T be an isometry that fixes the origin. Then

$$((Tu - Tv) \cdot (Tu - Tv)) = ((u - v) \cdot (u - v)), \tag{5.5}$$

for all u and v in $\mathbb{R}^n$. We substitute $v = 0$. Since $T0 = 0$, $(Tu \cdot Tu) = (u \cdot u)$. Similarly, $(Tv \cdot Tv) = (v \cdot v)$. Now ***(b)*** follows when we expand (5.5) and cancel $(u \cdot u)$ and $(v \cdot v)$ from both sides of the equation. □

Lemma 5.6. *Let x and y be vectors of $\mathbb{R}^n$. If $(x \cdot x) = (x \cdot y) = (y \cdot y)$, then $x = y$.*

Proof. Assume that $(x \cdot x) = (x \cdot y) = (y \cdot y)$. Then

$$((x - y) \cdot (x - y)) = (x \cdot x) - 2(x \cdot y) + (y \cdot y) = 0.$$

We get $|x - y| = 0$. So $x = y$. □

Corollary 5.7. *Every isometry f of $\mathbb{R}^n$ is the composition of an orthogonal linear operator and a translation. More precisely, if f is an isometry and if $f(0) = a$, then $f = t_a\varphi$, where t_a is a translation and φ is an orthogonal operator. This expression for f is unique.*

Proof. Let f be an isometry, let $a = f(0)$, and let $\varphi = t_{-a}f$. Then $t_a\varphi = f$. The corollary amounts to the assertion that φ is an orthogonal operator. Since φ is the composition of the isometries t_{-a} and f, it is an isometry. Also, $\varphi(0) = t_{-a}f(0) = t_{-a}(a) = 0$, so φ fixes the origin. By Theorem 5.3, φ is an orthogonal operator. The expression $f = t_a\varphi$ is unique because, since $\varphi(0) = 0$, we must have $a = f(0)$, and then $\varphi = t_{-a}f$. □

Corollary 5.8. *The set of all isometries of $\mathbb{R}^n$ forms a group, denoted by M_n, with composition of functions as its law of composition.*

Proof. The composition of isometries is an isometry, and the inverse of an isometry is an isometry too, because orthogonal operators and translations are invertible, and if $f = t_a\varphi$, then $f^{-1} = \varphi^{-1}t_a^{-1} = \varphi^{-1}t_{-a}$. This is a composition of isometries. □

Definition 5.9. An *orientation-preserving* (or *orientation-reversing*) *isometry* f is one such that, when it is written in the form $f = t_a\varphi$, the orthogonal operator φ is orientation-preserving (or orientation-reversing).

The isometries that fix the origin are the orthogonal operators, so when coordinates are chosen, the orthogonal group O_n becomes a subgroup of M_n. Specifically, it is the group of isometries of $\mathbb{R}^n$ that fix the origin.

We may also consider the subgroup of M_n that fix an arbitrary point other than the origin. The relationship of this group with the orthogonal group is given in the next proposition.

Proposition 5.10. *Assume that coordinates in $\mathbb{R}^n$ have been chosen, so that the orthogonal group O_n becomes the subgroup of M_n of isometries that fix the origin. Then the group of isometries that fix a point p of $\mathbb{R}^n$ is the conjugate subgroup $t_pO_nt_p^{-1}$.*

Proof. If an isometry m fixes p, then $t_p^{-1}mt_p$ fixes the origin: $t_p^{-1}mt_p0 = t_p^{-1}mp = t_p^{-1}p = 0$. Conversely, if m fixes 0, then $t_pmt_p^{-1}$ fixes p. □

One can visualize the rotation about a point p this way: First translate by t_p to move p to the origin, then rotate about the origin, then translate back to p.

6. Cyclic, Dihedral, Tetrahedral, Octahedral, Icosahedral Groups

In this section, we define the finite subgroups of SO_3. First, we describe the Platonic solids.

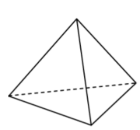
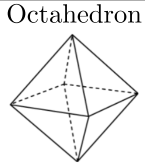
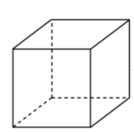
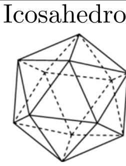
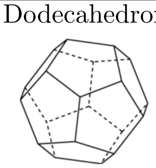
Definition 6.1. A *Platonic solid* is a convex polyhedron satisfying the following conditions:

- Its faces are congruent regular polygons.
- The intersection of two faces is either empty, or a single edge.
- The same number of faces meet at each vertex.

Each Platonic solid can be denoted by a symbol $\{p, q\}$, called the *Schläfli symbol*, where

- p is the number of edges (or, equivalently, vertices) of each face.
- q is the number of faces (or, equivalently, edges) that meet at each vertex.

There are five Platonic solids: the tetrahedron, the octahedron, the cube, the icosahedron, and the dodecahedron.

	Tetrahedron	Octahedron	Cube	Icosahedron	Dodecahedron
					
Vertices	4	6	8	12	20
Edges	6	12	12	30	30
Faces	4	8	6	20	12
$\{p, q\}$	$\{3, 3\}$	$\{3, 4\}$	$\{4, 3\}$	$\{3, 5\}$	$\{5, 3\}$

Definition 6.2. We have the following definitions of the finite subgroups of SO_3:

- The cyclic group C_k is the group of orientation-preserving isometries in $\mathbb{R}^2$ that preserve a regular k-gon.
- The dihedral group D_k is the group of orientation-preserving isometries in $\mathbb{R}^3$ that preserve a regular k-gon.
- The tetrahedral group T is the group of orientation-preserving isometries of $\mathbb{R}^3$ that preserve a tetrahedron.
- The octahedral group O is the group of orientation-preserving isometries of $\mathbb{R}^3$ that preserve an octahedron.
- The icosahedral group I is the group of orientation-preserving isometries of $\mathbb{R}^3$ that preserve an icosahedron.

Any isometry of $\mathbb{R}^n$ that fixes a regular k-gon, a tetrahedron, an octahedron, and an icosahedron has to fix the center of each of these shapes. Without loss of generality, we assume the center of each shape is at the origin, then we can define those groups in the following way:

Definition 6.3. Let α, β, γ, and δ be a regular k-gon, a tetrahedron, an octahedron, and an icosahedron, respectively. Then, we have the following definitions:

- The cyclic group $C_k = \{\rho \in SO_2 \mid \rho\alpha = \alpha\}$.
- The dihedral group $D_k = \{\rho \in SO_3 \mid \rho\alpha = \alpha\}$.
- The tetrahedral group $T = \{\rho \in SO_3 \mid \rho\beta = \beta\}$.
- The octahedral group $O = \{\rho \in SO_3 \mid \rho\gamma = \gamma\}$.
- The icosahedral group $I = \{\rho \in SO_3 \mid \rho\delta = \delta\}$.

Remark 6.4. When we say, for example, $\rho\alpha = \alpha$, we do not mean that ρ fixes every point of α. The only isometry that fixes every point of α is the identity. We mean that in permuting the set α, ρ carries α to itself. Specifically, $\rho\alpha = \{\rho x \mid x \in \alpha\}$.

Remark 6.5. The octahedron and the cube are *dual solids*, meaning that one can be constructed by connecting vertices that are placed at the centers of the other. In other words, they are pairs of solids with faces and vertices interchanged. Therefore, isometries that preserve a tetrahedron also preserve the cube – the octahedral group O also preserves the cube.

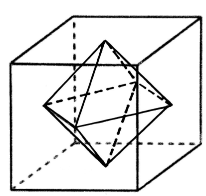


FIGURE 2. Cube frame with octahedron dual inside.

The icosahedron and the dodecahedron are dual solids. Therefore, the icosahedral group I also preserves the dodecahedron.

The tetrahedron is dual with itself.

Remark 6.6. The dihedral groups are usually presented as groups of symmetries of a regular polygon in the plane, where reflections reverse orientation. However, a reflection of a plane can be achieved by a rotation through the angle π in three-dimensional space, and in this way the symmetries of a regular polygon can be realized as rotations in $\mathbb{R}^3$. The dihedral group D_n can be generated by a rotation

x with angle $2\pi/n$ about the e_1-axis and a rotation y with angle π about the e_2-axis. With $c = \cos 2\pi/n$ and $s = \sin 2\pi/n$, the matrices that represent these rotations are

$$x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix}, \text{ and } y = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

7. Group action, Orbits, and Stabilizers

Groups become especially interesting when we think about the ways they "act" on other mathematical objects. This section deals with group actions and relevant concepts.

Definition 7.1. An *action* of a group G on a set S is a rule for combining an element g of G and an element s of S to get another element of S. In other words, it is a map $G \times S \to S$ satisfying the following conditions:

- $1 \cdot s = s$ for all s in S. (Here 1 is the identity of G.)
- *Associative law:* $(gg') \cdot s = g \cdot (g' \cdot s)$, for all g and g' in G and all s in S.

Definition 7.2. Given an action of a group G on a set S, an element s of S will be sent to various other elements by the group action. We collect together those elements, obtaining a subset called the *orbit* O_s of s:

$$O_s = \{s' \in S \mid s' = g \cdot s \text{ for some } g \text{ in } G\}.$$

We may also restrict an action of a group G to a subgroup H. By restriction, an action of G on a set S defines an action of H on S. The H-orbit of an element s will be contained in the G-orbit of s, so a single G-orbit will be partitioned into H-orbits.

Definition 7.3. The *stabilizer* of an element s of S is the set of group elements that leave s fixed. It is a subgroup of G that we often denote by G_s:

$$G_s = \{g \in G \mid g \cdot s = s\}.$$

Proposition 7.4. *Let S be a set on which a group G acts, let s be an element of S, and let H be the stabilizer of s.*

(a) *If a and b are elements of G, then $as = bs$ if and only if $a^{-1}b$ is in H, and this is true if and only if b is in the coset aH.*

(b) *Suppose that $as = s'$. The stabilizer H' of s' is a conjugate subgroup:*

$$H' = aHa^{-1} = \{g \in G \mid g = aha^{-1} \textit{for some } h \textit{ in } H\}.$$

Proof. **(a)** $as = bs$ if and only if $s = a^{-1}bs$.

(b) If g is in aHa^{-1} with h in H, then $gs' = (aha^{-1})(as) = ahs = as = s'$, so g stabilizes s'. This shows that $aHa^{-1} \subset H'$. since $s = a^{-1}s'$, we can reverse the roles of s and s', to conclude that $a^{-1}H'a \subset H$, which implies that $H' \subset aHa^{-1}$. Therefore $H' = aHa^{-1}$. □

Part ***(b)*** of the proposition shows the relationship between the orbit and the stabilizer: Points in the same orbit have conjugate stabilizers. Specifically, when $as = s'$, a group element g fixes s if and only if aga^{-1} fixes s'.

8. The Orbit-Stabilizer Theorem and the Counting Formula

In this section, we prove the Orbit-Stabilizer Theorem, from which we derive the Counting Formula – an important tool used extensively in the proof of the classification theorem.

Theorem 8.1. ***The Orbit-Stabilizer Theorem:*** *Let S be a set on which a group G acts, and let s be an element of S. Let G_s and O_s be the stabilizer and orbit of s, respectively. The correspondence $g(s) \to gG_s$ is a bijection between O_s and the set of left cosets of G_s in G for every element g in G.*

Proof. We will show that the correspondence $g(s) \to gG_s$ is bijective. It is clear that this correspondence is surjective because for all x in X, every gG_x corresponds to one $g(x)$. Let g and g' be elements in G. Assume $gG_x = g'G_x$. Then, there exist h_1 and h_2 in G_x such that $gh_1 = g'h_2$. Setting $h_2h_1^{-1} = h$, we get $g = g'h$ for some h in G_x. We have $h(x) = x$ since H is in G_x. Therefore, $g(x) = g'h(x) = g'(h(x)) = g'(x)$. The correspondence is also injective. □

Proposition 8.2. ***Counting Formula*** *Let S be a finite set on which a group G acts, and let G_s and O_s be the stabilizer and orbit of an element s of S. Then*

$$|G| = |G_s||O_s|, \text{ or} \tag{8.3}$$

$$\text{(order of } G) = \text{(order of stabilizer)(order of orbit).}$$

Proof. This follows directly from (2.8) and Theorem 8.1. □

9. Classifying The Finite Subgroups of SO_3

With the preliminary definitions and ancillary results presented in the previous sections, we are ready to prove the main result of this paper.

Theorem 9.1. *A finite subgroup of SO_3 is one of the following groups:*

- *C_k: the cyclic group of rotation by multiples of $2\pi/k$ about a line, with k arbitrary;*
- *D_k: the dihedral group of symmetries of a regular k-gon, with k arbitrary*
- *T: the tetrahedral group of 12 rotational symmetries of a tetrahedron;*
- *O: the octahedral group of 24 rotational symmetries of a cube or an octahedron;*
- *I: the icosahedral group of 60 rotational symmetries of a dodecahedron or an icosahedron.*

Proof. This proof proceeds in the following stages:

First, we show that a finite subgroup G of SO_3 acts on the set P of poles of G.

Then, we derive a relation between the number of orbits, the orders of the stabilizers, and the order of G, given by (9.4), showing that the action of G on P has either two or three orbits.

From the two-orbit case we get cyclic groups.

In the three-orbit case, we list the possibilities for the orders of the stabilizers. From these we get dihedral groups, the tetrahedral group, the octahedral group, and the icosahedral group.

Let G be a finite subgroup of SO_3, of order $N > 1$. We recall that a pole of an element g of G is a unit vector u that is fixed by g, and a rotation that is not the identity is described by a spin, which is a pair (u, θ), where u is a pole and θ is a nonzero angle of rotation. We will call a pole of an element $g \neq 1$ of G a *pole* of the group. Any rotation of $\mathbb{R}^3$ except the identity has two poles – the intersections of the axis of rotation with the unit sphere S^2. So a pole of G is a point on the 2-sphere that is fixed by a group element $g \neq 1$.

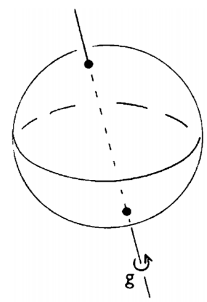


FIGURE 3. The two poles of g. [2]

Let P denote the set of all poles of a finite subgroup G. We will show that G acts on P.

Let p be a pole, say the pole of an element $g \neq 1$ in G, let h be another element of G, and let $q = hp$. We have to show that q is a pole, meaning that q is fixed by some element g' of G other than the identity. By Proposition 7.4, the required element is hgh^{-1}. This element is not equal to 1 because $g \neq 1$, and $hgh^{-1}q = hgp = hp = q$.

The stabilizer G_p of a pole p is the group of all of the rotations about p that are in G. It is a cyclic group, generated by the rotation of smallest positive angle θ. We will denote its order by r_p. Then $\theta = 2\pi/r_p$.

Since p is a pole, the stabilizer G_p contains an element besides 1, so $r_p > 1$. The set of elements of G with pole p is the stabilizer G_p, with the identity element ommited. So there are $r_p - 1$ group elements that have p as pole. Every group element g except one has two poles. Since $|G| = N$, there are $2N - 2$ spins. This gives us the relation

$$\sum_{p \in P} r_p - 1 = 2(N - 1). \tag{9.2}$$

We collect terms to simplify the left side of this equation: Let n_p denote the order of the orbit O_p of p. By Proposition 8.2,

$$r_p n_p = N. \tag{9.3}$$

If two poles p and p' are in the same orbit, their orbits are equal, so $n_p = n_{p'}$, and therefore $r_p = r_{p'}$. We label the various orbits arbitrarily, say as $O_1, O_2, ..., O_k$, and we let $n_i = n_p$ and $r_i = r_p$ for p in O_i, so that $n_i r_i = N$. Since the orbit O_i contains n_i elements, there are n_i terms equal to $r_i - 1$ on the left side of (9.3). We collect those terms together. This gives us the equation

$$\sum_{i=1}^{k} n_i(r_i - 1) = 2N - 2.$$

Dividing both sides by N, we get the formula

$$\sum_i (1 - \frac{1}{r_i}) = 2 - \frac{2}{N}. \tag{9.4}$$

The right side is between 1 and 2, while each term on the left is at least $\frac{1}{2}$. It follows that there can be at most three orbits.

The rest of the classification is made by listing the possibilities:

One orbit: (9.4) becomes $1-\frac{1}{r_1} = 2-\frac{2}{N}$. This is impossible, because $1-\frac{1}{r_1} < 1$, while $2 - \frac{2}{N} \geq 1$.

Two orbits: (9.4) becomes $(1 - \frac{1}{r_1}) + (1 - \frac{1}{r_2}) = 2 - \frac{2}{N}$, that is $\frac{1}{r_1} + \frac{1}{r_2} = \frac{2}{N}$.

Because r_i divides N, this equation holds only when $r_1 = r_2 = N$, and then $n_1 = n_2 = 1$. There are two poles p_1 and p_2, both fixed by every element of the group. So G is the cyclic group C_N of rotations whose axis of rotation is the line l through p_1 and p_2.

Three orbits: (9.4) becomes $(1 - \frac{1}{r_1}) + (1 - \frac{1}{r_2}) + (1 - \frac{1}{r_3}) = 2 - \frac{2}{N}$.

Since $\frac{2}{N}$ is positive, the formula implies that

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} > 1. \tag{9.5}$$

We arrange the r_i in increasing order. Then $r_1 = 2$ because if all r_i were at least 3, the left side of (9.5) would be ≤ 1.

Case 1: $r_1 = r_2 = 2$. The third order $r_3 = k$ can be arbitrary because (9.5) holds for any $k > 0$.

By (9.4), $N = 2k$. By (9.3), $n_i = k, k, 2$.

There is one pair of poles $\{p, p'\}$ making the orbit O_3. Half of the elements of G fix p, and the other half interchange p and p'. So the elements of G are rotations about the line l through p and p', or else they are rotations by π about a line perpendicular to l. The group G is the group of rotations fixing a regular k-gon $\triangle$, the dihedral group D_k. The polygon $\triangle$ lies in the plane perpendicular to l, and the vertices and the centers of faces of $\triangle$ correspond to the remaining poles. The bilateral symmetries of $\triangle$ in $\mathbb{R}^2$ have become rotations through the angle π in $\mathbb{R}^3$.

Case 2: $r_1 = 2$ and $2 < r_2 \leq r_3$. The equation $\frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1$ rules out the possibility that $r_2 \geq 4$. Therefore $r_2 = 3$. The equation $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$ rules out the possibility that $r_3 \geq 6$. The only three possibilities that remain are:

(a) $r_i = 2, 3, 3$.

By (9.4), $N = 12$. By (9.3), $n_i = 6, 4, 4$.

Let V be the orbit O_3 of order 4. Let z be one of the poles in V. Let H be the stabilizer of z. Since $r_3 = 3$, this is a cyclic group, generated by a rotation g about z with angle $2\pi/3$. Let $|a, b|$ denote the spherical distance between points a and b on the unit sphere. Choose an element u in V such that $0 < |u, z| < \pi$. In other words, u is not z or $-z$. Then, $u, g(u)$, and $g^2(u)$ are distinct. Since g preserves distance, they are equidistant from z and lie at the corners of an equilateral triangle. Similarly, if we look at the orbit starting from any other points, say u, we see that $z, g(u)$, and $g^2(u)$ are equidistant from u and lie at the corners of an equilateral triangle. Therefore, they are the vertices of a tetrahedron, which is sent to itself by every rotation in G, and G is the tetrahedral group.

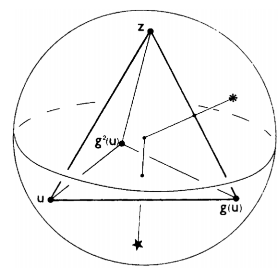


FIGURE 4. The rotational symmetries of a tetrahedron. [2]

(b) $r_i = 2, 3, 4$.

By (9.4), $N = 24$. By (9.3), $n_i = 12, 8, 6$.

Let V be the orbit O_3 of order 6. Let z be one of the poles in V. Let H be the stabilizer of z. Since $r_3 = 4$, this is a cyclic group, generated by a rotation g about z with angle $2\pi/4$. Choose an element u in V such that u is not z or $-z$. Then, $u, g(u), g^2(u)$, and $g^3(u)$ are distinct. Since g preserves distance, they are equidistant from z and lie at the corners of a square. The sixth point in V must be $-z$. Similarly, we know what $-u$ is in the orbit of u which is also V. We know $-u$ cannot be z or $-z$, and it cannot be $g(u)$ or $g^3(u)$ because $|u, g(u)| = |u, g^3(u)| < \pi$, i.e. those are adjacent to u. Therefore, $-u$ must be $g^2(u)$. Looking at the orbit starting from any of the six points, we always see four equidistant points from our starting point that form a square, and the antipode of our starting point. Therefore, they are the vertices of an octahedron, and G is the octahedral group.

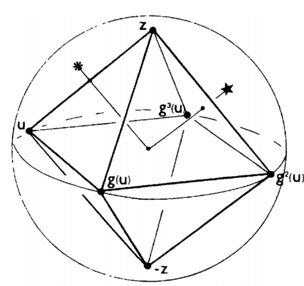


FIGURE 5. The rotational symmetries of an octahedron. [2]

(c) $r_i = 2, 3, 5$.

By (9.4), $N = 60$. By (9.3), $n_i = 30, 20, 12$.

Let V be the orbit O_3 of order 12. Let z be one of the poles in V. Let H be the stabilizer of z. Since $r_3 = 5$, this is a cyclic group, generated by a rotation g about z with angle $2\pi/5$. Choose two elements u and v in V such that $0 < |z, u| < |z, v| < \pi$. We will show that choosing such u and v is possible.

Consider the action of H on V. By Proposition 8.2, the order of any H-orbit divides the order of H, which is 5. Therefore, when we partition the set V of order 12 into H-orbits, we must find at least two H-orbits of order 1. Since g fixes only two points, namely z and $-z$, $-z$ is in V. So $\{z\}$ and $\{-z\}$ are the only H-orbits of order 1, and the remaining ten elements of V are partitioned into two H-orbits X and Y of order 5. Viewing z and $-z$ as the north and south poles respectively, we see that the orbits X and Y must be vertices of antipodal pentagons that are parallel to the equator. This is because V is closed under taking antipodal points, and any two points in X or in Y cannot be antipodal.

Let d be the spherical distance from z to X, and d' the spherical distance from z to Y. We claim that the spherical distance between any two distinct non-antipodal points of V is either d or d'.

Let q and q' be two distinct non-antipodal points of V. Since V is a G-orbit, there exists ρ in G such that $\rho(q) = z$. Now $\rho(q')$ is still distinct from $\rho(q)$ and not the antipode of $\rho(q)$. Therefore, $\rho(q')$ is either in X or Y, i.e. the spherical distance between $\rho(q) = z$ and $\rho(q')$ is either d or d'. Since ρ preserves spherical distance, the distance between q and q' is either d or d'.

Now, consider one of the pentagons, say X. We can find two different non-zero non-antipodal spherical distances here. These must be d and d'. Thus, $d \neq d'$. Without loss of generality, we assume $d < d'$. So we can choose u and v in V satisfying $0 < |z, u| < |z, v| < \pi$.

Now, we see that $u, g(u), g^2(u), g^3(u)$, and $g^4(u)$ are in X. They are all distinct, equidistant from z, and lie at the corners of a regular pentagon. Similarly, $v, g(v), g^2(v), g^3(v)$, and $g^4(v)$ are in Y. They are distinct, equidistant from z (further away from z than u), and lie at the corners of a regular pentagon. The twelfth point of V is $-z$. If we look at the orbit starting from u, we see that $-u$ is in the orbit of u which is also V. As $|u, -u| = \pi$, $-u$ must be one of the points $v, g(v), g^2(v), g^3(v)$, or $g^4(v)$. Relabelling if necessary, we can arrange that $-u = v$, when $-g^j(u) = g^j(v)$, $1 \leq j \leq 4$, as in Figure 6. Looking out from u we see eleven points, and the five points which are closest to u must be equidistant from u. These are $z, g(u), g^2(v), g^3(v)$, and $g^4(u)$. Similarly, looking at the orbit starting from any of the twelve points, we will see the same picture. Therefore, they are the vertices of an icosahedron, and G is the icosahedral group.

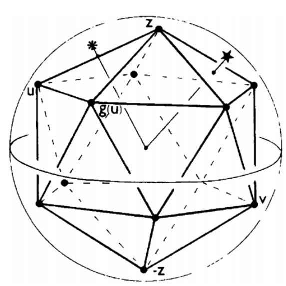


FIGURE 6. The rotational symmetries of an icosahedron. [2]

□

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