IMPOSSIBILITY OF THE CLASSIFICATION OF FINITE SPACES UP TO WEAK HOMOTOPY EQUIVALENCE

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Abstract. The word problem for finitely generated groups states that there exists no algorithmic way to determine whether a loop in the fundamental group of a finite space is nullhomotopic. However, we show that deciding whether such a loop is nullhomotopic is an equivalent problem to determining whether two particular finite spaces are weak equivalent. Thus, the undecidability of the word problem identifies two spaces that cannot be classified up to weak equivalence, disproving the existence of an algorithm. This then implies the impossibility of enumeration of weak equivalences of finite spaces and of simplicial complexes up to homotopy.

Contents

1. Introduction: Homotopy Theory on Finite Spaces

In classic homotopy theory, the category of simplicial sets admits a model structure that is Quillen equivalent to the familiar model structure on the category $\mathscr{T}op$ of topological spaces. That is to say, simplicial sets are useful and widely used combinatoric models of the homotopy theory of topological spaces. Perhaps lesser known however, is that the homotopy theory of $\mathscr{T}op$ can be also modelled by the category $\mathscr{P}os$ of partially-ordered sets (posets) [12]. This can be further extended to finite spaces by imposing upon the poset the structure of a topological space.

Of central interest to this paper is the notion of weak homotopy equivalence. Recall that a weak (homotopy) equivalence is a continuous mapping that induces an isomorphism on all homotopy groups.

Definition 1.1. Two spaces X and Y are weakly (homotopy) equivalent if there is a finite chain of weak homotopy equivalences $f_i : Z_i \to Z_{i+1}$ where

$$X = Z_0 \xrightarrow{f_0} Z_1 \xrightarrow{f_1} \dots \xrightarrow{f_{n-2}} Z_{n-1} \xrightarrow{f_{n-1}} Z_n = Y$$

As will be shown in the following background sections, the class of weak equivalences between finite spaces corresponds exactly to the class of weak equivalences between abstract simplicial complexes. When an abstract simplicial complex is realized however, the resulting polyhedra is a type of CW complex, and therefore this correspondence holds only in the strong case. That is to say that although it is always the case that a homotopy equivalence is a weak equivalence, there exists no distinction when considering maps between CW complexes.

Definition 1.2. We say that two spaces X and Y are *homotopy equivalent* if there are maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \sim id_X$ and $f \circ g \simeq id_Y$.

Since a weak equivalence between finite spaces induces a homotopy equivalence between their underlying polyhedra, the classification theorem for homotopy equivalences between finite spaces [13] does not extend, and further, the following sections prove that no such method for this purpose exists. In particular, the undecidability of the word problem for finitely generated groups implies that there is no algorithm to calculate whether two specific finite spaces are weak equivalent.

2. Background: Relationships Between F-Spaces and Other Categories

We begin by noticing that a finite space is an instance of an Alexandroff Space, wherein arbitary intersections of open sets are open. The Alexandroff topology embeds $\mathscr{P}os$ in $\mathscr{T}op$ as the full subcategory of Alexandroff T_0-spaces, which restricts to an isomorphism between finite posets and finite topological spaces.

Definition 2.1. The *Alexandroff topology* on a poset $\mathscr{P}$ gives the underlying set the topology $U_x = \{y|x \leq y\}$. The *specialization order* on a topological space X gives the underlying set the ordering $x \leq y$ if $U_x \subset U_y$.

Henceforth, we define $\mathscr{F} - spaces$ to be the category of finite T_0 Alexandroff spaces with objects called F-spaces.

One can also naturally define an Alexandroff topology on the poset of faces of an abstract simplicial complex, producing an ordered abstract simplicial complex, or order complex, which will be finite unless otherwise stated.

Definition 2.2. The *order complex functor* $\mathscr{K} : \mathscr{P}os \to \mathscr{O}\mathscr{S}\mathscr{C}$ associates to a finite space X the order complex $\mathscr{K}(X)$ whose simplices are the totally ordered subsets of X. [16]

The face poset of a simplicial complex is defined by the following construction.

Definition 2.3. The *face poset functor* $\mathscr{X} : \mathscr{S}\mathscr{C} \to \mathscr{P}os$ first forgets to $\mathscr{A}\mathscr{S}\mathscr{C}$ then takes the Posets of face simplices as objects and inclusions as orders.

Composing the previous two functors in either order does not return the original poset or simplicial complex, but rather defines the barycentric subdivision. This operation is further explored in section five.

Definition 2.4. Define a functor $i : \mathscr{O}\mathscr{S}\mathscr{C} \hookrightarrow \mathscr{S}set$ sending $K \mapsto K^s$ simply by allowing repetition. Let K_n be the set of n-simplices of an order complex. Then we define

$$K_n^s = \{v_0 \leq \cdots \leq v_n | \{v_0, \ldots, v_n\} \text{(may have repetition) is a simplex in } K\}$$

Face maps: $d_i : K^s_n \to K^s_{n-1}, 0 \leq i \leq n$ is by deleting v_i
Degeneracy maps: $s_i : K^s_n \to K^s_{n+1}, 0 \leq i \leq n$ is by repeating v_i. [16]

Definition 2.5. The *total singular complex functor* is defined as $S_n(X) = Map(\Delta^n, X)$ for a topological space X and geometric simplex $\Delta^n = \{(t_0, \ldots, t_n) | t_i \geq 0, \Sigma t_i = 1\}$. The *geometric realization functor* $|.| : \mathscr{S}set \to \mathscr{T}op$ defines

$$|\mathscr{K}(X)| := \coprod_{i \geq 0} \mathscr{K}(X)_i \times \Delta^i / (k, \sigma_i t) \sim (s_i k, t), (k, \Delta_i s) \sim (d_i k, s),$$

for $k \in \mathscr{K}(X)_n$, $t \in \Delta^{n+1}$, and $s \in \Delta^{n-1}$. [16] We refer to $|\mathscr{K}(X)|$ as a polyhedron.

3. The Word Problem for Finitely Presented Groups

We embark upon a brief interlude outlining the word problem for finitely presented groups.

Definition 3.1. A *presentation* for a group G consists of a set of generators so that every group element can be written as a product of powers of some of these generators and a set of relations among those generators. A group is *finitely presented* if both the set of generators and the set of relations are finite.

The word problem is an algorithmic problem that decides whether two words W, V in the generators of a group are equivalent. By setting $W := WV^{-1}$, this is equivalent to asking whether a specific word W is equivalent to the identity.
Because Tietze transformations give an effective way of changing between two presentations, having a solvable word problem is a group-theoretic property, and thus is independent of any specific presentation. Thus, the discovery of a finitely presented group with an unsolvable word problem by Novikov and Boone [3] answers the word problem in the negative, proving that given an arbitrary finite presentation, there is very little we can know about the group or even about any given element. The following theorem, which we will henceforth refer to as the word problem (for finitely generated groups), formalizes this.

Theorem 3.2. *There is no algorithm that, given a finitely presented group G, decides whether or not a given word is equivalent to the identity in G.* [3]

The following lemma motivates a useful corollary of the word problem.

Lemma 3.3. *Given a finitely presentable group G, there exists a finite space X so that $\pi_1(X) \simeq G$.*

Proof. For any finitely presented group G, there exists a finite CW complex Y so that $\pi_1(Y) \simeq G$ [7, pf. of Cor. 1.28, p.52]. Further, since every CW complex is homotopy equivalent to a simplicial complex (which can be chosen to be of the same dimension as X [7, thm. 2C.5, p.182]), X is homotopy equivalent to a finite simplicial complex K. Then from the previously mentioned result, K is homotopy equivalent to an associated finite space, X. □

Thus, we can equally determine whether an element (loop) of the fundamental group of a finite space is equivalent to the identity of the group (the constant map).

Theorem 3.4. *There is a finite space X with a presentation $G \simeq \pi_1(X)$ such that there is no algorithm that determines whether or not a given loop $f \in G \simeq \pi_1(X)$ is nullhomotopic.*

4. Computing Whether Two General Spaces are Weak Equivalent

Recall that the fundamental group of a space can be represented as based homotopy classes of based maps $[S^1, X]_*$, with the natural basepoint being the homotopy class of the constant map from S^1 to the basepoint $x_0 \in X$. We will eventually prove that if a specific loop $f : S^1 \to X$ in $\pi_1(X)$ of a finite space X is nullhomotopic, then two adjunction spaces in $\mathscr{T}op$ can be shown to be weak equivalent. We begin however, by proving the more general version of the result concerning an arbitrary continuous map $f : X \to Y$ between spaces. We will begin with a discussion of a cofibers, which motivates the eventual construction of these spaces. A homotopy cofiber in $\mathscr{T}op$ is the analogue of the construction of a cokernel $Y/Im(f)$ of a homomorphism $f : X \to Y$ in $\mathscr{A}b$, the category of abelian groups, where a map being zero is replaced by a map being nullhomotopic. As should become clear, in the homotopy cofiber of a map $f : X \to Y$, the image $f(X) \in Y$ is removed up to homotopy.

In the eventual construction of the cofiber, we introduce several adjunction spaces, which are examples of pushouts in $\mathscr{T}op$. Geometrically, if A is a subspace of X and $i : A \hookrightarrow X$ is an inclusion map, the pushout of two spaces X and Y is can be constructed by "gluing" along a A using an attaching map $f : A \to Y$. The following commutative diagram should clarify this notion.

Definition 4.1. Let $A \subset X$ be a subspace. Then the *pushout* of a given map $f : A \to Y$ along the inclusion $i : A \hookrightarrow X$, is defined as the quotient of the coproduct $X \coprod Y$ under the identifications $i(a) \sim f(a)$, written

$$X \cup_f Y = (X \coprod Y)/(a \sim f(a))$$

$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ {\scriptstyle i}\downarrow & & \downarrow \\ X & \longrightarrow & X \cup_f Y \end{array}$$

The *wedge sum* $X \vee Y$ is the pushout $(Y \leftarrow \cdot \to X)$ in $\mathscr{T}op$.

Definition 4.2. The *suspension* of X is the pushout

$$\begin{array}{ccc} X & \xrightarrow{f} & C(X) \\ {\scriptstyle i}\downarrow & & \downarrow \\ C(X) & \longrightarrow & S(X) \end{array}$$

Definition 4.3. The *cone* on X is the pushout

$$\begin{array}{ccc} X & \xrightarrow{f} & * \\ {\scriptstyle i}\downarrow & & \downarrow \\ X \times I & \longrightarrow & C(X) \end{array}$$

where $*$ is a disjoint basepoint that is a one-point space.

The map $X \mapsto C(X)$ induces a functor $C : \mathscr{T}op \to \mathscr{T}op$, which we can apply to a morphism between topological spaces.

Definition 4.4. The *homotopy cofiber* of a continuous map $f : X \to Y$ is the pushout

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle i}\downarrow & & \downarrow \\ C(X) & \longrightarrow & C(f) \end{array}$$

The homotopy cofiber of a topological map has extremely convenient properties. Notice that the composition of the *unpointed homotopy cofiber sequence* $X \xrightarrow{f} Y \xrightarrow{i_0} C(f)$ is nullhomotopic by design. Further, since $C(X)$ is contractible, $q : C(f) \to C(f)/C(X) \cong Y/f(X)$ is a homotopy equivalence, and thus this sequence leads to the use of $C(f)$ as a "homotopy cokernel."

Note that although in the following theorem the unreduced suspension $S(X)$ is based, it should not be confused with the reduced suspension $\Sigma(X)$. We prefer the unreduced suspension because $S(X) = C(X \to *)$ and lose no generality by assuming a basepoint.

Theorem 4.5. *If $f : X \to Y$ is homotopic to the constant map at a point $y_0 \in Y$, then $C(f)$ is homotopy equivalent to the wedge sum $S(X) \vee Y$ where the point of $S(X)$ is identified with y_0.*

Proof. Assume that $f : X \to Y$ is the constant map where $f(x) = y_0$ for every $x \in X$. Then $C(f) = C(X) \cup_f Y = (C(X) \coprod Y)/\sim$ with respect to the equivalence relation $(x, 0) \sim f(x) = y_0$, just by definition. It's clear that when identified, this simplifies to the wedge sum $C(f) = S(X) \vee Y = (C(X) \coprod Y)/\sim$ where $(x, 0) \sim y_0, (x, 1) \sim y_0$ for all $x \in X$. $\square$

Then considering a loop in the fundamental group as a specific map $f : S^1 \to X$ where X is a topological space or $A-$space we obtain the following corollary which the word problem implies in undecidable.

Corollary 4.6. *If $f : S^1 \to X$ is homotopic to the constant loop at a point $x_0 \in X$, then $C(f)$ is homotopy equivalent to the wedge sum $S(S^1) \vee X$ where the point of $S(S^1)$ is identified with x_0.*

5. Computing Whether Two Finite Spaces are Weak Equivalent

The interpretation of the prior theorem requires the derivation of finite analogues of the topological constructions introduced. We define the coproduct in $\mathscr{F}-spaces$ in the following way.

Definition 5.1. For a set $\{X_i | i \in I\}$ of finite spaces, the topology of the union on the disjoint union $\coprod_{i \in I} X_i$ has as open sets the unions of open sets $U_i \subset X_i$.

The cone and suspension can be constructed in a finite setting using the following operation.

Definition 5.2. The ordinal sum $X \odot Y$ of two finite T_0-spaces X and Y is the disjoint union $X \coprod Y$ keeping the given ordering within X and Y and setting $x \leq y$ for every $x \in X$ and $y \in Y$.

Definition 5.3. We now define the following adjunction spaces.

- Define the non-Hausdorff cone as $\mathbb{C}(X) := X \odot *$.
- Define the non- Hausdorff suspension as $\mathbb{S}(X) := X \odot S_0$ [2].
- Give the non-Hausdorff mapping cone $\mathbb{C}(f)$ the underlying set $\mathbb{C}(X) \coprod Y$ and for $x \in \mathbb{C}(X)$ and $y \in Y$, set $x \leq y$ in $\mathbb{C}(f)$ if $f(x) \leq y$ in Y.

Unfortunately, the construction of $\mathbb{C}(f)$ does not apply to the map $f' : S^1 \to X$ but rather requires a map $f : Z \to X$ where both spaces are F-spaces. In order to "replace" S^1 with a finite space, we must therefore show that the map $f' : S^1 \to X$ factors through an F-space Z up to homotopy.

As shown in reference [9], $\mathbb{S}(S^1)$ presents a weak equivalent "finite analogue" of S^1 that presents as a reasonable candidate for Z. This is the four-point circle, ordered $a, b \leq c, d$. As representing complicated loops with only four points presents a rarely feasible challenge, we introduce the barycentric subdivision of F-space as a method of introducing more points to the space without changing its weak homotopy type.

Definition 5.4. The *subdivision* $Sd(X)$ of an F-space X is the composition $\mathscr{X}(\mathscr{K}(X))$. The n^{th} iteration of this process is denoted $Sd^n(X)$ or $Sd^n(K)$.

Intuition suggests that replacing the map $Sd^n(\mathbb{S}(S^1)) \to X$ should be homotopy equivalent to $S^1 \to X$ for a sufficiently large number of subdivisions. This is formalized by the simplicial approximation theorem (Hardie-Vermeulen).

Theorem 5.5. *If $f : |\mathscr{K}(X)| \to |\mathscr{K}(Y)|$ is any continuous map where X is an F-space, and Y is an arbitrary A-space, then there exists an $n \geq 0$ and a map $g : Sd^n X \to Y$ so that $|\mathscr{K}(g)| \sim f$.*

Thus, we have a way of representing continuous maps up to homotopy as maps between F-spaces. In the establishing homotopy classes of maps between F-spaces, we define the conditions under which two F-space maps $f, g : X \to Y$ have homotopic geometric realizations. Notice that this comparison is between associated order complexes, and therefore subdivision is considered to be the composite $\mathscr{K}(\mathscr{X}(i(K)))$ for some simplicial complex K. Further note that since that the terminal object in a poset (if it exists) is the supremum, we can define the sup map for an F-space X, so if $\sigma = \{x_0, x_1, \ldots, x_k\} \in Sd(X)$, then $\sup(\sigma) = x_k$.

Definition 5.6. Two simplicial maps $g : \mathscr{K}(X) \to \mathscr{K}(Y)$ and $f : Sd^n(\mathscr{K}(X)) \to \mathscr{K}(Y)$ are contiguous if for each simplex $\sigma \in Sd^n(\mathscr{K}(X))$, there is a simplex $\tau \in \mathscr{K}(Y)$ such that $f(\sup(n)(\sigma)) \subset \tau$ and $g(\sigma) \subset \tau$.

The following lemma clarifies that homotopic maps between F-spaces are homotopic when realized.

Lemma 5.7. *Suppose f and g are maps from X to Y, where X and Y are F-spaces. If $f \simeq g$, then $|\mathscr{K}(f)| \simeq |\mathscr{K}(g)|$.*

Proof. Since $f \simeq g$, we can show $f(x) = g(x)$ for all but one x', where $f(x') \leq g(x')$. For a simplex $\sigma \in X$, $x' \notin \sigma$ implies $f(\sigma) = g(\sigma)$, which is clearly contained in some simplex of Y. If $x \in \sigma$, then $x = x_i$ for some i and after removing repetitions, $f(\sigma)$ and $g(\sigma)$ are contained in $\{f(x_0), f(x_1), \ldots, f(x'), g(x'), g(x_{i+1}), \ldots, g(x_k)\}$. Because they are simplicially close [?], $\mathscr{K}(f) : \mathscr{K}(X) \to \mathscr{K}(Y)$ and $\mathscr{K}(g) : \mathscr{K}(Sd^n X) \to \mathscr{K}(Y)$ are contiguous. □

The previous facts in tandem with the prior lemma lend to the follow colimit construction.

Theorem 5.8. *If X and Y are F-spaces, there is a natural bijection between $[|\mathscr{K}(X)|, |\mathscr{K}(Y)|]$ and the colimit of the system*

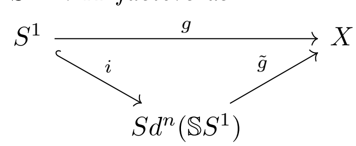
$$[X,Y] \xrightarrow{\sup^*} [SdX, Y] \xrightarrow{\sup^*} [Sd^2X, Y] \xrightarrow{\sup^*} \dots$$

This bijection, which we denote $K : colim_n[Sd^nX, Y] \to [|\mathscr{K}(X)|, |\mathscr{K}(Y)|]$, maps $[f] \in [Sd^iX, Y]$ to $[|\mathscr{K}(f)|] \in [|\mathscr{K}(X)|, |\mathscr{K}(Y)|]$.

Proof. Lemma 5.7 tells us K is well-defined. If $[f] \in [Sd^iX, Y]$ and $[g] \in [Sd^jX, Y]$ are identified in the colimit, then there exists an N so that $f \circ \sup^{(N-i)} \simeq g \circ \sup^{(N-j)}$. From lemma 5.7, $|\mathscr{K}(f \circ \sup^{(N-i)})| \simeq |\mathscr{K}(g \circ \sup^{(N-j)})|$, and since $f \circ \sup^{(Ni)}$ is contiguous to f, $|\mathscr{K}(f)| \simeq |\mathscr{K}(g \circ \sup^{(N-i)})|$. Thus, $|\mathscr{K}(f) \simeq \mathscr{K}(g)$. Each homotopy class of maps has at least one associated A-space approximation so K is surjective. It is also injective because if $|\mathscr{K}(f)| \simeq |\mathscr{K}(g)|$, then f and g are A-space approximations. Then for $[f] \in [Sd^iX, Y]$ and $[g] \in [Sd^jX, Y]$, if $g : Sd^iX \to Y$ and $g' : Sd^jX \to Y$ are both A-space approximations of f, g and g' are contiguous. Thus, f and g are contiguous and $[f] = [g]$ in $colim[Sd^nX, Y]$. □

Since the map K is surjective, only a finite number of subdivisions on the level of F-spaces are necessary to model any continuous map up to homotopy. Therefore we have the desired result.

Theorem 5.9. *Any map $S^1 \xrightarrow{g} X$ factors as*



up to homotopy.

Thus, there exists no loss of generality when considering a map from $\tilde{g} : Sd^n(\mathbb{S}S^1) \to X$, and this implies that $C(\tilde{g})$ is weak equivalent to $X \vee S(Sd^n(\mathbb{S}S^1))$ for some n. We turn our attention toward showing that $C(\tilde{g})$ is weak equivalent to $\mathbb{C}(\tilde{g})$. The following theorem is a useful tool towards this goal, and states that if two spaces are locally weak equivalent, they are so globally as well.

Lemma 5.10. *Let $p : E \to B$ be a map and $\mathscr{O}$ an open cover of B where:*

(1) If $x \in U \cap V \in \mathscr{O}$, then there exists some $W \in \mathscr{O}$ with $x \in W \subset U \cap V$.
(2) For each $U \in \mathscr{O}$, the restriction $p : p^{-1}(U) \to U$ is a weak equivalence.

Then p is a weak homotopy equivalence. [9]

The following relation holds between a classical cone and its finite analogue.

Corollary 5.11. *If $f : X \to Y$ is a map between finite spaces, then the map $\gamma : C(f) \to \mathbb{C}(f)$ is weak equivalence.*

Proof. Take the subspaces X, Y and $X \cup \{*\}$ as the open cover of $\mathbb{C}(f)$. Notice that these subsets satisfy the conditions of Lemma 5.10. Let γ be the map sending the cone point to $\{*\}$, $X \times (0,1)$ to X and Y to itself. If we restrict γ to each of these subspaces, γ becomes a homotopy equivalence and, therefore, a weak homotopy equivalence. Thus, γ is a weak homotopy equivalence. □

In pursuit of extending Corollary 4.6 to F-spaces, we derive one last weak equivalence.

Corollary 5.12. *Let X and Z be finite spaces where Z is weak equivalent to S^1. Then if the basepoint x of X is chosen so that $\{x\}$ is an open set and the basepoint z of Z is chosen so that for a weak equivalence $f : S^1 \to Z$, the image $z = f(x)$ is an open point and the preimage is contractible, the wedge sum $X \vee Z$ using these two basepoints is weak equivalent to $X \vee S^1$.*

Proof. From the given assumptions, we obtain a map $h : X \vee S^1 \to X \vee Z$ restricting to the identity on X and to the weak equivalence f on S^1. We can cover $X \vee Z$ with the open cover $\{X, Z, \{*\}\}$, where $*$ is the basepoint where x and z are identified. Since $h^{-1}(X) \to X$ is the identity, $h^{-1}(Z) \to Z$ is f, and $h^{-1}(*) \to *$ from the assumption that $f^{-1}(*)$ is contractible, by lemma 5.10, h is a weak equivalence. □

It's worth noting that while usually it's more useful for basepoints to be closed (along with a slightly stronger condition) in order to get a "well-pointed space", in this case open points work better.

6. Proof

Theorem 6.1. *There is no algorithm that decides whether two finite spaces are weak equivalent.*

Proof. Let G be a finitely presented group with generators $g_1, \dots g_n$ and W and V are words in the g_i. We can assume $V = id_G$, so we're interested in whether $W = V^{-1} = id_G$.

We know there exists a finite space X so that $\pi_1(X) = G$. Then, the word problem we're trying to solve is equivalent to asking if a specific map $f : S^1 \to X$ is nullhomotopic, and therefore whether $X \vee S(S^1)$ is weak equivalent to $C(f)$. We can replace this map with a map $\tilde{g} : Sd^n(\mathbb{S}S^1) \to X$. Note that $\mathbb{S}^n(X)$ is weak equivalent to $S^n(X)$ for any finite space X[9, p.25]. Then since $X \vee \mathbb{S}(\mathbb{S}S^1)$ is weak equivalent $X \vee Sd^n(\mathbb{S}S^1)$ and $\mathbb{C}(f)$ is weak equivalent to $\mathbb{C}(\tilde{g})$, if $\tilde{g}$ is nullhomotopic, then $\mathbb{C}(\tilde{g})$ is weak equivalent to $X \vee \mathbb{S}(Sd^n\mathbb{S}S^1)$.

And that solves the word problem. Since it's known that the word problem can't be solved by algorithms, this is a contradiction and our hope of telling if two finite spaces were weak equivalent seems impossible. □

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