CUBIC GRAPH SYMMETRIES

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Abstract. We prove a theorem of Frucht, which states that there exists a finite, connected cubic graph with exactly n vertices for any $n > 2$. We use basic properties of graphs, graph symmetries and the automorphism group of graphs in an attempt towards an intuitive proof of the theorem.

Contents

1. Introduction

Graph theory is the study of structures that represent how objects are related, namely how vertices are connected by edges. Since a defining feature of graphs is the relations that they represent, an interesting question to think about is how can a given set of relations be portrayed in a graph?

Many graphs represent the same information. One can see that rescaling or rotating a graph does not change the information represented, nor do any other artistic redrawings so long as the underlying relations are preserved: vertices which were related via an edge before remain related after. One calls the family of graphs that represent the same set of relations an isomorphism class of graphs.

In this paper, we turn our attention to the automorphisms of graphs. These hold particular interest because though the general appearance of a graph and its underlying relations remain unchanged, the vertices are nevertheless reordered, producing a different graph. We will work towards a result Robert Frucht proved in 1939: For any $n > 2$, one may construct a cubic graph with exactly n symmetries.

However, it is not immediately obvious how one would solve this problem. Graph symmetries can be computationally difficult to count, since a graph with vertex set of size m has $m!$ permutations, or relabelings of vertices. Moreover, to verify that any one of these relabelings is a graph symmetry, one would have to check that each vertex is next to the same set of vertices as before for every vertex.

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Instead of analysing each graph permutation as whole, however, we will simplify the problem by looking at invariants of a graph— information that is preserved under graph symmetries. We will see that well-chosen invariants can greatly reduce the pool of permutations that represent possible symmetries. In addition, we will draw graphs that are geometrically symmetric so symmetries are easier to identify. By the end, we will see that thinking of graph symmetries principally through how its invariants are affected is a strong way to understand graph structure, and will lead us to Frucht's construction of the proof.

To start, let us go through groundwork definitions of graph theory and group theory needed for this paper.

2. Graphs

Definition 2.1. A graph $\Gamma = (V, E)$ is a pair of a vertex set V and an edge set E. Elements of V are called vertices of the graph Γ, and elements of E are 2-element subsets of V, which are called edges. If two vertices are connected by an edge, we say that the vertices are *adjacent* or that they are *neighbors*. Moreover, if a vertex is contained in an edge, we say the vertex and edge are *incident*. We denote $V(\Gamma)$ as the vertex set of the graph Γ, and $E(\Gamma)$ as its edge set.

Definition 2.2. The *degree* of a vertex is the number of edges incident to it. If v is a vertex, we write its degree as $\deg(v)$.

Definition 2.3. A *k-regular* graph is a graph whose vertices are all of degree k. If a graph has such properties, we call it a *regular graph*. A *cubic* graph is a 3-regular graph.

Definition 2.4. A graph is *connected* if any vertex may be reached by any other by travelling along edges.

Definition 2.5. A *cycle* on a graph Γ is defined as an ordered sequence of alternating vertices $(v_0, \ldots, v_n)$ where sequential vertices form distinct edges, $(v_{i-1}, v_i) \in E(\Gamma)$ for $1 \leq i \leq n$, and $v_0 = v_n$.

Definition 2.6. A graph is *finite* if it has a finite number of vertices. For this paper, we discuss finite graphs unless otherwise stated.

Definition 2.7. A *graph isomorphism* from Γ to Λ is a bijection $f : V(\Gamma) \to V(\Lambda)$ such that edges are preserved, $\{v, w\} \in E(\Gamma) \Leftrightarrow \{f(v), f(w)\} \in E(\Lambda)$.

Definition 2.8. A graph *symmetry* is an isomorphism from a graph to itself, also known as an *automorphism*. It is a bijection $f : V(\Gamma) \to V(\Gamma)$ such that edges are preserved, $\{v, w\} \in E(\Gamma) \Leftrightarrow \{f(v), f(w)\} \in E(\Gamma)$.

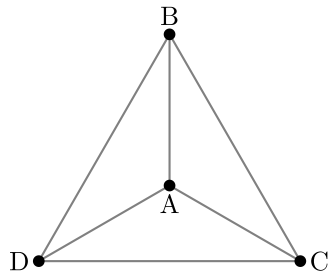


FIGURE 1. A K_4 Graph

As an example of the definitions, the graph above is cubic, finite, connected, and has $4! = 24$ symmetries, corresponding to all permutations of the vertices.

If we collect all the symmetries of a graph, they form a mathematical structure called a group. In the following section, we explore basic definitions of group theory.

3. Groups

Definition 3.1. A *group* is a set G equipped with an operation, $*$, which combines elements of G, a and b, to make another element $a * b$. A group $(G, *)$ must satisfy the group axioms:

(1) Closure: If elements a, b in G are combined with the operation $*$, then $a * b$ is also in G.
(2) Associativity: For all elements a, b, c in G, $(a * b) * c = a * (b * c)$.
(3) Identity Element: There exists an identity in G, e, such that for any a in G, we have $e * a = a * e = a$.
(4) Inverse Element: For any a in G, there is an element in G, denoted a^{-1} such that $a * a^{-1} = a^{-1} * a = e$.

Note that from these axioms, one can show that the identity element and the inverse for each element are unique. From now, we will refer to a group simply as G with the operation implied.

Example 3.2. The set of symmetries of a graph Γ, $\mathrm{Aut}(\Gamma)$, form a group with composition.

Remark 3.3. $\mathrm{Aut}(K_4)$ is the symmetric group S_4 on four letters (see Figure 1).

Definition 3.4. The *order* of a group G is the number of elements in G, denoted $|G|$.

Definition 3.5. A *subgroup* H of a group G, denoted $H \leq G$, is a non-empty subset that forms a group under the same group operation. In other words, H is closed under multiplication and taking inverses.

Remark 3.6. The most apparent subgroups of any group G are the group itself, G and the subset $\{e\}$, also called the *trivial* subgroup. These groups may in fact be identical. A subgroup H is a *proper* subgroup if it is a proper subset.

Definition 3.7. A *left coset* of a subgroup H of a group G is a set obtained by multiplying each element of H by a fixed element $g \in G$. It is denoted $gH = \{gh : h \in H\}$.

Now that we have listed our definitions, let us clarify the main theorem to be shown. We want to show by some construction of graphs that for any $n > 2$, we may provide a graph with n symmetries, and as an added restriction the graph must be cubic. In addition, the graphs should be constructed systematically so the drawing of the graph with arbitrarily many symmetries is obvious. We turn our attention to invariants which provide a useful way to think about symmetries.

4. Type of a Vertex

In this section we introduce the idea of invariants of a graph symmetry– properties that do not change undergoing a graph symmetry. These help serve as markers. Consider the following graph.

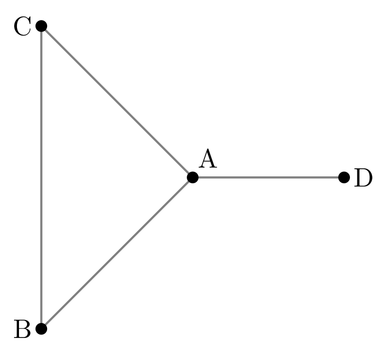


Figure 2

One good way to avoid checking every permutation is to look at the degrees on each vertex. Degrees must be preserved under symmetries since symmetries define a bijection between the neighbours for any vertex, and thus the number of adjacent vertices must be the same before and after symmetry. In the above example, this means that since B and C are the only two vertices of the same degree, they are the only ones that may be permuted by a symmetry. Thus the symmetries of Figure 1 are the identity, and that which swaps vertices B and C.

Already we see that the idea of invariants is quite strong. We have reduced the candidates of symmetries from the $4! = 24$ permutations to 2. We will attach another invariant for cubic graphs, since degrees provide no information in this case. Robert Frucht identified what he called a vertex's *type*– a triple that records the shortest cycles of a vertex depending on the three unique ways in and out of the vertex. As an example, see the types listed for Figure 2.

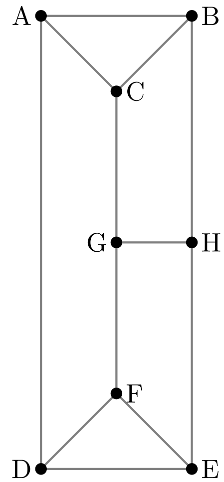


FIGURE 3

We see that for vertex A, the shortest cycle that contains the incident edges AB and AC is the path $ABCA$. The shortest cycle containing AB and AD is $ABHEDA$, and the shortest cycle containing AC and AD is $ACGFDA$. Thus, we may assign the triple $(3, 5, 5)$ to A representing the lengths of the shortest cycles containing two of the three edges to which A is incident, arranging the triple in ascending order. Repeating the process, we get the following types for the other vertices.

- $B, C, E, F ...(3, 4, 5)$
- $A, D ...(3, 5, 5)$
- $G, H ...(4, 4, 5)$

The following lemma proves that a vertex's type is indeed an invariant.

Lemma 4.1. *Let* Γ *be a cubical graph. Then for any graph symmetry* f*, the type of a vertex as defined above remains unchanged. In short,* $\text{type}(v) = \text{type}(f(v))$.

Proof. Let $\Gamma = (V, E)$. Suppose edges e_1, e_2 are incident to v_0. Then define $C(e_1, e_2)$ to be the set of cycles that include e_1 and e_2. Next, define a function $\tilde{f}$ which maps a cycle to the image cycle after f is applied. In other words, if we have a cycle $(v_0, \ldots, v_n)$, then $\tilde{f}((v_0 \ldots v_k)) \coloneqq (f(v_0), \ldots, f(v_k))$. We can denote the mapping as $\tilde{f} : C(e_1, e_2) \to C(f(e_1), f(e_2))$.

Now, $\tilde{f}$ has an inverse which reverses this, as f has an inverse to undo itself. In particular, $\tilde{f}^{-1}((v_0 \ldots v_k)) \coloneqq (f^{-1}(v_0), \ldots, f^{-1}(v_k))$, and it maps $\tilde{f}^{-1} : C(f(e_1), f(e_2)) \to C(e_1, e_2)$. Thus, $\tilde{f}$ is bijective. As a result, the minimum length of the preimage cycle in $C(e_1, e_2)$ is the minimum length of the image cycle in $C(f(e_1), f(e_2))$.

Thus, the type of a vertex (κ, λ, μ) is the same before and after a symmetry is applied. □

Now, we may apply this lemma to identify the symmetries of any cubical graph. As an example, let us examine Figure 2.

Finding the possible symmetries now amounts to case work. Let σ be a symmetry that maps B to B. By the edge-preserving assumption, BA, BC, BH must also be edges in $\sigma(V(B))$. But since the types of A, C and H are distinct from each other, then by the lemma the locations of A, C, H must also be fixed. If C and H are fixed, then G is fixed since G is incident to both. Similarly, since B and C are fixed, A is fixed. The remaining vertices E, F, D are all of distinct types so σ may not permute those three vertices. Thus we have the identity.

Continuing this case work, now suppose σ maps B to C (one of the four vertices B can map to due to shared type). Then since BC is an edge, the only vertex C can map to is B given that A and G are of different types than B. Now, B, C are fixed so their shared neighbor A is also fixed. Thus D is fixed since there are no other openings of its type. Now, BH is an edge, so H must map to G, and thus G to H. And thus, E maps to F and F to E.

The remaining two symmetries are simply reflections of the previous two across the vertical midline of the graph, corresponding to exchanging B for E and B for F. Thus we have a total of four symmetries.

As a side-note, one can explicitly write the symmetry using permutation notation. For the symmetry reflecting Figure 2 across its midline, we have:

$$\sigma = \begin{pmatrix} A & B & C & D & E & F & G & H \\ D & E & F & A & B & C & H & G \end{pmatrix}$$

Now we see how powerful the notion of a type is. From a total of $8! = 40,320$ permutations, we have systematically reduced to find four symmetries. An important observation is that in this previous case, we explicitly wrote out where vertices would go, but deducing symmetries was made easier by choices that naturally constrained us. Neighbours of vertices must remain neighbours, so when choosing the 3 neighbours for any vertex, our previous choices as well as different types constrain us. Using this idea, we can begin a construction of Frucht's proof of a 3-regular graph with arbitrarily many symmetries.

5. Construction

Before we move on to the method of constructing a sequence of graphs with $2, 5, 19$ symmetries, it is worth considering what type of symmetry we want our graphs to have. Assuming that the graphs are drawn so that symmetries are most apparent (in other words, we consider the isomorphisms of graphs that have reasonable looking shapes), then the two most apparent visual symmetries are reflection and rotation.

A big idea behind Frucht's proof is that a systematic construction of graphs that have an arbitrary number of symmetries must proceed via rotations. The formal proof for why reflections are not a suitable candidate is given later, but let us work through an example to see why this could be the case. Say our goal was to draw a graph (not necessarily cubic) with exactly three symmetries that has a reflective symmetry.

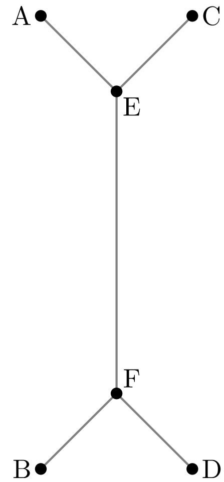


FIGURE 4

The attempt above has four symmetries: the identity, the horizontal reflection, the vertical reflection, and a rotation. But this result seems to have been forced upon us; include only one reflection, and the graph has two symmetries. Include two, and there are four symmetries. It seems that the composition of two reflections make a rotational symmetry, which rounds the total number of symmetries to four. We may thus conjecture that it is impossible to have an odd number of symmetries with reflections. The following proposition and corollary show this is indeed the case.

Proposition 5.1. *(Lagrange's Theorem) Let G be a finite group, and H be a subgroup. Then $|H|$ divides $|G|$.*

Proof. We will show that cosets of H partition G into sets of size $|H|$.

First, we prove that distinct elements of G form disjoint cosets. Suppose by contradiction we had $g' \in G \setminus gH$, but the cosets $g'H$ and gH overlapped. Then there would be an element $k \in g'H \cap gH$. Thus, we have the equation $gh_i = g'h_j$, for elements $h_i, h_j \in H$. Therefore, $g(h_i {h_j}^{-1}) = g'$. By closure, the left hand side is an element of gH, but this contradicts our assumption that g' is not an element of gH. Thus we have $g' \in G \setminus gH$ implies $g'H$ and gH are disjoint (*).

Now, suppose $g_1H, g_2H, \ldots, g_nH$ are disjoint cosets for $n \geq 1$. If an element g_{n+1} does not lie in any of these cosets, then $g_1H, g_2H, \ldots g_nH, g_{n+1}H$ are all disjoint, by (*) applied to each coset. By finiteness, this partitioning exhausts G and we have some fixed total number of cosets.

All that is left to show is the cosets are of the same cardinality. To see this, note that the map $h_i \mapsto gh_i$ from $H \to gH$ is a surjection by definition of gH. In addition, since g is in a group, there is an inverse map $h_j \mapsto g^{-1}h_j$ from $gH \to H$. Thus the map is bijective, and cosets are of equal cardinality.

Hence, the cosets partition G into finitely many sets of size $|H|$, so $|H|$ divides $|G|$. □

Corollary 5.2. *Graphs with reflective symmetries have an even total number of symmetries.*

Proof. First note that reflective symmetries form an order two subgroup of Aut(Γ) made up of the identity permutation and the reflective permutation. One can verify that this by the group axioms– the identity automorphism is contained; the two elements are each others' inverse; compositions of either element result in either element and thus the set is closed under composition, and the mappings are associative since only the reflective automorphism changes a graph.

The order of a reflective symmetry subgroup is 2 since it contains only two elements: the identity, and the automorphism which corresponds to visually reflecting a graph. Thus, by Lagrange's theorem, a graph that admits a drawing with reflective symmetry has an even number of symmetries. □

From this corollary, the sequence of graphs we shall construct for the main theorem will have only rotational symmetries. But before moving to the main proof, it is worth considering what other attributes we want the graph to have.

The simplest cubical graph that has rotational symmetries is the tetrahedron, (Figure 1). However, it has too many symmetries— in fact, since all vertices are of the same type and connected, this graph has $4! = 24$ symmetries. One can picture the symmetries this way: Label a vertex at the center (4 possibilities), then rotate the other 3 labelings around (times 3 possibilities), and reflect those symmetries across lines bisecting the triangle (times 2 possibilities).

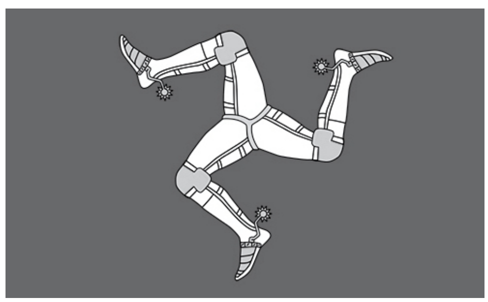
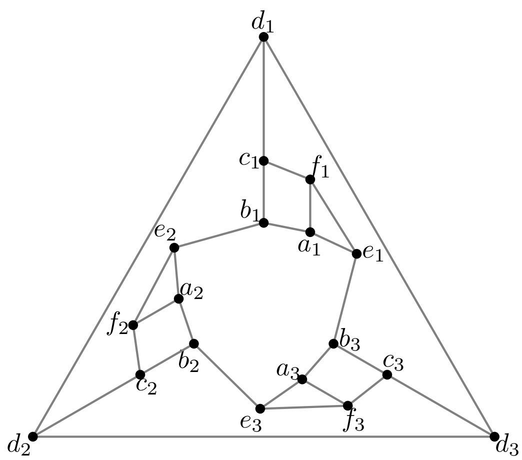


Figure 5

In contrast, a visual image such as the Isle of Man Flag recommended by mentor Peter Huxford has only 3 rotational symmetries. The reason this may appear obvious is because visually, the legs contain a lot of information– the plating on the knees and toes are gold whereas the rest are white, there are lines on the backside of the leg, etc. Thus, it seems clear that rotational symmetries are the only possible symmetries since a foot can clearly not substitute for a knee, nor a calf segment for a thigh; parts must lock into the same parts. Anatomy aside, the takeaway is that in order to have our sequence of graphs build up by one symmetry at a time, the segment which is rotated must contain enough information so that rotations are the only types of symmetries which lock in place. A type is information that can be assigned to each vertex, so we will endeavour to create a graph which has vertices of distinct types.

Theorem 5.3. *For every number of symmetries greater than 2, there exists a 3-regular graph with exactly that many symmetries.*

Proof. Now, the proof amounts to finding a cubical graph with a rotational block composed of vertices of sufficiently diverse types that an additional block increases the symmetry count by only one. There are many ways to do this, but we will replicate the structure Frucht constructed in his original paper.



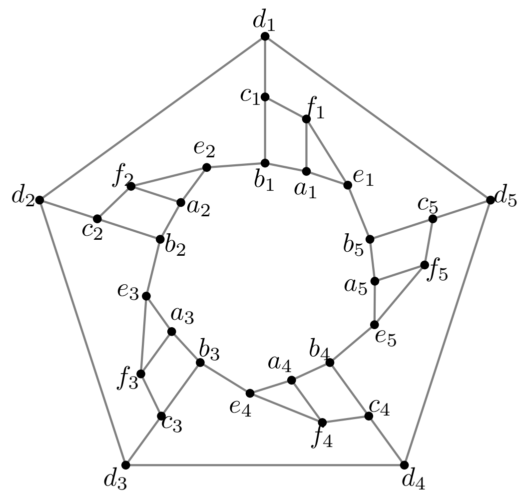


FIGURE 6. For cases $n = 3$ and $n = 5$

From this figure, we have the following types:

- a_i, f_i...(3, 4, 5)
- e_i...(3, 7, 8)
- c_i...(4, 6, 7)
- b_i...(4, 7, 9)
- d_i...(5, 7, 7)

Now, we want to show that the rotational symmetries are the only symmetries present. First, let Γ be the general n-sided drawing of the graph. We claim that any symmetry $\varphi \in \mathrm{Aut}(\Gamma)$ which fixes b_1 is the identity. One can see this by process of elimination: Suppose b_1 is fixed. Since a_1, c_1 and e_2 must remain adjacent and are all of different types, they must also stay in place. Since d_1 and f_1 are of different types but both neighbour c_1 which is fixed, then d_1 and f_1 must also remain fixed. Now, a_1 and f_1 remain in place so e_1 also remains in place. Similarly, e_1 neighbours b_n, and since e_1 differs in type from a_n and c_n, b_n remains fixed. Thus, we have shown that fixing b_1 fixes $a_1, c_1, d_1, e_1, f_1, b_n$.

Since each rotational arm is the same except for indices, we can say that fixing b_i fixes $a_i, c_i, d_i, e_i, f_i, b_{i-1}$, where the index 0 corresponds to n. Thus, a symmetry φ that fixes b_1 also fixes the arm of index 1 and the vertex b_n. Since b_n is fixed, then the arm of index n is fixed and so is the vertex b_{n-1}. Proceeding inductively, we get that a symmetry which fixes b_1 must be the identity.

This fact becomes useful in the following. Let $\phi \in \mathrm{Aut}(\Gamma)$ be a symmetry. By the preceding discussion we know types must exchange for the same types so $\phi(b_1) = b_k$ for some $1 \leq k \leq n$. Let $r \in \mathrm{Aut}(\Gamma)$ be the rotation that sends $b_1 \mapsto b_k$, where it is visually clear why such a rotational symmetry exists. Thus, we have the following.

$$\begin{aligned} \phi(b_1) &= b_k \\ r(b_1) &= b_k \end{aligned}$$

Since r is in the automorphism group, it has an inverse. Thus,

$$\begin{aligned} (r^{-1} \circ \phi)(b_1) &= r^{-1}(\phi(b_1)) \\ &= r^{-1}(b_k) \\ &= b_1 \end{aligned}$$

By the previous fact, $r^{-1} \circ \phi$ is the identity. Hence, $\phi = r$, and it must be a rotational symmetry.

Thus, we have given a sufficient construction to the problem.

□

6. Acknowledgments

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