Mathematics REU 2020: Examples of minimal surfaces

The University of Chicago

July 8

The catenoid

- **Catenoid:**

$$\left\{(x, y, z) \mid x^2 + y^2 = \cosh^2(z)\right\}$$

(Euler 1744, Meusnier 1776).

- **Surface of revolution**: rotation of

$$\{(x, 0, z) \mid x = \cosh(z)\}$$

around the z axis.

- Unique **non-planar** minimal surface of revolution.

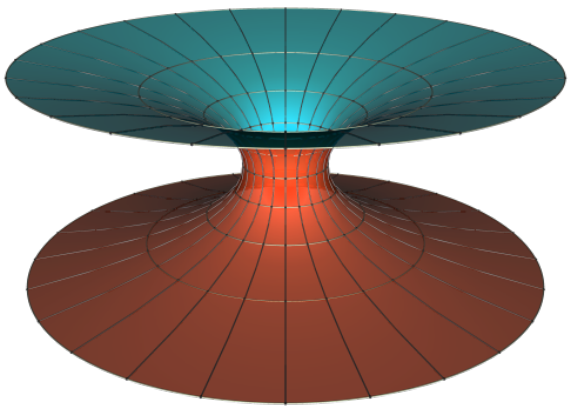

Figure: A catenoid, by Matthias Weber, indiana.edu/∼minimal/archive

The helicoid

- **Helicoid:**

$$\Sigma = \{(t\cos s, t\sin s, s) \mid s, t \in \mathbb{R}\}$$

(Meusnier 1776).

- **Ruled surface**: $\Sigma \cap \{z = s\}$ are lines.
- Unique **non-planar** ruled minimal surface (Catalan 1830).

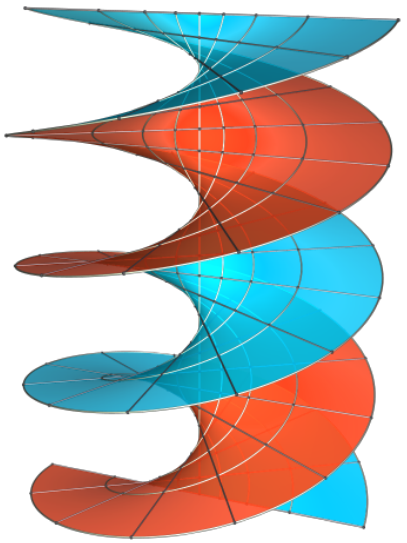

Figure: A helicoid, by Matthias Weber, indiana.edu/∼minimal/archive

Scherk's surface

- **Scherk's surface:**

$$\left\{(x, y, z) \mid z = \log \frac{\cos y}{\cos x}\right\}$$

(Scherk 1835).

- Graph defined over "checkerboard" in $\mathbb{R}^2$.
- **Doubly-periodic** surface: invariant by translations.

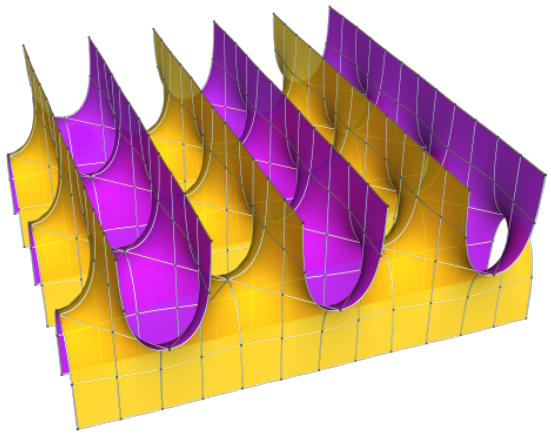

Figure: Scherk's surface, by Matthias Weber, indiana.edu/∼minimal/archive

Scherk's surface

- Related graph (in the *hyperbolic plane* $\mathbb{H}$) was used by H. Rosenberg and P. Collins, in 2007, to construct a harmonic diffeomorphism $f : \mathbb{C} \to \mathbb{H}$

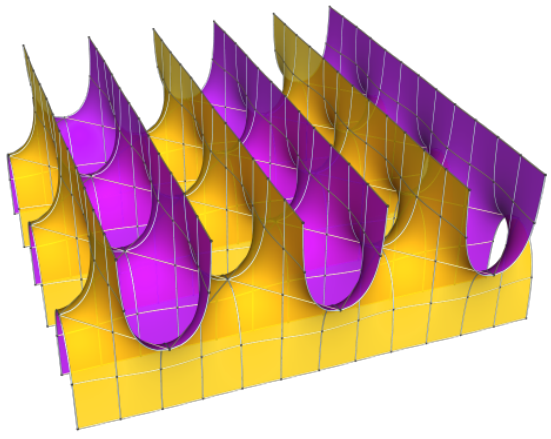

Figure: Scherk's surface, by Matthias Weber, indiana.edu/∼minimal/archive

Scherk's singly-periodic surface

- **Scherk's singly-periodic surface:**

$$\{(x, y, z) \mid \sin z = (\sinh x)(\sinh y)\}$$

(Scherk 1835).

- **Q:** How to make the union of two planes into a minimal surface?
- **Desingularization:** Important technique to construct examples (N. Kapouleas).

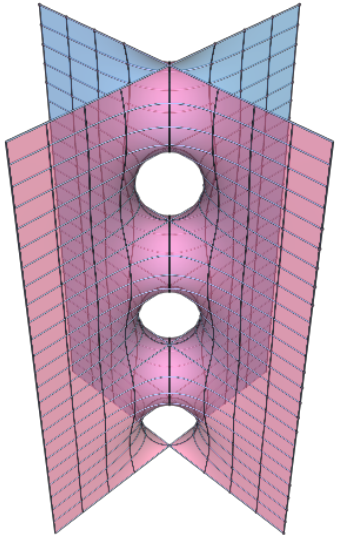

Figure: Scherk's singly-periodic surface, by Matthias Weber, indiana.edu/∼minimal/archive

Some remarks

- Many other examples on Matthias Weber's page: indiana.edu/∼minimal/archive.
- The investigation of minimal surfaces is closely related to the development of **Calculus of Variations** and **Geometric Measure Theory**.
- Even though minimal surfaces and hard to construct, recent progress shows that they are actually **abundant! Variational techniques**: Marques, Neves, Song, Liokumovich, Irie, Zhou.
- Some references:
  (1) **R. Osserman**, *A survey of minimal surfaces*, Dover.
  (2) **T. Colding, W. Minicozzi**, *A Course in Minimal Surfaces*, AMS (Graduate Studies in Mathematics).
  (3) **F. Morgan**, *Geometric Measure Theory: a beginner's guide*, Academic Press.