Localization and completion

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T-local abelian groups

Let T be a set of primes. An abelian group B is said to be T-local if it admits a structure of $\mathbb{Z}_T$-module, necessarily unique, where $\mathbb{Z}_T$ is the subring of $\mathbb{Q}$ of fractions with denominators prime to T. Equivalently, the multiplication map $q\colon B \longrightarrow B$ is an isomorphism for all primes q not in T. We have the following observation.

Lemma

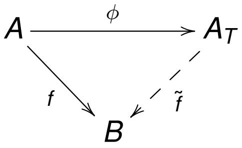
Let

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

be a short exact sequence of abelian groups. If any two of A', A, and A'' are T-local, then so is the third.

Localization of abelian groups

The localization at T of an abelian group A is a map $\phi\colon A \longrightarrow A_T$ to a T-local abelian group A_T which is universal among such maps. This means that any homomorphism $f\colon A \longrightarrow B$, where B is T-local, factors uniquely through ϕ. Thus there is a unique homomorphism $\tilde{f}$ that makes the following diagram commute.



We can define ϕ explicitly by setting $A_T = A \otimes \mathbb{Z}_T$ and letting $\phi(a) = a \otimes 1$. Clearly A is T-local if and only if ϕ is an isomorphism. Since $\mathbb{Z}_T$ is a torsion free abelian group, it is a flat $\mathbb{Z}$-module. We record the following important consequence.

Lemma

Localization is an exact functor from abelian groups to $\mathbb{Z}_T$-modules.

We shall focus on cohomology when defining localizations of spaces. In preparation for this, we describe the homological behavior of localization of abelian groups.

Theorem

The induced map

$$\phi_* \colon H_*(A;\mathbb{Z}_T) \longrightarrow H_*(A_T;\mathbb{Z}_T)$$

is an isomorphism for all abelian groups A. If B is T-local, then the homomorphism

$$\tilde{H}_*(B;\mathbb{Z}) \longrightarrow \tilde{H}_*(B;\mathbb{Z}_T)$$

induced by the homomorphism $\mathbb{Z} \longrightarrow \mathbb{Z}_T$ *is an isomorphism and thus* $\tilde{H}_*(B;\mathbb{Z})$ *is T-local in every degree.*

Proof. For a group G, a commutative ring R, and an R-module M,

$$H_*(G;M) = Tor_*^{R[G]}(R,M) \cong H_*(K(G,1);M),$$

so we can prove this using algebra, topology, or a combination.

Any module over a PID R is the colimit of its finitely generated submodules, and any finitely generated R-module is a finite direct sum of cyclic R-modules. We apply this with $R = \mathbb{Z}$ and $R = \mathbb{Z}_T$. In these cases, the finite cyclic modules can be taken to be of prime power order, using only primes in T in the case of $\mathbb{Z}_T$, and the infinite cyclic modules are isomorphic to $\mathbb{Z}$ or to $\mathbb{Z}_T$.

The localization functor commutes with colimits since it is a left adjoint, and the homology of a colimit of abelian groups is the colimit of their homologies. To see this topologically, for example, one can use the standard simplicial construction of classifying spaces to give a construction of $K(A,1)$'s that commutes with colimits, and one can then use that homology commutes with colimits.

Finite sums of abelian groups are finite products, and

$$K(A,1)\times K(A',1)\simeq K(A\times A',1).$$

By the Künneth theorem, the conclusions of the theorem hold for a finite direct sum if they hold for each of the summands. This reduces the problem to the case of cyclic R-modules.

One can check the cyclic case directly, but one can decrease the needed number of checks by using the Lyndon-Hochschild-Serre (LHS) spectral sequence. That spectral sequence allows one to deduce the result for cyclic groups of prime power order inductively from the result for cyclic groups of prime order.

Thus suppose first that A is cyclic of prime order q. Since $\mathbb{Z}_T$ is T-local, so are both the source and target homology groups. We focus on the reduced homology groups since $K(\pi, 1)$'s are connected and the zeroth homology group with coefficients in R is always R.

If $q = 2$, $K(A, 1) = \mathbb{R}P^\infty$ and if q is odd, $K(A, 1)$ is the analogous lens space S^∞/A. In both cases, we know the integral homology explicitly (e.g., by an exercise in Concise, p. 103). The non-zero reduced homology groups are all cyclic of order q, that is, copies of A. Taking coefficents in $\mathbb{Z}_T$ these groups are zero if $q \notin T$ and A if $q \in T$, and of course $A_T = 0$ if $q \notin T$ and $A_T = A$ if $q \in T$. Thus the conclusions hold in the finite cyclic case.

Finally, consider $A = \mathbb{Z}$, so that $A_T = \mathbb{Z}_T$. The circle S^1 is a $K(\mathbb{Z}, 1)$. Our first example of a localized space is S^1_T, which not surprisingly turns out to be $K(\mathbb{Z}_T, 1)$.

A_T can be constructed as the colimit of copies of $\mathbb{Z}$ together with the maps induced by multiplication by the primes not in T. For example, if we order the primes q_i not in T by size and define r_n inductively by $r_1 = q_1$ and $r_n = r_{n-1}q_1 \cdots q_n = q_1^n \ldots q_n$, then $\mathbb{Z}_T$ is the colimit over n of the maps $r_n \colon \mathbb{Z} \longrightarrow \mathbb{Z}$.

We can realize these maps on $\pi_1(S^1)$ by using the $r_n{}^{\text{th}}$ power map $S^1 \longrightarrow S^1$. Using the telescope construction (Concise, p. 113) to convert these multiplication maps into inclusions and passing to colimits, we obtain a space $K(\mathbb{Z}_T, 1)$; the van Kampen theorem gives that the colimit has fundamental group $\mathbb{Z}_T$, and the higher homotopy groups are zero because a map from S^n into the colimit has image in a finite stage of the telescope, which is equivalent to S^1.

The commutation of homology with colimits gives that the only non-zero reduced integral homology group of $K(\mathbb{Z}_T, 1)$ is its first, which is $\mathbb{Z}_T$ by the Hurewicz theorem.

An alternative proof in this case uses the LHS spectral sequence of the quotient group $\mathbb{Z}_T/\mathbb{Z}$. Groups such as this will play an important role in completion theory and will be discussed later. The spectral sequence has the form

$$E^2_{p,q} = H_p(\mathbb{Z}_T/\mathbb{Z}; H_q(\mathbb{Z}; \mathbb{Z}_T)) \Longrightarrow H_{p+q}(\mathbb{Z}_T; \mathbb{Z}_T).$$

The group $\mathbb{Z}_T/\mathbb{Z}$ is local away from T, hence the terms with $p > 0$ are zero, and the spectral sequence collapses to the edge isomorphism

$$\phi_*\colon H_*(\mathbb{Z}; \mathbb{Z}_T) \longrightarrow H_*(\mathbb{Z}_T; \mathbb{Z}_T).$$

Corollary

The induced map

$$\phi^*\colon H^*(A_T;B) \longrightarrow H^*(A;B)$$

is an isomorphism for all $\mathbb{Z}_T$-modules B.

On H^1, by the representability of cohomology and the topological interpretation of the cohomology of groups, this says that

$$\phi^*\colon [K(A_T,1),K(B,1)] \longrightarrow [K(A,1),K(B,1)]$$

is an isomorphism. On passage to fundamental groups, this recovers the defining universal property of localization. For any groups G and H, passage to fundamental groups induces a bijection

$$[K(G,1),K(H,1)] \cong \mathrm{Hom}(G,H).$$

(This is an exercise in Concise, p. 119). In fact, the classifying space functor from groups to Eilenberg–Mac Lane spaces (Concise, p. 126) gives an inverse bijection to π_1.

Localizations of spaces

We take all spaces to be path connected.

Definition

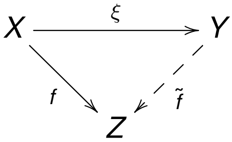
A map $\xi\colon X \longrightarrow Y$ is a $\mathbb{Z}_T$-equivalence if the induced map $\xi_*\colon H_*(X;\mathbb{Z}_T) \longrightarrow H_*(Y;\mathbb{Z}_T)$ is an isomorphism.

It is equivalent that the induced map $\xi^*\colon H^*(Y;B) \longrightarrow H^*(X;B)$ is an isomorphism for all $\mathbb{Z}_T$-modules B.

Definition

A space Z is T-local if $\xi^*\colon [Y,Z] \longrightarrow [X,Z]$ is a bijection for all $\mathbb{Z}_T$-equivalences $\xi\colon X \longrightarrow Y$.

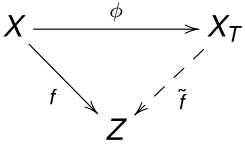
Diagrammatically, this says that for any map $f: X \longrightarrow Z$, there is a map $\tilde{f}$, unique up to homotopy, that makes the following diagram commute up to homotopy.



Definition

A map $\phi: X \longrightarrow X_T$ from X into a T-local space X_T is a localization at T if ϕ is a $\mathbb{Z}_T$-equivalence.

This prescribes a universal property. If $f\colon X \longrightarrow Z$ is any map from X to a T-local space Z, then there is a map $\tilde{f}$, unique up to homotopy, that makes the following diagram commute.



Therefore localizations are unique up to homotopy if they exist.

Remark

On the full subcategory of connected spaces in $Ho\mathscr{T}$ that admit localizations at T, localization is functorial on the homotopy category. For a map $f\colon X \longrightarrow Y$, there is a unique map $f_T\colon X_T \longrightarrow Y_T$ in $Ho\mathscr{T}$ such that $\phi \circ f = f_T \circ \phi$ in $Ho\mathscr{T}$, by the universal property.

When specialized to Eilenberg–Mac Lane spaces $K(A, 1)$, these definitions lead to alternative topological descriptions of T-local abelian groups and of the algebraic localizations of abelian groups at T. The proofs are exercises in the use of the representability of cohomology.

Proposition

An abelian group B is T-local if and only if the space $K(B, 1)$ is T-local.

Proof. If B is T-local and $\xi\colon X \longrightarrow Y$ is a $\mathbb{Z}_T$-equivalence, then

$$\xi^*\colon H^1(Y; B) \longrightarrow H^1(X; B)$$

is an isomorphism. Since this is the map

$$\xi^*\colon [Y, K(B, 1)] \longrightarrow [X, K(B, 1)],$$

$K(B, 1)$ is T-local.

Conversely, if $K(B,1)$ is T-local, then the identity map of $K(B,1)$ is a $\mathbb{Z}_T$-equivalence to a T-local space and is thus a localization at T. However, the map $\phi\colon K(B,1) \longrightarrow K(B_T,1)$ that realizes $\phi\colon B \longrightarrow B_T$ on fundamental groups is also a $\mathbb{Z}_T$-equivalence. Therefore ϕ is also a localization at T. By the uniqueness of localizations, ϕ must be an equivalence and thus $\phi\colon B \longrightarrow B_T$ must be an isomorphism.

Corollary

An abelian group B is T-local if and only if the homomorphism $\xi^*\colon H^*(Y;B) \longrightarrow H^*(X;B)$ induced by any $\mathbb{Z}_T$-equivalence $\xi\colon X \longrightarrow Y$ is an isomorphism.

Proof. If B has the cited cohomological property, then $K(B,1)$ is T-local by the representability of cohomology, hence B is T-local. The converse holds by the definition of a $\mathbb{Z}_T$-equivalence.

Proposition

A homomorphism $\phi\colon A \longrightarrow B$ of abelian groups is an algebraic localization at T if and only if the map, unique up to homotopy,

$$\phi\colon K(A,1) \longrightarrow K(B,1)$$

that realizes ϕ on π_1 is a topological localization at T.

Proof. If $\phi\colon A \longrightarrow B$ is an algebraic localization at T, then $\phi\colon K(A,1) \longrightarrow K(B,1)$ is a $\mathbb{Z}_T$-equivalence. Since $K(B,1)$ is T-local, this proves that ϕ is a topological localization. Conversely, if $\phi\colon K(A,1) \longrightarrow K(B,1)$ is a localization at T and C is any T-local abelian group, then the isomorphism

$$\phi^*\colon H^1(K(B,1),C) \longrightarrow H^1(K(A,1),C)$$

translates by representability and passage to fundamental groups into the isomorphism

$$\phi^*\colon \mathrm{Hom}(B,C) \longrightarrow \mathrm{Hom}(A,C)$$

which expresses the universal property of algebraic localization.

Localization of Eilenberg–Mac Lane spaces $K(A, n)$.

Theorem

If B is a T-local abelian group, then $K(B, n)$ is a T-local space and $\tilde{H}_*(K(B, n); \mathbb{Z})$ is T-local in each degree. For any abelian group A, the map $\phi\colon K(A, n) \longrightarrow K(A_T, n)$, unique up to homotopy, that realizes the localization $\phi\colon A \longrightarrow A_T$ on π_n is a localization at T.

Proof. If $\xi\colon X \longrightarrow Y$ is a $\mathbb{Z}_T$-equivalence, then

$$\xi^*\colon [Y, K(B, n)] \longrightarrow [X, K(B, n)]$$

is the isomorphism induced on the nth cohomology group. Thus $K(B, n)$ is T-local.

From here we proceed by induction. For $n \geq 2$, we may write $\Omega K(A,n) = K(A,n-1)$. Then ϕ induces a map of path space fibrations

$$\begin{array}{ccccc} K(A,n-1) & \longrightarrow & PK(A,n) & \longrightarrow & K(A,n) \\ \downarrow{\scriptstyle \Omega\phi} & & \downarrow & & \downarrow{\scriptstyle \phi} \\ K(A_T,n-1) & \longrightarrow & PK(A_T,n) & \longrightarrow & K(A_T,n). \end{array}$$

This induces a map of Serre spectral sequences converging to the homologies of contractible spaces. The map $\Omega\phi$ on fibers realizes ϕ on π_{n-1}, and we assume inductively that it is a $\mathbb{Z}_T$-equivalence. By the comparison theorem for spectral sequences, it follows that the map ϕ on base spaces is also a $\mathbb{Z}_T$-equivalence.

Finally, taking $A = B$ to be T-local, we prove that $\tilde{H}_*(K(B,n),\mathbb{Z})$ is T-local by inductive comparison of the $\mathbb{Z}$ and $\mathbb{Z}_T$ homology Serre spectral sequences of the displayed path space fibrations. Starting with the case $n = 1$, we find that

$$\tilde{H}_*(K(B,n),\mathbb{Z}) \cong \tilde{H}_*(K(B,n),\mathbb{Z}_T).$$

Localizations of simple spaces

Our construction is based on a special case of the dual Whitehead theorem. Take $\mathscr{A}$ to be the collection of $\mathbb{Z}_T$-modules. Then that result takes the following form, which generalizes the fact that $K(B, n)$ is a T-local space if B is a T-local abelian group.

Theorem

Every $\mathbb{Z}_T$-tower is a T-local space.

Theorem

Every simple space X admits a localization $\phi\colon X \longrightarrow X_T$.

Proof. Let $A_n = \pi_n(X)$. We may assume without loss of generality that X is a Postnikov tower $\lim X_n$ constructed from k-invariants $k^n\colon X_n \longrightarrow K(A_{n+1}, n+2)$. Here $X_0 = *$, and we let $(X_0)_T = *$. Assume inductively that a localization $\phi_n\colon X_n \longrightarrow (X_n)_T$ has been constructed.

Consider the following diagram, in which $K(A_n + 1, n+1) = \Omega K(A_{n+1}, n+2)$.

$$\begin{array}{ccccccc}
K(A_{n+1}, n+1) & \longrightarrow & X_{n+1} & \longrightarrow & X_n & \xrightarrow{k^{n+2}} & K(A_{n+1}, n+2) \\
\downarrow{\scriptstyle \Omega\phi} & & \downarrow{\scriptstyle \phi_{n+1}} & & \downarrow{\scriptstyle \phi_n} & & \downarrow{\scriptstyle \phi} \\
K((A_{n+1})_T, n+1) & \longrightarrow & (X_{n+1})_T & \longrightarrow & (X_n)_T & \xrightarrow[(k^{n+2})_T]{} & K((A_{n+1})_T, n+2)
\end{array}$$

Since ϕ_n is a $\mathbb{Z}_T$-equivalence and $K((A_{n+1})_T, n+2)$ is a $\mathbb{Z}_T$-local space there is a map $(k^{n+2})_T$, unique up to homotopy that makes the right square commute up to homotopy. The space X_{n+1} is the fiber Fk^{n+2}, and we define $(X_{n+1})_T$ to be the fiber $F(k^{n+2})_T$.
There is a map ϕ_{n+1} that makes the middle square commute and the left square commute up to homotopy. Then $(X_{n+1})_T$ is T-local since it is a $\mathbb{Z}_T$-tower. We claim that ϕ_{n+1} induces an isomorphism on homology with coefficients in $\mathbb{Z}_T$ and is thus a localization at T.

Applying the Serre spectral sequence to the displayed fibrations, we obtain spectral sequences

$$E^2_{p,q} \cong H_p(X_n; H_q(K(A_{n+1}, n+1); \mathbb{Z}_T)) \Longrightarrow H_{p+q}(X_{n+1}; \mathbb{Z}_T)$$

$$E^2_{p,q} \cong H_p((X_n)_T; H_q(K((A_{n+1})_T, n+1); \mathbb{Z}_T)) \Longrightarrow H_{p+q}((X_{n+1})_T; \mathbb{Z}_T)$$

and a map between them. The induced map on the homology of fibers is an isomorphism. Since ϕ_n is a localization at T and thus a $\mathbb{Z}_T$-equivalence, the map on E^2 terms is an isomorphism. It follows that ϕ_{n+1} is a $\mathbb{Z}_T$-equivalence, as claimed.

Let $X_T = \lim (X_n)_T$ and $\phi = \lim \phi_n \colon X \longrightarrow X_T$. Then ϕ is a $\mathbb{Z}_T$-equivalence and is thus a localization of X at T.

Completions of abelian groups

We develop completion at T for abelian groups and simple spaces. There is a choice here. It is usual to focus on a single prime p, and there is no loss of information in doing so since completion at T turns out to be the product over $p \in T$ of the completions at p, and similarly for all relevant algebraic invariants. More Concise chose to work with sets of primes, but you may prefer to concentrate on a single fixed prime, as in all other sources.

In contrast to localization, completions of abelian groups can sensibly be defined in different ways, and the most relevant definitions are not standard fare in basic graduate algebra courses, so I'll try to go slow.

p-adic completion

It is usual to define the completion of an abelian group A at a given prime p to be the p-adic completion

$$\hat{A}_p = \lim(A/p^r A),$$

where the limit is defined with respect to the evident quotient homomorphisms $q_r \colon A/p^{r+1}A \longrightarrow A/p^r A$. The limit can be displayed in the short exact sequence

$$0 \longrightarrow \hat{A}_p \longrightarrow \times_r A/p^r A \xrightarrow{\iota^*} \times_r A/p^r A \longrightarrow 0,$$

where ι^* is the difference of the identity map and the map whose r^{th} coordinate is the composite of the projection to $A/p^{r+1}A$ and q_r. (Since the q_r are epimorphisms, ι^* is an epimorphism; see Concise p. 147.)

This definition will not fully serve our purposes since p-adic completion is neither left nor right exact in general, and exactness properties are essential to connect up with topology. The Artin-Rees lemma implies the following result.

Lemma

When restricted to finitely generated abelian groups, the p-adic completion functor is exact.

When $A = \mathbb{Z}$, we write $\mathbb{Z}_p$ instead of $\hat{\mathbb{Z}}_p$ for the ring of p-adic integers; we abbreviate $\mathbb{Z}/n\mathbb{Z}$ to $\mathbb{Z}/n$.

The p-adic completion functor takes values in the category of $\mathbb{Z}_p$-modules. The action is given by the evident natural maps

$$\lim \mathbb{Z}/p^r \otimes \lim A/p^r A \longrightarrow \lim(\mathbb{Z}/p^r \otimes A/p^r A) \cong \lim A/p^r A.$$

When A is finitely generated, p-adic completion is given by the map $\psi\colon A \longrightarrow A \otimes \mathbb{Z}_p$ specified by $\psi(a) = a \otimes 1$, this again being a consequence of the Artin-Rees lemma. In this case, the alternative notion of completion at p that we shall give shortly agrees with p-adic completion. Since $\mathbb{Z}_p$ is torsion free, it is a flat $\mathbb{Z}$-module, in line with the previous lemma.

Even if we restrict to finitely generated abelian groups, we notice one key point of difference between localization and completion. While a homomorphism of abelian groups between p-local groups is necessarily a map of $\mathbb{Z}_{(p)}$-modules, a homomorphism of abelian groups between p-adically complete abelian groups need not be a map of $\mathbb{Z}_p$-modules.

Derived functors of p-adic completion

To overcome the lack of exactness of p-adic completion in general, we consider the left derived functors of the p-adic completion functor. Left derived functors are usually defined only for right exact functors, in which case the 0^{th} left derived functor agrees with the given functor.

However, the definition still makes sense for functors that are not right exact. We shall not go into the general theory of derived functors since, for our present purposes, the abstract theory is less useful than a concrete description of the specific example at hand.

The left derived functors of p-adic completion are given on an abelian group A by first taking a free resolution

$$0 \longrightarrow F' \longrightarrow F \longrightarrow A \longrightarrow 0$$

of A, then applying p-adic completion, and finally taking the homology of the resulting length two chain complex $\hat{F}'_p \longrightarrow \hat{F}_p$.

Thus the left derived functors of p-adic completion are defined by

$$L_0(A) = \operatorname{coker}(\hat{F}'_p \longrightarrow \hat{F}_p) \quad \text{and} \quad L_1(A) = \ker(\hat{F}'_p \longrightarrow \hat{F}_p).$$

These groups are independent of the choice of resolution, as one checks by comparing resolutions, and they are functorial in A. The higher left derived functors are zero. We have a map of exact sequences

$$\begin{array}{ccccccccccc} & & 0 & \longrightarrow & F' & \longrightarrow & F & \longrightarrow & A & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L_1A & \longrightarrow & \hat{F}'_p & \longrightarrow & \hat{F}_p & \longrightarrow & L_0A & \longrightarrow & 0. \end{array}$$

It induces a natural map

$$\phi\colon A \longrightarrow L_0A.$$

Since kernels and cokernels of maps of $\mathbb{Z}_p$-modules are $\mathbb{Z}_p$-modules, since a free abelian group is its own free resolution, and since p-adic completion is exact when restricted to finitely generated abelian groups, we have the following observations.

Proposition

The functors L_0 and L_1 take values in $\mathbb{Z}_p$-modules. If A is either a finitely generated abelian group or a free abelian group, then $L_0 A = \hat{A}_p$, $L_1 A = 0$, and $\phi\colon A \longrightarrow L_0 A$ coincides with p-adic completion.

We usually work at a fixed prime, but we write L_0^p and L_1^p when we need to record the dependence of the functors L_i on the chosen prime p.

Definition

Fix a prime p. We say that the completion of A at p is defined if $L_1 A = 0$, and we then define the completion of A at p to be the homomorphism $\phi\colon A \longrightarrow L_0 A$. We say that A is p-complete if $\phi\colon A \longrightarrow L_0 A$ is an isomorphism. As we shall see shortly, if A is p-complete, then $L_1 A = 0$.

Example

We have seen that finitely generated and free abelian groups are completable and their completions at p coincide with their p-adic completions.

Example

$\mathbb{Z}_p \otimes \mathbb{Z}_p$ *and* $\mathbb{Z}/p^\infty$ *(see below) are* $\mathbb{Z}_p$*-modules that are* not *p-complete.*

The essential exactness property of our derived functors, which is proven in the same way as the long exact sequences for Tor and Ext, reads as follows.

Lemma

For a short exact sequence of abelian groups

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0,$$

there is a six term exact sequence of $\mathbb{Z}_p$-modules

$$0 \longrightarrow L_1A' \longrightarrow L_1A \longrightarrow L_1A'' \longrightarrow L_0A' \longrightarrow L_0A \longrightarrow L_0A'' \longrightarrow 0.$$

This sequence is natural with respect to maps of short exact sequences.

Reinterpretation in terms of Hom and Ext

These derived functors give a reasonable replacement for p-adic completion, but they may seem unfamiliar and difficult to compute. However, they can be replaced by isomorphic functors that are more familiar and sometimes more easily computed.

Define $\mathbb{Z}/p^\infty$ to be the colimit of the groups $\mathbb{Z}/p^r$ with respect to the homomorphisms $p\colon \mathbb{Z}/p^r \longrightarrow \mathbb{Z}/p^{r+1}$ given by multiplication by p.

Exercize

Verify that $\mathbb{Z}/p^\infty \cong \mathbb{Z}[p^{-1}]/\mathbb{Z}$.

Notation

For a prime p and an abelian group A, define $\mathbb{E}_pA$ *to be* $\mathrm{Ext}(\mathbb{Z}/p^\infty, A)$ *and define* $\mathbb{H}_pA$ *to be* $\mathrm{Hom}(\mathbb{Z}/p^\infty, A)$.

$\mathbb{E}_p A = 0$ if A is a divisible or equivalently injective abelian group!

Write $\mathrm{Hom}(\mathbb{Z}/p^r, A) = A_r$ for brevity. We may identify A_r with the subgroup of elements of A that are annihilated by p^r.

Proposition

There is a natural isomorphism

$$\mathbb{H}_p A \cong \lim A_r,$$

where the limit is taken with respect to the maps $p\colon A_{r+1} \longrightarrow A_r$*, and there is a natural short exact sequence*

$$0 \longrightarrow \lim{}^1 A_r \longrightarrow \mathbb{E}_p A \xrightarrow{\xi} \hat{A}_p \longrightarrow 0.$$

Proof.

The exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{p^r} \mathbb{Z} \longrightarrow \mathbb{Z}/p^r \longrightarrow 0$$

displays a free resolution of $\mathbb{Z}/p^r$, and the sum of these is a resolution of $\oplus_r \mathbb{Z}/p^r$. Since $\mathrm{Hom}(\mathbb{Z}, A) \cong A$, we may identify $\mathrm{Ext}(\mathbb{Z}/p^r, A)$ with $A/p^r A$. Moreover, we have maps of free resolutions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{p^{r+1}} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/p^{r+1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle p} & & \| & & \downarrow{\scriptstyle q} & & \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow[p^r]{} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/p^r & \longrightarrow & 0. \end{array}$$

The colimit $\mathbb{Z}/p^\infty$ fits into a short exact sequence

$$0 \longrightarrow \oplus_r \mathbb{Z}/p^r \xrightarrow{\iota} \oplus_r \mathbb{Z}/p^r \longrightarrow \mathbb{Z}/p^\infty \longrightarrow 0.$$

Writing 1_r for the image of 1 in $\mathbb{Z}/p^r$, $\iota(1_r) = 1_r - p1_{r+1}$. The resulting six term exact sequence of groups $\mathrm{Ext}(-, A)$ takes the form

$$0 \longrightarrow \mathbb{H}_p A \longrightarrow \times_r A_r \xrightarrow{\iota^*} \times_r A_r \longrightarrow \mathbb{E}_p A \longrightarrow \times_r A/p^r A \xrightarrow{\iota^*} \times_r A/p^r A \longrightarrow 0$$

The first map ι^* is the difference of the identity map and the map whose r^{th} coordinate is $p\colon A_{r+1} \longrightarrow A_r$. Its kernel and cokernel are $\lim A_r$ and $\lim^1 A_r$. The second map ι^* is the analogous map whose kernel and cokernel are $\hat{A}_p$ and 0.

Example

Any torsion abelian group A with all torsion prime to p satisfies $\mathbb{H}_p A = 0$ *and* $\mathbb{E}_p A = 0$.

Example

$\mathbb{H}_p(\mathbb{Z}/p^\infty)$ *is a ring under composition, and by inspection it is isomorphic to the ring* $\mathbb{Z}_p$.

$\mathbb{E}_p(\mathbb{Z}/p^\infty) = 0$ *since it is a quotient of* $\mathrm{Ext}(\mathbb{Z}/p^\infty, \mathbb{Z}[p^{-1}]) = 0$.

The following observation shows that $\mathbb{E}_p A$ is isomorphic to $\hat{A}_p$ in the situations most often encountered in algebraic topology.

Corollary

If the p-torsion of A is of bounded order, then $\mathbb{H}_p A = 0$ *and* $\xi\colon \mathbb{E}_p A \longrightarrow \hat{A}_p$ *is an isomorphism.*

Example

If $A = \oplus_{n\geq 1}\mathbb{Z}/p^n$, *then* $\mathbb{E}_p A$ *is* not *a torsion group, the map* $\xi\colon \mathbb{E}_p A \longrightarrow \hat{A}_p$ *is* not *an isomorphism, and A is* not *p-complete.*

Proposition

There are natural isomorphisms

$$L_0(A) \cong \mathbb{E}_p A \quad \textit{and} \quad L_1(A) \cong \mathbb{H}_p A$$

Moreover $\phi\colon A \longrightarrow L_0 A$ *coincides with the connecting homomorphism*

$$\delta\colon A \cong \mathrm{Hom}(Z, A) \longrightarrow \mathrm{Ext}(\mathbb{Z}/p^\infty, A) = \mathbb{E}_p A.$$

associated with the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}[p^{-1}] \longrightarrow \mathbb{Z}/p^\infty \longrightarrow 0.$$

Proof.

Let

$$0 \longrightarrow F' \longrightarrow F \longrightarrow A \longrightarrow 0$$

be a free resolution of A. From this sequence we obtained the exact sequence

$$0 \longrightarrow L_1 A \longrightarrow \hat{F'}_p \longrightarrow \hat{F}_p \longrightarrow L_0 A \longrightarrow 0.$$

Since $\mathbb{H}_p F = 0$, we also have the exact sequence of Ext groups

$$0 \longrightarrow \mathbb{H}_p A \longrightarrow \mathbb{E}_p F' \longrightarrow \mathbb{E}_p F \longrightarrow \mathbb{E}_p A \longrightarrow 0.$$

Since $\mathbb{E}_p F \cong \hat{F}_p$ for free abelian groups F, we may identify these two exact sequences. The last statement follows by a comparison of exact sequences.

Proposition

Let A be an abelian group and let B be any of $\hat{A}_p$, $\mathbb{H}_pA$, and $\mathbb{E}_pA$. Then $\mathbb{H}_pB = 0$ and $\delta\colon B \longrightarrow \mathbb{E}_pB$ is an isomorphism. Equivalently, $L_1B = 0$ and $\phi\colon B \longrightarrow L_0B$ is an isomorphism. Therefore, if $\phi\colon A \longrightarrow L_0A$ is an isomorphism, then $L_1A \cong L_1L_0A = 0$.

Proof. Using the six term sequence of groups $\mathrm{Ext}(-, B)$ associated to the short exact sequence $0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}[p^{-1}] \longrightarrow \mathbb{Z}/p^\infty \longrightarrow 0$, we see that $\mathbb{H}_pB = 0$ and $\delta\colon B \longrightarrow \mathbb{E}_pB$ is an isomorphism if and only if

$$\mathrm{Hom}(\mathbb{Z}[p^{-1}], B) = 0 \quad \text{and} \quad \mathrm{Ext}(\mathbb{Z}[p^{-1}], B) = 0. \tag{1}$$

Certainly (1) holds if $p^r B = 0$ for any r, so it holds for all A_r and $A/p^r A$. If (1) holds for groups B_i, then it holds for their product $\times_i B_i$. Suppose given a short exact sequence $0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$. If (1) holds for B, then

$$\mathrm{Hom}(\mathbb{Z}[p^{-1}], B') = 0$$

$$\mathrm{Hom}(\mathbb{Z}[p^{-1}], B'') \cong \mathrm{Ext}(\mathbb{Z}[p^{-1}], B')$$

$$\mathrm{Ext}(\mathbb{Z}[p^{-1}], B'') = 0.$$

If (1) holds for B'', then it holds for B' if and only if it holds for B. Now the short exact sequence defining $\hat{A}_p$ implies that (1) holds for $B = \hat{A}_p$, and the four short exact sequences into which the six term exact sequence for Hom and Ext breaks up by use of kernels and cokernels implies that (1) holds for $B = \mathbb{E}_p A$ and $B = \mathbb{H}_p A$.

As promised, this shows that p-complete groups are completable at p.

The generalization to sets of primes

Definition

Fix a nonempty set of primes T and recall that $\mathbb{Z}[T^{-1}]$ is obtained by inverting the primes in T, whereas $\mathbb{Z}_T$ is obtained by inverting the primes not in T. Define

$$\mathbb{H}_T A = \mathrm{Hom}(\mathbb{Z}[T^{-1}]/\mathbb{Z}, A) \quad \text{and} \quad \mathbb{E}_T A = \mathrm{Ext}(\mathbb{Z}[T^{-1}]/\mathbb{Z}, A).$$

We say that the completion of A at T is defined if $\mathbb{H}_T A = 0$, and we then define the completion of A at T to be the connecting homomorphism

$$\phi\colon A \cong \mathrm{Hom}(\mathbb{Z}, A) \longrightarrow \mathbb{E}_T A$$

that arises from the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}[T^{-1}] \longrightarrow \mathbb{Z}[T^{-1}]/\mathbb{Z} \longrightarrow 0.$$

We say that B is T-complete if ϕ is an isomorphism.

Remark

The short exact sequence above gives rise to an exact sequence

$$0 \longrightarrow \mathbb{H}_T A \longrightarrow \mathrm{Hom}(\mathbb{Z}[T^{-1}], A) \longrightarrow A \longrightarrow \mathbb{E}_T A \longrightarrow \mathrm{Ext}(\mathbb{Z}[T^{-1}], A) \longrightarrow 0;$$

A is completable at T if $\mathrm{Hom}(\mathbb{Z}[T^{-1}], A) = 0$ *and B is T-complete if and only if*

$$\mathrm{Hom}(\mathbb{Z}[T^{-1}], B) = 0 \quad \textit{and} \quad \mathrm{Ext}(\mathbb{Z}[T^{-1}], B) = 0. \tag{2}$$

The inclusion $\mathbb{Z}[T^{-1}] \longrightarrow \mathbb{Q}$ induces an isomorphism

$$\mathbb{Z}[T^{-1}]/\mathbb{Z} \longrightarrow \mathbb{Q}/\mathbb{Z}_T,$$

and $\mathbb{Q}/\mathbb{Z}_T$ is isomorphic to the T-torsion subgroup of $\mathbb{Q}/\mathbb{Z}$. In turn, $\mathbb{Q}/\mathbb{Z}$ is isomorphic to the direct sum over all primes p of the groups $\mathbb{Z}/p^\infty$. These statements are well-known in the theory of infinite abelian groups, and I invite you to check them for yourself. It follows that the definitions above generalize those given when T is a single prime. Indeed, we have the chains of isomorphisms

$$\begin{aligned}
\mathbb{H}_T A &= \operatorname{Hom}(\mathbb{Z}[T^{-1}]/\mathbb{Z}, A) \\
&\cong \operatorname{Hom}(\oplus_{p\in T}\mathbb{Z}[p^{-1}]/\mathbb{Z}, A) \\
&\cong \times_{p\in T} \operatorname{Hom}(\mathbb{Z}[p^{-1}]/\mathbb{Z}, A) \\
&= \times_{p\in T}\mathbb{H}_p A
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}_T A &= \operatorname{Ext}^1(\mathbb{Z}[T^{-1}]/\mathbb{Z}, A) \\
&\cong \operatorname{Ext}^1(\oplus_{p\in T}\mathbb{Z}[p^{-1}]/\mathbb{Z}, A) \\
&\cong \times_{p\in T} \operatorname{Ext}^1(\mathbb{Z}[p^{-1}]/\mathbb{Z}, A) \\
&= \times_{p\in T}\mathbb{E}_p A.
\end{aligned}$$

Analogously, we define

$$\hat{A}_T = \times_{p\in T}\hat{A}_p.$$

All of these are modules over the ring $\hat{\mathbb{Z}}_T = \times_{p\in T}\mathbb{Z}_p$.

The results we have proven for a single prime p carry over to sets of primes.

Proposition

If A is a torsion free or finitely generated $\mathbb{Z}_S$-module for any set of primes $S \supset T$, then $\mathbb{H}_T A = 0$ and the canonical map $\mathbb{E}_T A \longrightarrow \hat{A}_T$ is an isomorphism; its inverse can be identified with the map $A \otimes \hat{\mathbb{Z}}_T \longrightarrow \mathbb{E}_T A$ induced by the action of $\hat{\mathbb{Z}}_T$ on $\mathbb{E}_T A$. In particular, $\mathbb{E}_T$ restricts to an exact functor from finitely generated $\mathbb{Z}_T$-modules to $\hat{\mathbb{Z}}_T$-modules.

Proposition

For any abelian group A, the groups $\hat{A}_T$, $\mathbb{H}_T A$, and $\mathbb{E}_T A$ are T-complete.

Completions of spaces

We take all spaces to be path connected. Recall that $\mathbb{F}_T = \times_{p \in T} \mathbb{F}_p$. We have the following three basic definitions, which are written in precise parallel to the definitions in the case of localization.

Definition

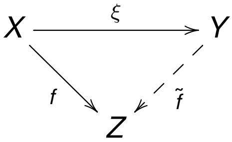
A map $\xi \colon X \longrightarrow Y$ is said to be an $\mathbb{F}_T$-equivalence if $\xi_* \colon H_*(X; \mathbb{F}_p) \longrightarrow H_*(X; \mathbb{F}_p)$ is an isomorphism for all primes $p \in T$.

It is equivalent that the induced map $\xi^* \colon H^*(Y; B) \longrightarrow H^*(X; B)$ is an isomorphism for all $\mathbb{F}_T$-modules B.

Definition

A space Z is T-complete if $\xi^* \colon [Y, Z] \longrightarrow [X, Z]$ is a bijection for all $\mathbb{F}_T$-equivalences $\xi \colon X \longrightarrow Y$.

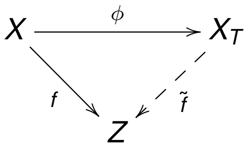
Diagrammatically, this says that for any map $f\colon X \longrightarrow Z$, there is a map $\tilde{f}$, unique up to homotopy, that makes the following diagram commute up to homotopy.



Definition

A map $\phi\colon X \longrightarrow \hat{X}_T$ from X into a T-complete space $\hat{X}_T$ is a completion at T if ϕ is an $\mathbb{F}_T$-equivalence.

This prescribes a universal property. If $f\colon X \longrightarrow Z$ is any map from X to a T-complete space Z, then there is a map $\tilde{f}$, unique up to homotopy, that makes the following diagram commute.



Therefore completions are unique up to homotopy if they exist.

Remark

On the full subcategory of connected spaces in $Ho\mathscr{T}$ *that admit completions at* T*, completion is functorial on the homotopy category. For a map* $f\colon X \longrightarrow Y$*, there is a unique map* $f_T\colon \hat{X}_T \longrightarrow \hat{Y}_T$ *in* $Ho\mathscr{T}$ *such that* $\phi \circ f = f_T \circ \phi$ *in* $Ho\mathscr{T}$*, by the universal property.*

The topological definitions just given do not mention any of the algebraic notions that we discussed Friday.

Let $\mathscr{A}_T$ denote the collection of all abelian groups that are completable at T and let $\mathscr{B}_T \subset \mathscr{A}_T$ denote the collection of all T-complete abelian groups. The topological definitions lead directly to consideration of $\mathscr{B}_T$.

Theorem

The $\mathbb{F}_T$-equivalences coincide with the maps that induce isomorphisms on cohomology with coefficients in all groups in $\mathscr{B}_T$, and $\mathscr{B}_T$ is the largest collection of abelian groups for which this is true.

Proof. Let $\mathscr{C}_T$ denote the collection of all abelian groups C such that

$$\xi^*\colon H^*(Y;C) \longrightarrow H^*(X;C)$$

is an isomorphism for all $\mathbb{F}_T$-equivalences $\xi\colon X \longrightarrow Y$. Our claim is that $\mathscr{B}_T = \mathscr{C}_T$. The collection $\mathscr{C}_T$ has the following closure properties.

(i) If two terms of a short exact sequence of abelian groups are in $\mathscr{C}_T$, then so is the third term since a short exact sequence gives rise to a natural long exact sequence of cohomology groups.

(ii) If $p \in T$ and $p^r C = 0$, then $C \in \mathscr{C}_T$, as we see by (i) and induction on r; the case $r = 1$ holds by the definition of an $\mathbb{F}_T$-equivalence.

(iii) Any product of groups in $\mathscr{C}_T$ is also in $\mathscr{C}_T$ since $H^*(X; \times_i C_i)$ is naturally isomorphic to $\times_i H^*(X; C_i)$.

(iv) By (i), the limit of a sequence of epimorphisms $f_i \colon C_{i+1} \longrightarrow C_i$ between groups in $\mathscr{C}_T$ is a group in $\mathscr{C}_T$ since we have a natural short exact sequence

$$0 \longrightarrow \lim C_i \longrightarrow \times_i C_i \longrightarrow \times_i C_i \longrightarrow 0;$$

the lim one error term is 0 on the right because the f_i are epimorphisms.

(v) All groups $\hat{A}_T$ are in $\mathscr{C}_T$, as we see by (ii), (iii), and (iv).

(vi) $\mathbb{E}_T A$ is in $\mathscr{C}_T$ if A is completable at T, as we see by (i), the defining exact sequence for L_0 and L_1, and the isomorphism $L_1 A \cong \mathbb{E}_T A$.

(vii) A is in $\mathscr{C}_T$ if A is T-complete since A is then completable and isomorphic to $\mathbb{E}_T A$.

This proves that $\mathscr{B}_T \subset \mathscr{C}_T$. For the opposite inclusion, we observe first that the unique map from $K(\mathbb{Z}[T^{-1}], 1)$ to a point is an $\mathbb{F}_T$-equivalence.

Indeed, $K(\mathbb{Z}[T^{-1}], 1)$ is a localization of $S^1 = K(\mathbb{Z}, 1)$ away from T. Its only non-zero reduced homology group is

$$\tilde{H}_1(K(\mathbb{Z}[T^{-1}], 1); \mathbb{Z}) \cong \mathbb{Z}[T^{-1}].$$

Since multiplication by $p \in T$ is an isomorphism on this group, the universal coefficient theorem implies that $\tilde{H}_*(\mathbb{Z}[T^{-1}], \mathbb{F}_p) = 0$ for $p \in T$.

For $C \in \mathscr{C}_T$, we conclude that $\tilde{H}^*(\mathbb{Z}[T^{-1}], C) = 0$. By the universal coefficient theorem again,

$$\mathrm{Hom}(\mathbb{Z}[T^{-1}], C) \cong H^1(\mathbb{Z}[T^{-1}], C) = 0$$

and

$$\mathrm{Ext}(\mathbb{Z}[T^{-1}], C) \cong H^2(\mathbb{Z}[T^{-1}], C) = 0.$$

This means that C is T-complete.

A remark on profinite p-complete groups

We understand a profinite abelian group B to be the limit of a (filtered) diagram $\{B_d\}$ of finite abelian groups. Say that a profinite group $B = \lim B_d$ is T-profinite if the B_d are T-torsion groups. The previous proof implies the following observation.

Corollary

All T-profinite abelian groups are in $\mathscr{B}_T$.

Returning to topology, we can now relate $\mathscr{B}_T$ to Eilenberg-Mac Lane spaces.

Corollary

If B is T-complete, then $K(B, n)$ is T-complete for all $n \geq 1$.

Proof. If $\xi\colon X \longrightarrow Y$ is an $\mathbb{F}_T$-equivalence, then

$$\xi^*\colon H^*(Y; B) \longrightarrow H^*(X; B)$$

is an isomorphism and thus

$$\xi^*\colon [Y, K(B, n)] \longrightarrow [X, K(B, n)]$$

is an isomorphism by the representability of cohomology.

By analogy with our topological description of T-local abelian groups, we have an alternative topological description of the collection $\mathscr{B}_T$ of T-complete abelian groups. Just as for localization, we cannot prove this without first doing a little homological calculation, but we defer that for the moment.

Proposition

An abelian group B is T-complete if and only if the space $K(B,1)$ is T-complete.

Proof. We have proven that if B is T-complete, then $K(B,1)$ is T-complete. Conversely, suppose that $K(B,1)$ is T-complete. Then the identity map of $K(B,1)$ is a completion at T. Moreover,

$$\xi^*\colon 0 = [*, K(B,1)] \longrightarrow [K(\mathbb{Z}[T^{-1}],1), K(B,1)]$$

is an isomorphism since $K(\mathbb{Z}[T^{-1}],1) \longrightarrow *$ is an $\mathbb{F}_T$-cohomology isomorphism, as we observed in a previous proof. Using the representability of cohomology, this gives that

$$\mathrm{Hom}(\mathbb{Z}[T^{-1}], B) \cong H^1(\mathbb{Z}[T^{-1}], B) = 0.$$

This implies that $\mathbb{H}_T B = 0$, so that B is completable at T.

We defer the proof of the following result.

Lemma

The map $\phi\colon K(B,1) \longrightarrow K(\mathbb{E}_T B,1)$ *that realizes* $\phi\colon B \longrightarrow \mathbb{E}_T B$ *on fundamental groups is an* $\mathbb{F}_T$*-equivalence.*

Here $K(\mathbb{E}_T B,1)$ is T-complete since $\mathbb{E}_T B$ is T-complete. Thus ϕ is also a completion of $K(B,1)$ at T. By the uniqueness of completion, ϕ must be an equivalence and thus $\phi\colon B \longrightarrow \mathbb{E}_T B$ must be an isomorphism.

We generalize the lemma to the following result.

Theorem

For each abelian group A and each $n \geq 1$, there is a completion $\phi\colon K(A,n) \longrightarrow K(A,n)^{\wedge}_{T}$. The only non-zero homotopy groups of $K(A,n)^{\wedge}_{T}$ are

$$\pi_n(K(A,n)^{\wedge}_{T}) = \mathbb{E}_T A$$

and

$$\pi_{n+1}(K(A,n)^{\wedge}_{T}) = \mathbb{H}_T A,$$

and $\phi_*\colon \pi_n(K(A,n)) \longrightarrow \pi_n(K(A,n)^{\wedge}_{T})$ coincides with $\phi\colon A \longrightarrow \mathbb{E}_T A$.

Proof. First consider a free abelian group F. Here we have $\mathbb{H}_T F = 0$ and $\mathbb{E}_T F \cong \hat{F}_T$. We claim that the map

$$\phi\colon K(F,n) \longrightarrow K(\hat{F}_T, n)$$

that realizes $\phi\colon F \longrightarrow \hat{F}_T$ is a completion at T. We have already seen that $\hat{F}_T$ is T-complete and therefore $K(\hat{F}_T, n)$ is T-complete. Thus we only need to prove that ϕ_* is an isomorphism on mod p homology for $p \in T$. We proceed by induction on n. First consider the case $n = 1$.

The projection $\hat{F}_T \longrightarrow \hat{F}_p$ induces an isomorphism on mod p homology since its kernel is local away from p. We now use the LHS spectral sequence of the quotient group $\hat{F}_p/F$. The spectral sequence has the form

$$E^2_{p,q} = H_p(\hat{F}_p/F; H_q(F; \mathbb{F}_p)) \Longrightarrow H_{p+q}(\hat{F}_p; \mathbb{F}_p).$$

The group $\hat{F}_p/F$ is uniquely p-divisible. One can see this, for example, by noting that the canonical map $F \longrightarrow \hat{F}_p$ is a monomorphism of torsion free abelian groups that induces an isomorphism upon reduction mod p.

Alternatively, writing elements of F in terms of a basis for F and writing integer coefficients in p-adic form, we see that elements of F/p^rF can be written in the form $f + pg$, where the coefficients appearing in f satisfy $0 \leq a < p$. For an element

$$(f_r + pg_r) \in \lim F/p^rF \subset \times_r F/p^rF$$

with components so written, compatibility forces (f_r) to come from an element $f \in F$, hence our given element is congruent to $p(g_r)$ mod F.

It follows that the terms with $p > 0$ are zero and the spectral sequence collapses to the edge isomorphism

$$\phi_*\colon H_*(F;\mathbb{F}_p) \cong E^2_{0,*} = E^\infty_{0,*} \cong H_*(\hat{F}_p;\mathbb{F}_p).$$

For $n > 1$, take $K(F, n-1) = \Omega K(F, n)$ and consider the map of path space fibrations

$$\begin{array}{ccccc} K(F,n-1) & \longrightarrow & PK(F,n) & \longrightarrow & K(F,n) \\ \downarrow & & \downarrow & & \downarrow \\ K(\hat{F}_T,n-1) & \longrightarrow & PK(\hat{F}_T,n) & \longrightarrow & K(\hat{F}_T,n). \end{array}$$

By the Serre spectral sequence, the induction hypothesis, and the comparison theorem, the map

$$\phi_*\colon H_q(K(F,n),\mathbb{F}_p) \longrightarrow H_q(K(\hat{F}_T,n),\mathbb{F}_p)$$

is an isomorphism and therefore ϕ is a completion at T.

Now consider a general abelian group A. Write A as a quotient F/F' of free abelian groups and let $i\colon F' \longrightarrow F$ be the inclusion. We construct a map of fibration sequences

$$\begin{array}{ccccccc}
K(F,n) & \longrightarrow & K(A,n) & \longrightarrow & K(F',n+1) & \xrightarrow{\ i\ } & K(F,n+1) \\
\downarrow{\scriptstyle \Omega\phi} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\
K(\hat{F}_T,n) & \longrightarrow & K(A,n)^{\wedge}_T & \longrightarrow & K(\hat{F}'_T,n+1) & \xrightarrow{\ i^{\wedge}_T\ } & K(\hat{F}_T,n+1).
\end{array}$$

Here the map i realizes the algebraic map i on passage to π_{n+1} and can be viewed as the map from the fiber to the total space of a fibration with base space $K(A,n+1)$. We take $K(A,n)$ to be the fiber Fi and take $K(F,n) = \Omega K(F,n+1)$. The two completion maps on the right have been constructed, and that on the left is the loops of that on the right.

The map $i_T^\wedge$ is the map, unique up to homotopy, that makes the right square commute up to homotopy, and it realizes the algebraic map $i_T^\wedge$ on passage to π_{n+1}. We define $K(A,n)_T^\wedge$ to be its fiber.

There is a dotted arrow map ϕ that makes the middle square commute and the left square commute up to homotopy. This map induces an isomorphism on mod p homology for $p \in T$ by the map of Serre spectral sequences induced by the map of fibrations given by the left two squares.

To show that ϕ is a completion of $K(A,n)$ at T it remains to show that $K(A,n)_T^\wedge$ is complete, which will follow once we know its homotopy groups are complete. The bottom fibration sequence gives a long exact sequence

$$\dots \longrightarrow \pi_{n+1}(K(\hat{F}_T, n)) \longrightarrow \pi_{n+1}(K(A,n)^{\wedge}_T) \longrightarrow \pi_{n+1}(K(\hat{F}'_T, n+1))$$

$$\longrightarrow \pi_n(K(\hat{F}_T, n)) \longrightarrow \pi_n(K(A,n)^{\wedge}_T) \longrightarrow \pi_n(K(\hat{F}'_T, n+1)) \longrightarrow \dots.$$

By the case of free abelian groups this simplifies to

$$0 \longrightarrow \pi_{n+1}(K(A,n)^{\wedge}_T) \longrightarrow \hat{F}'_T \xrightarrow{i^{\wedge}_T} \hat{F}_T \longrightarrow \pi_n(K(A,n)^{\wedge}_T) \longrightarrow 0.$$

The map $i^{\wedge}_T$ is the product over $p \in T$ of the maps $i^{\wedge}_p$, and our algebraic definitions and results give exact sequences

$$0 \longrightarrow \mathbb{H}_p A \longrightarrow \hat{F}'_p \longrightarrow \hat{F}_p \longrightarrow \mathbb{E}_p A \longrightarrow 0.$$

The product over $p \in T$ of these exact sequences is isomorphic to the previous exact sequence, and this gives the claimed identification of homotopy groups. Comparison of diagrams shows that the map on n^{th} homotopy groups induced by ϕ is the algebraic map ϕ.

Note that we have an interesting explicit example where homotopy groups shift dimension.

Example

For a prime p, $K(\mathbb{Z}/p^\infty, n)^\wedge_p$ *is an Eilenberg–Mac Lane space* $K(\mathbb{Z}_p, n+1)$.

Analogous dimension shifting appeared in comparing the algebraic K-theory of an algebraically closed field, which is concentrated in odd degrees, to topological K-theory, which is concentrated in even degrees

Completions of simple spaces

We construct completions here, We should record the relevant special case of a result called the dual Whitehead theorem. Taking $\mathscr{A} = \mathscr{B}_T$ in that general result, it takes the following form, which generalizes the fact that $K(B, n)$ is T-complete if B is T-complete.

Theorem

Every $\mathscr{B}_T$-tower is a T-complete space.

We use this result to construct completions of simple spaces. The generalization from Eilenberg–Mac Lane spaces to nilpotent spaces works in precisely the same way as the construction of localizations. We need only replace the localizations $K(A_T, n)$ by the completions $K(A, n)^{\wedge}_T$. The fact that the latter are not Eilenberg–Mac Lane spaces does not change the details of the construction.

Theorem

Every simple space X admits a completion $\phi\colon X \longrightarrow \hat{X}_T$.

Here $X_0 = *$, and we let $(X_0)^\wedge_T = *$. Assume that a completion $\phi_n \colon X_n \longrightarrow (X_n)^\wedge_T$ has been constructed and consider the following diagram, in which $K(A_n + 1, n+1) = \Omega K(A_{n+1}, n+2)$.

$$\begin{array}{ccccccc}
K(A_{n+1}, n+1) & \longrightarrow & X_{n+1} & \longrightarrow & X_n & \xrightarrow{k^{n+2}} & K(A_{n+1}, n+2) \\
\downarrow{\scriptstyle \Omega\phi} & & \downarrow{\scriptstyle \phi_{n+1}} & & \downarrow{\scriptstyle \phi_n} & & \downarrow{\scriptstyle \phi} \\
K((A_{n+1})^\wedge_T, n+1) & \longrightarrow & (X_{n+1})^\wedge_T & \longrightarrow & (X_n)^\wedge_T & \xrightarrow[(k^{n+2})^\wedge_T]{} & K((A_{n+1})^\wedge_T, n+2)
\end{array}$$

Since ϕ_n is a $\mathbb{Z}^\wedge_T$-equivalence and $K((A_{n+1})^\wedge_T, n+2)$ is a $\mathbb{F}_T$-complete space there is a map $(k^{n+2})^\wedge_T$, unique up to homotopy that makes the right square commute up to homotopy. The space X_{n+1} is the fiber Fk^{n+2}, and we define $(X_{n+1})^\wedge_T$ to be the fiber $F(k^{n+2})^\wedge_T$.

There is a map ϕ_{n+1} that makes the middle square commute and the left square commute up to homotopy. $(X_{n+1})^{\wedge}_{T}$ is T-complete since it is a $\mathscr{B}_T$-tower. To see that ϕ_{n+1} is a completion at T it remains to show that it induces an isomorphism on homology with coefficients in $\mathbb{F}_p$ for $p \in T$. The proof is a comparison of Serre spectral sequences exactly like that in the proof for localization. We define $X^{\wedge}_{T} = \lim (X_n)^{\wedge}_{T}$ and $\phi = \lim \phi_n \colon X \longrightarrow X^{\wedge}_{T}$. Then ϕ is an $\mathbb{F}_T$-equivalence and is thus a completion of X at T.