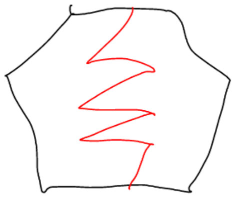
Scissors congruence

Area = "that which does not change under cutting and moving"

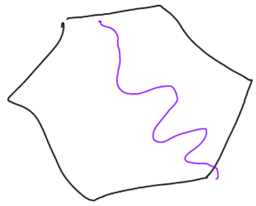
- for polygons (2-dimensional shapes)

Only "straight" cuts (in whatever dimension)

- $\boxed{\text{piecewise linear}}$



good



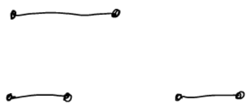
bad

- area does not change under decomposition and isometry. } ← Euclidean transformations these are the moves we are interested in

$P = P_1 \cup P_2$ ← polygons

$\text{meas}(P_1 \cap P_2) = 0$ $\qquad \underbrace{\mathring{P}_1 \cap \mathring{P}_2 = \emptyset}_{\text{interior-disjoint}}$

dim: 1 what is a polygon?



← is this a polygon? Yes

n-Polytope: finite union of n-simplices

n-Simplex: convex hull of n+1 points

1-polytope: finite set of segments.

"area" = length

dim 2: 2-polytope: finite union of triangles

"area" = ? Spoiler: "area" = area

Pf: Goal: two polygons P, Q have

$P = \bigcup_{i=1}^{n} P_i$ with $Q = \bigcup_{i=1}^{n} Q_i$ $\quad P_i \cong Q_i$

iff $\text{Area}(P) = \text{Area}(Q)$

definition of scissors congruence

$\Rightarrow$: by definition.

$\Leftarrow$: • STS P is s.c. to a $1 \times \mathrm{Area}(P)$ rectangle.

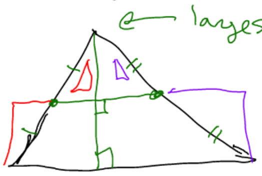
• STS a triangle Δ is s.c. $1 \times \mathrm{Area}(\Delta)$ rectangle

• 2 things:

(a) a triangle is s.c. to a rectangle

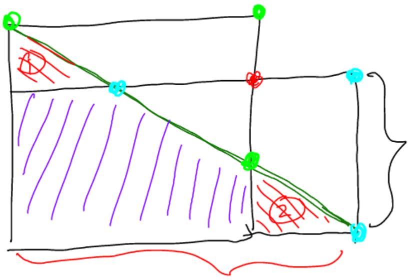
(b) a rectangle $\square$ is s.c. to a $1 \times \mathrm{Area}(\square)$ rectangle

Pf: (a): $\leftarrow$ largest angle



✓

(b)



a, 1, b, ab

$2a > b$
$a > 1$
$b > a$

$\Rightarrow$ green line is below red point

Facts:

"✓"

P, Q polyhedra

dim 3: Want: $\mathrm{Vol}(P) = \mathrm{Vol}(Q) \Longleftrightarrow P = \bigcup_{i=1}^{n} P_i \quad Q = \bigcup_{i=1}^{n} Q_i$ with $P_i \cong Q_i$

$D(P) = D(Q)$

true now? (1965 Sydler) yes

Again: $\Longleftarrow$ by definition

Gauss: $\Rightarrow$ if $n = \infty$ not true in general

Hilbert's 3rd problem: $\Rightarrow$? give counterexample.

Dehn (1901): cube, regular tetrahedron.

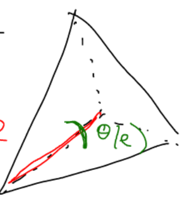
$$D : \{\text{polyhedra}\}/_{\text{s.c.}} \longrightarrow \mathbb{R} \otimes \mathbb{R}/_{2\pi\mathbb{Z}}$$

really hard to work in

invented in 1938

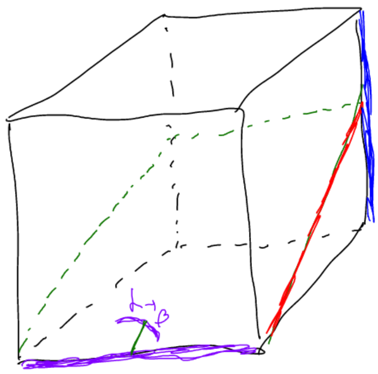
$$\text{Dehn}: \{\text{polyhedra}\}/_{\text{s.c.}} \longrightarrow \mathbb{R} \qquad s(x) := \begin{cases} q & \text{if } x = q \cdot \cos^{-1}\left(\frac{1}{3}\right) \ q \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

$$P \longmapsto \sum_{\text{edges } e \text{ of } P} \mathrm{length}(e) \cdot s(\theta(e))$$



e, $\theta(e)$

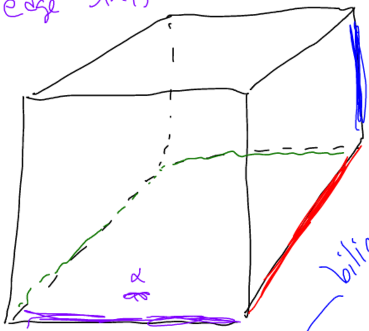
cube: $\theta(e) = 0 \ \forall e$ tetrahedron: nonzero. $\cos^{-1}\left(\frac{1}{3}\right) \neq \frac{p}{q}\pi$.

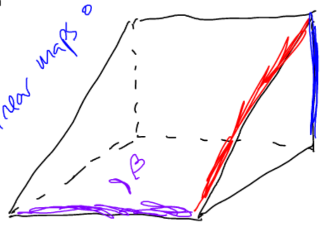


angle stays same
edge is sliced

new edge
angles around
it add up to
π

angle is sliced
edge stays same





bilinear maps out of this

$$\underbrace{\mathbb{R}}_{\text{edges}} \otimes \underbrace{\mathbb{R}/\pi\mathbb{Z}}_{\text{angles}}$$

want a function
(edges, angles)

$\Downarrow$ stay same

- edge cut, angle constant
- edge constant, angle cut

$\Big]$ biadditivity

- edge constant, $\underbrace{\text{angles add up to } \pi}$

$= 0$

biadditivity
$\iff$ bilinearity over $\mathbb{Z}$

want: a map
polytopes $\longrightarrow$?
which is bilinear over $\mathbb{Z}$
in edge lengths and angle
measures mod π