Scissors congruence

Area = “that which does not change under cutting and moving”
- for polygons (2-dimensional shapes)

Only “straight” cuts (in whatever dimension)
  * piecewise linear

- area does not change under decomposition and isometry

\[ P = P_1 \cup P_2 \]
\[ \text{mes}(P_1 \cap P_2) = 0 \]
\[ P_1 \cap P_2 = \emptyset \]

interior - disjoint

**dim:** 1  what is a polygon?

- Polytope: finite union of \( n \)-simplices
- Simplex: convex hull of \( n+1 \) points

- polytope: finite set of segments
  \( “area“ = \text{length} \)

**dim 2:** 2-polytope: finite union of triangles
  \( “area“ = ? \)
  **Spoiler:** \( “area“ = \text{area} \)

**Pf:** Goal: two polygons \( P, Q \) have

\[ \text{Pf: Goal: two polygons } P, Q \text{ have} \]
\[ \text{if } \text{Area}(P) = \text{Area}(Q), \]

\[ P = \bigcup_{i=1}^{n} P_i \]
\[ Q = \bigcup_{i=1}^{n} Q_i \]

\( P_i \cong Q_i \) with \( P_i \cong Q_i \)

**definition of scissors congruence**
\[ \text{by definition} \]

\[ \text{STS } P \text{ is s. c. to a } 1 \times \text{Area}(P) \text{ rectangle.} \]

\[ \text{STS a triangle } A \text{ is s. c. } 1 \times \text{Area}(A) \text{ rectangle} \]

- 2 things:
  (a) a triangle is s. c. to a rectangle
  (b) a rectangle is s. c. to a \( 1 \times \text{Area}(D) \) rectangle.

Proof:

(a) \[ \text{largest angle} \]

(b) \[ \text{green line is below red point} \]

\[ \text{Facts: } \sqrt{2} \leq \text{Volume} \]

\[ \sqrt{2} \approx 1.41 \]

\[ \text{dim } 3: \text{ Want: } \text{Vol}(P) = \text{Vol}(Q) \iff P = \bigcup_{i=1}^{n} P_i \text{ and } Q = \bigcup_{i=1}^{n} Q_i. \]

\[ D(P) = D(Q) \text{ true now? with } P_i \subseteq Q_i \]

\[ \text{Again: } \leq \text{ by definition (1965 Synder)} \]

\[ \text{Gauss: } \Rightarrow \text{ if } n=\infty \text{ not true in general} \]

\[ \text{Hilbert's 3rd problem: } \Rightarrow ?? \text{ give counterexample.} \]

\[ \text{Dehn (1901): cube, regular tetrahedron.} \]

\[ D: \{ \text{polyhedra} \}_{/ \text{s.c.}} \rightarrow \mathbb{R} \oplus \mathbb{R}/\mathbb{Z} \]

\[ \text{Dehn: } \{ \text{polyhedra} \}_{/ \text{s.c.}} \rightarrow \mathbb{R} \]

\[ P_i \rightarrow \sum \text{length}(e) \cdot s(\theta(e)) \]

\[ \text{cube: } \theta(e) = 0 \text{ for edges } \theta \text{ have non-zero. } \cos^{-1}(\frac{1}{2}) = \frac{\pi}{3}. \]
Want a function \((\text{edges, angles}) \rightarrow \mathbb{R}^2\) with:

- biadditivity
- bilinearity over \(\mathbb{Z}\)

Want: a map polytopes \(\rightarrow \mathbb{Z}^2\) which is bilinear over \(\mathbb{Z}\) in edge lengths and angle measures mod \(\pi\).