NOTES ON FIBER BUNDLES

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ABSTRACT. These are notes on fiber bundles and principal bundles, especially over CW complexes and spaces homotopy equivalent to them. They are meant to supplement a first-year graduate course on Algebraic Topology given at the University of Chicago in Fall 2013.

CONTENTS

These notes provide a bare outline of some of the foundations of the theory of fiber bundles and principal bundles, especially over CW complexes. Some basic references for the theory of fiber bundles are Husemöller [2], and Hatcher [1] § 4.2.

1. FIBER BUNDLES

Let's fix a topological space F and a group G acting on F by homeomorphisms; informally we say that G is the *structure group* of F. Informally, an F-bundle over X is a space E mapping to X in such a way that E looks *locally* (in X) like the product of X with F. The *fibers* are the preimages in E of points in X; they should all be isomorphic (as G spaces) to F but not *canonically* so.

Definition 1.1. If X is a topological space, an F-bundle over X is a space E and a surjective map $\pi : E \to X$ with the property that X admits an open cover by sets U_i such that

(1) **(locally trivial)** for each index i, there is a homeomorphism $\varphi_i : \pi^{-1}(U_i) \to F \times U_i$ for which $p_2 \circ \varphi_i = \pi$ on $\pi^{-1}(U_i)$ (where p_2 is the projection from $F \times U_i$ to U_i) and such that
(2) **(structure group G)** for each pair of indices i, j the transition functions

$$\varphi_j^{-1} \circ \varphi_i : F \times (U_i \cap U_j) \to F \times (U_i \cap U_j)$$

are of the form $(f, x) \to (\psi_{ij}(x)f, x)$ for some $\psi_{ij} : U_i \cap U_j \to G$.

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Sometimes an F bundle over X with total space E is denoted by a sequence $F \to E \to X$.

Lemma 1.2. *To give the data of an F-bundle over X it suffices to give an open cover U_i of X and*

(1) **(transition functions)** *for each i, j a map $\psi_{ij} : U_i \cap U_j \to G$ satisfying*
(2) **(cocycle condition)** *for each i, j, k the composition $\psi_{jk} \circ \psi_{ij} = \psi_{ik}$ on $U_i \cap U_j \cap U_k$*

Proof. The functions ψ_{ij} promised by the definition of an F bundle satisfy the cocycle condition. Conversely, given a collection of transition functions, we can glue products $F \times U_i$ together along $F \times (U_i \cap U_j)$, identifying fibers pointwise in X by the value of the transition function; these gluings are compatible if they are compatible on triple intersections, which is exactly what the cocycle condition guarantees. □

The data of an F-bundle in terms of transition functions lets us define (in a natural way) an F' bundle for any space F' with a G action. Taking $F' = G$ and the G action to be left multiplication, one arrives at the idea of a *principal G-bundle*.

Definition 1.3. Two F bundles $\pi : E \to X$ and $\pi' : E' \to X$ over X are *isomorphic* if there is a homeomorphism $\psi : E \to E'$ with $\pi' \circ \psi = \pi$ such that when restricted to each open set over which both bundles are locally trivial $\psi|_{F \times U_i} : F \times U_i \to F \times U_i$ the map $F \to F$ on each fiber is obtained by the action of some $g \in G$.

If ψ_{ij} is a collection of transition functions satisfying the cocycle condition and defining an F bundle $\pi : E \to X$, then if $\phi_i : U_i \to G$ is any collection of functions, the transition functions $\psi'_{ij} := \phi_j^{-1} \psi_{ij} \phi_i$ also satisfy the cocycle conditions, and are said to be *cohomologous* to the ψ_{ij}.

A collection of transition functions defined with respect to an open cover U_i defines a collection of transition functions with respect to any refinement of the cover.

Lemma 1.4. *Two collections of transitions functions ψ_{ij}, $\psi'_{\alpha\beta}$ with respect to open covers U_i and V_α on X define isomorphic F bundles if and only if there are refinements of the open covers over which the transition functions are defined such that the transition functions obtained by restriction are cohomologous.*

If X is a topological space, we can define the sheaf $\mathcal{G}$ whose values over an open set U are continuous functions from U to G. The transition functions defining a principal G bundle are thus an example of a 1-cochain with values in this sheaf; and the cocycle condition is precisely the condition that this cochain is a 1-cocycle. Different choices of transition functions define cohomologous 1-cocycles, and passing to the direct limit over refined coverings determines a class in Čech cohomology; thus F bundles up to isomorphism are classified by $H^1(X; \mathcal{G})$ — i.e. by 1-dimensional Čech cohomology classes with coefficients in $\mathcal{G}$.

2. Classifying spaces

Definition 2.1. If $\pi : E \to X$ is an F bundle, and $g : Y \to X$ is a map, then there is a *pullback F bundle* $g^*\pi : g^*E \to Y$ over Y whose fiber over each point y of Y is identified with the fiber of E over $g(y)$.

In terms of transition functions, if U_i is an open cover of X with transition functions ψ_{ij} on the overlaps, then $V_i := g^{-1}(U_i)$ is an open cover of Y and we can define transition functions $\phi_{ij} : V_i \cap V_j \to G$ by $\phi_{ij} = \psi_{ij} \circ g$.

The key property of fiber bundles (and their cousins fibrations) is the *homotopy lifting* property. In order to streamline the discussion of this property, we make the following two simplifications in the sequel. First, we deal with principal G bundles for simplicity. Second, we use the language of connections. A connection on a bundle E over X is a canonical choice, for each path $\gamma : I \to X$, of a trivialization of the pullback bundle $\gamma^* E$ over I, which varies continuously with γ. Equivalently (for a principal G bundle), for every $\gamma : I \to X$ and every lift $\tilde{\gamma}(0) : 0 \to E$ lifting the initial point, there is given a lift $\tilde{\gamma} : I \to X$ extending the lift of the initial point, and such that different lifts of the initial point give lifts of I which differ by (global) multiplication by some $g \in G$.

If X is a smooth manifold, and G is a (possibly disconnected) Lie group, any principal G bundle E is a smooth manifold, and admits a connection, which can be determined by splitting the tangent bundle of E into a vertical direction (tangent to the fibers) and a G-invariant horizontal direction (tangent to the trivialization over each path in the base, infinitesimally).

Using this language simplifies the discussion and streamlines the proofs in the cases of most interest in geometry and topology; the proofs in the more general case are not significantly more difficult, but are more messy. For instance, the proofs of the following two lemmas become completely transparent:

Lemma 2.2. *Let X and Y be CW complexes. Homotopic maps $g, h : Y \to X$ pull back a bundle $\pi : E \to X$ to isomorphic bundles g^*E, h^*E over Y.*

Proof. Let $H : Y \times I \to X$ be a homotopy from g to h. Then the bundle H^*E over $Y \times I$ restricts to g^*E and h^*E over $Y \times 0$ and $Y \times 1$. Using a connection, we can trivialize H^*E over each path $y \times I$; this family of trivializations defines an isomorphism between h^*E and g^*E. ☐

Lemma 2.3. *Every bundle over a contractible CW complex X is isomorphic to a trivial bundle.*

Proof. Choose a homotopy from X to a point $x \in X$. Using a connection, the track of the homotopy defines an identification of the fiber over x with the fiber over any other point in X; this family of identifications defines a global trivialization of the bundle. ☐

If $\pi : E \to X$ is a principal G bundle, then a trivialization of E over some subset $Y \subset X$ is nothing more than a section $\sigma : Y \to \pi^{-1}(Y)$. For, using the action of the structure group of each fiber, given a point on a fiber there is a unique homeomorphism from that fiber to G (respecting the structure) and taking that point to the identity element. The family of homeomorphisms over Y determined by a section gives a trivialization of E over Y.

Definition 2.4. A principal G bundle $\pi : EG \to BG$ (with the homotopy type of a CW complex) is a *universal* principal G bundle (for CW complexes) if for all CW complexes X, there is a natural bijection between the set of principal G bundles over X up to isomorphism, and the set $[X, BG]$ of homotopy classes of maps from X to BG.

Theorem 2.5 (Classifying spaces)**.** *Let G be a group.*

1. *A principal G bundle $EG \to BG$ is universal if and only if EG is contractible.*
2. *For any topological group G with the homotopy type of a CW complex some principal bundle exists, and is unique up to homotopy.*

Proof. Let $EG \to BG$ be a principal G bundle with EG contractible. Let $\pi : E \to X$ be a principal G bundle over a CW complex X. Suppose we have built a map $f^k : X^k \to BG$ which pulls back EG to $E^k := \pi^{-1}(X^k)$, where X^k denotes the k-skeleton. Let D be a $(k+1)$-cell of X. The restriction of E to D is trivial because D is contractible, and any trivialization defines a section over ∂D. This section defines a lift of $\partial D \to X^k \to BG$ to $\partial D \to EG$, and since EG is contractible, this lift extends over D. This defines an extension of f^k over D by projecting to BG, and by construction the pullback of this map over D agrees with the trivialization of E over D. So by induction we get a map from X to BG pulling back EG to E.

An isomorphism of two bundles E_0, E_1 over X defines an E bundle over $X \times I$; a map of $X \times \{0,1\}$ to BG pulling back EG to the E_i can be extended over $X \times I$ cell by cell as above. This proves that any principal G bundle $EG \to BG$ with EG contractible is universal.

If $EG \to BG$ and $EG' \to BG'$ are both universal, there are maps from BG to BG' and back pulling back EG' to EG and conversely; composing these maps defines maps from BG to BG pulling back EG to itself (and similarly for BG'); such maps are homotopic to the identity by the definition of universal bundle, so any two principal bundles are homotopy equivalent (as bundles).

Finally we show that there is a principal G bundle with EG contractible; this will complete the proof of the theorem. Define EG to be the infinite join $EG := G * G * G * \cdots$. Taking joins raises the connectivity, so EG is contractible. On the other hand, G acts on EG by the diagonal action on the factors, and the quotient is BG. □

Remark 2.6. Here is a more geometric way to think about the finite join $G * G * \cdots * G$ ($n+1$ factors) which is a universal bundle for principal G bundles over CW complexes X with all cells of dimension $< n$.

Think of the n-simplex Δ^n as the space of $(n+1)$-tuples $(t_0, \cdots, t_n)$ in $\mathbb{R}^{n+1}$ with each $t_i \in [0,1]$, and $\sum t_i = 1$. Over each point $t \in \Delta^n$ put a copy of G^{n+1}, and scale each factor G_i by t_i, so that when $t_i = 0$ the factor G_i is crushed to a point.

We can think of this space as being obtained from the product $G^{n+1} \times \Delta^n$ by crushing each G_i factor to a point over the face of Δ^n where the corresponding coordinate t_i is equal to 0. In other words, each tuple $(g_0, \cdots, g_n)$ determines an n-simplex (where we think of the vertices decorated by the g_i, and we glue two such simplices along their i-th faces $(g_0, \cdots, \hat{g}_i, \cdots, g_n)$ if all but their ith coordinate agree. This way of thinking about EG and its quotient space BG makes it seem very similar to the construction of a $K(G,1)$ using the bar complex, except that there are many different n-simplices in BG with a given $(n+1)$-tuple as label (corresponding to the choice of a tuple of indices t_j for $0 \le j < \infty$ with $t_j \ne 0$) instead of just one in the bar complex.

This construction of BG is due to Milnor.

3. Fibrations

The key property of principal fiber bundles is the ability to continuously find trivializations (equivalently: sections) over paths in the base; i.e. the path lifting property.

Definition 3.1. A map $\pi : E \to B$ has the *homotopy lifting property* with respect to X if, given any homotopy $H : X \times I \to B$, and a lift $\tilde{h} : X \to E$ lifting $h := H|X \times 0$, there is an extension of $\tilde{h}$ to a lift $\tilde{H} : X \times I \to B$. A map $\pi : E \to B$ with the homotopy lifting property for disks D is a *Serre fibration*.

Fiber bundles are Serre fibrations.

Proposition 3.2 (Long exact sequence)**.** *Let $\pi : E \to B$ be a Serre fibration, let $b \in B$, and let $x \in F := \pi^{-1}(b)$. There is a long exact sequence*

$$\cdots \to \pi_n(F, x) \to \pi_n(E, x) \to \pi_n(B, b) \to \pi_{n-1}(F, x) \to \cdots \to \pi_0(E, x) \to 0$$

The boundary maps $\pi_n(B, b) \to \pi_{n-1}(F, x)$ are defined by using the fibration property to lift a map of pairs $(D^n, S^{n-1}) \to (B, b)$ to (E, F) extending the lift of basepoints $s \to b$ lifted to $s \to x$. Exactness at each term is proved easily by the homotopy lifting property for disks and spheres.

From the long exact sequence it follows that if $EG \to BG$ is a universal bundle, then $\pi_{i+1}(BG) = \pi_i(G)$ for all i. Thus $\pi_i(\Omega BG) = \pi_i(G)$ for all i, and in fact ΩBG is homotopy equivalent to G.

The process of replacing a space G by BG is called *delooping*; the space G does not need to be literally a group, but only an H-space (i.e. a space with a multiplication map and an identity element satisfying the group axioms *up to homotopy*); for instance ΩX for any CW complex X. So for any X we have $B\Omega X$ is homotopy equivalent to X.

One way to connect this picture up to the description of universal bundles is to use the *path space* construction. Let's let X be a path-connected space with a basepoint x, and let $\mathcal{P}X$, the *path space of* X, be the space of maps $\gamma : I \to X$ with $\gamma(0) = x$. There is a projection $\pi : \mathcal{P}X \to X$ defined by $\pi(\gamma) = \gamma(1)$.

Theorem 3.3. *The map $\pi : \mathcal{P}X \to X$ is a (Serre) fibration, whose total space is contractible, and whose fibers are homotopy equivalent to ΩX.*

Proof. We first show that $\pi : \mathcal{P}X \to X$ has the path lifting property with respect to all spaces. Suppose $\gamma : I \to X$ is a path, and $\tilde{\gamma}(0) : 0 \to \mathcal{P}X$ is a lift of the initial point. Then $\tilde{\gamma}(0)$ is (by definition) a path from x to $\gamma(0)$. So we can define $\tilde{\gamma}(t)$ to be the concatenation of $\tilde{\gamma}(0)$ with the path $\gamma|[0, t]$, parameterized by $[0, 1]$ in the obvious way. This defines *canonical* homotopy lifting for points, and (because it depends continuously on γ and $\tilde{\gamma}(0)$) thereby homotopy lifting for all spaces mapping to X.

Thus $\mathcal{P}X \to X$ is a fibration. On the other hand, $\mathcal{P}X$ is contractible, since every path can be contracted (at linear speed) to its initial point, which is the constant map to x. Finally, the fiber over x is by definition equal to ΩX. □

The path space construction has the following useful generalization. If $f : Y \to X$ is any map of spaces, let E_f be the space of pairs (y, γ) with $y \in Y$ and $\gamma : I \to X$ with $\gamma(0) = f(y)$. Then the map $\pi : E_f \to X$ sending (y, γ) to $\gamma(1)$ is a fibration. Moreover,

E_f is homotopy equivalent to Y (as can be seen by contracting (y,γ) to $(y,f(y))$ where $f(y): I \to X$ is the constant map to the point $f(y)$) and the map $Y \to E_f \to X$ agrees with f. In other words, any map between spaces can be replaced by a fibration, up to homotopy. The fiber F_f of $E_f \to X$ is called the *homotopy fiber*, and is unique up to homotopy equivalence (at least if X and Y are CW complexes).

4. Examples

Example 4.1 (G discrete). If G is discrete, then $\pi_i(BG) = G$ if $i = 1$ and 0 otherwise; thus BG is just a $K(G,1)$. A principal G bundle over X is a regular covering space $\hat{X}$ with deck group equal to G. Such bundles are classified by homomorphisms from $\pi_1(X)$ to G up to conjugacy, equivalently by $[X, K(G,1)]$.

Example 4.2 ($G = \mathbb{Z}$). When $G = \mathbb{Z}$ with its discrete topology, $B\mathbb{Z} = K(\mathbb{Z},1) = S^1$. Thus $[X, S^1]$ classifies principal $\mathbb{Z}$ bundles over X. But also $[X, S^1]$ classifies $H^1(X;\mathbb{Z})$.

Example 4.3 ($G = \mathbb{C}^*$). In this case $BG = \mathbb{CP}^\infty$ and $EG = \mathbb{C}^\infty - 0$. Since $\mathbb{C}^*$ is homotopy equivalent to S^1, we see that $\mathbb{CP}^\infty$ is a $K(\mathbb{Z},2)$. Thus: principal $\mathbb{C}^*$ bundles over X up to isomorphism are in natural bijection with $H^2(X;\mathbb{Z})$, and both are identified with $[X, \mathbb{CP}^\infty]$.

Now consider the space $S^1 * \cdots * S^1/S^1$ ($n+1$ factors in the product). This space maps naturally to Δ^n in such a way that the preimage of the vertices are points, the preimage of the edges are (open) cylinders $S^1 \times I$, the preimages of the 2-simplices are disk bundles over tori $T^2 \times D^2$, and so on. Geometrically, this space is $\mathbb{CP}^n$, and the projection is

$$[z_0 : z_1 : \cdots : z_n] \to \frac{1}{\sum |z_i|^2}(|z_0|^2, |z_1|^2, \cdots, |z_n|^2)$$

We can think of $\mathbb{CP}^n$ as a quotient in the following way. The unitary group $\mathrm{U}(n+1)$ acts on the unit sphere in $\mathbb{C}^{n+1}$ with point stabilizers the conjugates of $\mathrm{U}(n)$, so the coset space $\mathrm{U}(n+1)/\mathrm{U}(n) = S^{2n+1}$. The center of $\mathrm{U}(n+1)$ is isomorphic to S^1, and consists of matrices of the form $\lambda \cdot \mathrm{Id}$ where $|\lambda| = 1$; thus $\mathbb{CP}^n = S^1 \backslash \mathrm{U}(n+1)/\mathrm{U}(n)$. The circle S^1 is a subgroup of a torus T^{n+1} consisting of matrices whose diagonal entries have absolute value 1 and whose off-diagonal entries are 0, and the map $\mathbb{CP}^n \to \Delta^n$ is just the quotient

$$S^1 \backslash \mathrm{U}(n+1)/\mathrm{U}(n) \to T^{n+1} \backslash \mathrm{U}(n+1)/\mathrm{U}(n)$$

An S^1 bundle over X gives rise to a long exact sequence in homotopy groups. If $\pi_2(X)$ is trivial (and therefore $H^2(X;\mathbb{Z}) = H^2(\pi_1(X);\mathbb{Z})$) this long exact sequence contains the short exact sequence $0 \to \mathbb{Z} \to \pi_1(E) \to \pi_1(X) \to 0$. Since the S^1 bundle is principal, it is oriented, and the conjugation action of $\pi_1(X)$ acts trivially on $\mathbb{Z} = \pi_1(S^1)$. Thus such a structure determines a $\mathbb{Z}$ central extension of $\pi_1(X)$. In fact, $H^2(\pi_1(X);\mathbb{Z})$ classifies $\mathbb{Z}$ central extensions.

Example 4.4 (Vector bundles). If $F = \mathbb{R}^n$ and $G = \mathrm{GL}(n,\mathbb{R})$, an F bundle is called a *real* (n-dimensional) *vector bundle*; similarly, if $F = \mathbb{C}^n$ and $G = \mathrm{GL}(n,\mathbb{C})$ an F bundle is a *complex* (n-dimensional) *vector bundle*. The case of $n = 1$ are called (real or complex) *line bundles*.

Since $\mathrm{GL}(1,\mathbb{R}) = \mathbb{R}^*$ which is homotopy equivalent to ± 1, real line bundles are classified by maps to $\mathbb{RP}^\infty = K(\mathbb{Z}/2\mathbb{Z},1)$. Equivalently, they are classified by $H^1(X;\mathbb{Z}/2\mathbb{Z})$; the class in cohomology associated to a real line bundle is w_1, the *first Stiefel-Whitney class*. It pairs nontrivially with loops in X around which the bundle is nonorientable.

Since $\mathrm{GL}(1,\mathbb{C}) = \mathbb{C}^*$, complex line bundles are classified by maps to $\mathbb{CP}^\infty$; the class in $H^2(X;\mathbb{Z})$ associated to a complex line bundle is c_1, the *first Chern class*.

The space $B\mathrm{GL}(n,\mathbb{R})$ is the *real Grassmannian of n-planes*, whose points correspond to n-dimensional real subspaces of $\mathbb{R}^\infty$. The total space of the principal $\mathrm{GL}(n,\mathbb{R})$ bundle over this space can be identified (by the orbit map) with the space of n-tuples of linearly independent vectors $(v_1, v_2, \cdots, v_n)$ in $\mathbb{R}^\infty$; by Gram-Schmidt, we may identify this space up to homotopy with collections of basis vectors for which each v_i has length 1, and distinct v_i, v_j are orthogonal (i.e. the space of n-dimensional *orthonormal frames*); this space is naturally the total space of a principal $\mathrm{O}(n,\mathbb{R})$ bundle, and Gram-Schmidt shows that $\mathrm{O}(n,\mathbb{R})$ is homotopy equivalent to $\mathrm{GL}(n,\mathbb{R})$.

The space of n-dimensional orthonormal frames in $\mathbb{R}^\infty$ is contractible. For, the first vector defines a map to S^∞, the next vector defines a map to the S^∞ contained in the $\mathbb{R}^\infty$ perpendicular to the first vector, and so on; thus this total space is an iterated bundle where each fiber is contractible. This demonstrates that the Grassmannians are classifying spaces.

The space $B\mathrm{GL}(n,\mathbb{C})$ is the *complex Grassmannian of n-planes*, whose points correspond to n-dimensional complex subspaces of $\mathbb{C}^\infty$. This can be proved by the same argument as above, using the homotopy equivalence between $\mathrm{GL}(n,\mathbb{C})$ and $\mathrm{U}(n)$, and a similar argument with (complex) frames (note that there are also homotopy equivalences $\mathrm{Sp}(2n,\mathbb{R}) \simeq \mathrm{U}(n)$, which explain why a smooth manifold admits an almost symplectic structure if and only if it admits an almost complex structure).

There is an isomorphism as rings $H^*(B\mathrm{GL}(n,\mathbb{C});\mathbb{Z}) = \mathbb{Z}[c_1, c_2, \cdots, c_n]$ where each c_i is of degree $2i$. Thus a complex n-dimensional bundle over X gives rise to *Chern classes* $c_i \in H^{2i}(X;\mathbb{Z})$.

There is an isomorphism as rings $H^*(B\mathrm{GL}(n,\mathbb{R});\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[w_1, w_2, \cdots, w_n]$ where each w_i is of degree i. Thus a real n-dimensional bundle over X gives rise to *Stiefel-Whitney classes* $w_i \in H^i(X;\mathbb{Z}/2\mathbb{Z})$. A real n-dimensional bundle can be complexified to a complex n-dimensional bundle; the corresponding Chern classes are 2-torsion for i odd; the classes c_{2i} determine classes p_i in $H^{4i}(X;\mathbb{Z})$ called the *Pontriagin classes*.

Example 4.5 (Spin). The special orthogonal group includes as the component of the identity in the full orthogonal group $\mathrm{SO}(n,\mathbb{R}) \to \mathrm{O}(n,\mathbb{R})$. This corresponds to a (connected) double covering of classifying spaces $B\mathrm{SO}(n,\mathbb{R}) \to B\mathrm{O}(n,\mathbb{R})$. Since $B\mathrm{O}(n,\mathbb{R}) \simeq B\mathrm{GL}(n,\mathbb{R})$, this double cover corresponds to the kernel of w_1. Thus w_1 is the obstruction to orienting a real vector bundle.

There is also a connected double covering space of groups $\mathrm{Spin}(n) \to \mathrm{SO}(n,\mathbb{R})$, where $\mathrm{Spin}(n)$ is the universal cover for $n \geq 3$. This gives rise to a fibration $B\mathrm{Spin}(n) \to B\mathrm{SO}(n,\mathbb{R})$ with (homotopy) fiber $\mathbb{RP}^\infty$ $(= B\mathbb{Z}/2\mathbb{Z})$; thus the homotopy groups of these classifying spaces are the same outside dimension 2 where $\pi_2(B\mathrm{SO}(n,\mathbb{R})) = \pi_1(\mathrm{SO}(n,\mathbb{R})) = \mathbb{Z}/2\mathbb{Z}$ for $n \geq 3$. Thus w_2 is the obstruction to putting a spin structure on an oriented real vector bundle. Since $B\mathrm{Spin}(n)$ is 3-connected, a real vector bundle over a CW complex can be given a spin structure if and only if it can be (stably) trivialized over the 2 skeleton.

In low dimensions one has $\mathrm{Spin}(2) = S^1$ double covering $\mathrm{SO}(2,\mathbb{R}) = \mathbb{RP}^1$, $\mathrm{Spin}(3) = S^3$ double covering $\mathrm{SO}(3,\mathbb{R}) = \mathbb{RP}^3$, and $\mathrm{Spin}(4) = S^3 \times S^3$.

Example 4.6 (Flat bundles). If G acts on a space F by homeomorphisms, then we can give G the discrete topology (to distinguish it from some other topology we denote it G^δ). An F bundle with structure group G^δ is determined by transition functions $\psi_{ij} : U_i \cap U_j \to G^\delta$, which are necessarily locally constant. We call such an F bundle a *flat* bundle. An F bundle can be given the structure of a flat bundle if and only if the classifying map $X \to BG$ admits a lift to $X \to BG^\delta$, where the identity map $G^\delta \to G$ (which is continuous, but *not* a homeomorphism if G is not discrete) determines $BG^\delta \to BG$.

Let x and y be points in some component of $U_i \cap U_j$. The local trivializations in U_i and U_j each let us identify the fiber over x with the fiber over y, and these identifications are the same (because ψ_{ij} is constant in the component of $U_i \cap U_j$ containing x and y). So by following a path from chart to chart, each path $\gamma : [0,1] \to X$ determines a canonical isomorphism from the fiber over $\gamma(0)$ to the fiber over $\gamma(1)$.

Since homotopies between paths with the same endpoints can be factorized into homotopies supported in a single chart, it follows that there is a *holonomy representation* $\rho : \pi_1(X,x) \to G^\delta$, well-defined up to conjugacy.

If $\tilde{X}$ denotes the universal cover of X, the product $\tilde{X} \times G$ carries a product foliation by leaves $\tilde{X} \times g$, transverse to the copies of G in this product. The group $\pi_1(X)$ acts on $\tilde{X} \times G$ by the diagonal action: on the left as a deck group, and on the right by ρ. The quotient is an F bundle $E \to X$ together with a foliation transverse to the fibers. Conversely, *if F is compact*, a foliated F bundle $E \to X$ (i.e. a bundle with a foliation transverse to the fibers) can be given the structure of a flat bundle.

Example 4.7. It can happen that for some space F the structure group G might contain a subgroup H such that the inclusion of H into G is a homotopy equivalence. Then any F bundle with structure group G can be given the structure group H, since BG is homotopic to BH. However, typically G^δ is not equal to H^δ, so a *flat* F bundle with structure group G can *not* usually be given the structure of a *flat* F bundle with structure group H.

For example, taking $F = S^1$, and $G = \mathrm{PSL}(2,\mathbb{R})$ and $H = \mathrm{SO}(2,\mathbb{R})$. The unit tangent bundle of a hyperbolic surface admits an $\mathrm{SO}(2,\mathbb{R})$ structure (coming from a Riemannian metric) but *not* a flat $\mathrm{SO}(2,\mathbb{R})$ structure. However, it does admit a flat $\mathrm{PSL}(2,\mathbb{R})$ structure.

Example 4.8 (Smale conjecture). There are homotopy equivalences (induced by inclusion in the obvious way) between $\mathrm{O}(2,\mathbb{R})$ and $\mathrm{Homeo}(S^1)$.

There are homotopy equivalences

$$S^1 = \mathrm{SO}(2,\mathbb{R}) \simeq \mathrm{PSL}(2,\mathbb{R}) \simeq \mathrm{Diff}^+(S^1) \simeq \mathrm{Homeo}^+(S^1) \simeq \mathrm{Homeo}^+(\mathbb{R}^2)$$

There are homotopy equivalences

$$\mathrm{SO}(3,\mathbb{R}) \simeq \mathrm{PSL}(2,\mathbb{C}) \simeq \mathrm{Diff}^+(S^2) \simeq \mathrm{Homeo}^+(S^2)$$

The equivalence of $\mathrm{SO}(3,\mathbb{R})$ and $\mathrm{Diff}^+(S^2)$ is a theorem of Smale.

There are homotopy equivalences

$$\mathrm{SO}(4,\mathbb{R}) \simeq \mathrm{Diff}^+(S^3) \simeq \mathrm{Homeo}^+(S^3)$$

The first equivalence was the *Smale Conjecture*, proved by Hatcher (the latter equivalence was known earlier). This pattern does *not* persist in higher dimensions; in general $\text{Diff}(S^n) \simeq \text{O}(n) \times \text{Diff}(D^n, \partial D^n)$ but this latter group is not even connected whenever there are exotic $(n+1)$-spheres (e.g. $n = 6$).

Example 4.9 (Hopf fibrations). There are *Hopf fibrations* $S^0 \to S^1 \to S^1$, $S^1 \to S^3 \to S^2$ and $S^3 \to S^7 \to S^4$; these are the first examples in families $S^0 \to S^n \to \mathbb{RP}^n$, $S^1 \to S^{2n+1} \to \mathbb{CP}^n$ and $S^3 \to S^{4n+3} \to \mathbb{HP}^n$. There is also a Hopf fibration $S^7 \to S^{15} \to S^8$ which is not part of an infinite family.

Example 4.10 (Bott Periodicity). There are fiber bundles $\text{O}(n-1) \to \text{O}(n) \to S^n$, $\text{U}(n-1) \to \text{U}(n) \to S^{2n-1}$ and $\text{Sp}(n-1) \to \text{Sp}(n) \to S^{4n-1}$ (here Sp denotes the compact group of norm-preserving automorphisms of quaternionic space, and not the noncompact symplectic group as above). Thus in every case the inclusions $\text{O}(n-1) \to \text{O}(n)$, $\text{U}(n-1) \to \text{U}(n)$ and $\text{Sp}(n-1) \to \text{Sp}(n)$ induce isomorphisms on homotopy groups in dimensions small compared to n. The limiting spaces $\text{O}(\infty)$, $\text{U}(\infty)$, $\text{Sp}(\infty)$ thus have well-defined homotopy groups. The *Bott Periodicity Theorem* says that these spaces have periodic homotopy groups; in fact, $\Omega^2\text{U}(\infty) \simeq \text{U}(\infty)$ while $\Omega^4\text{O}(\infty) \simeq \text{Sp}(\infty)$ and $\Omega^4\text{Sp}(\infty) \simeq \text{O}(\infty)$.

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