$\Sigma \subset \mathbb{R}^3$ smooth surface, $p_0 \in \Sigma$

$\exists$ open $U \subset \mathbb{R}^3$, $p_0 \in U$, s.t. there is a unit normal vector field $N : \Sigma \cap U \to \mathbb{R}^3$ (e.g. if $\Sigma \cap U = \text{Graph}\, u$, let $N(p) = N_u(x,y)$, where $p = \varphi(x,y)$) which is differentiable everywhere.

$N_u(x,y) = \left(-\frac{\partial u}{\partial x}, -\frac{\partial u}{\partial y}, 1\right)$

CLAIM: For any $p \in \Sigma \cap U$ and $v \in T_p\Sigma$, it holds $DN(p)(v) \in T_p\Sigma$. $\quad DN(p) : T_p\Sigma \to \mathbb{R}^3$

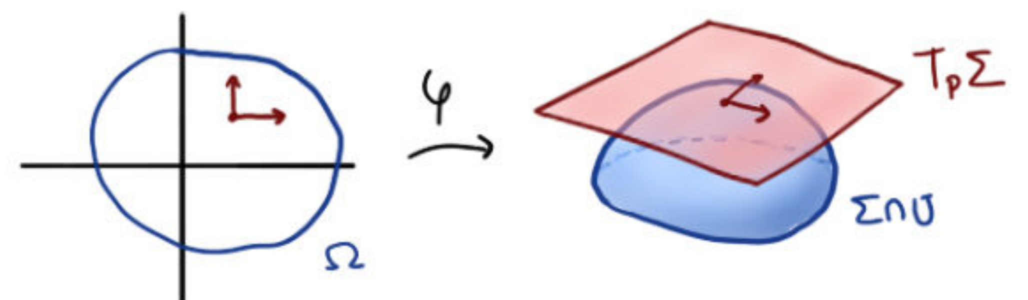
Pf: For $\gamma : (-\delta, \delta) \to \Sigma$ s.t. $\gamma(0) = p$ and $\gamma'(0) = v$, we have

$$\langle \underline{DN(p)(v)}, N(p) \rangle = \left\langle \frac{d}{dt} N(\gamma(t)), N(\gamma(t)) \right\rangle \Big|_{t=0} = \frac{1}{2} \frac{d}{dt}\Big|_{t=0} \cancel{\|N(\gamma(t))\|^2}^{=1} = 0 \quad \blacksquare$$

The second fundamental form of Σ at p is the bilinear map $\alpha_p : T_p\Sigma \times T_p\Sigma \to \mathbb{R}$ given by

$$\alpha_p(v,w) = -\langle \underline{DN(p)v}, w \rangle$$

CLAIM: α is symmetric, that is $\alpha_p(v,w) = \alpha_p(w,v)$.



Recall that $T_p\Sigma = D\varphi(p')(\mathbb{R}^2)$, where $\varphi(p') = p$. If $X_1, X_2 : \Omega \to \mathbb{R}^3$ are

$$X_1(x,y) = \frac{\partial \varphi}{\partial x}(x,y) = D\varphi(x,y)(e_1) \quad \text{and} \quad X_2(x,y) = \frac{\partial \varphi}{\partial y}(x,y) = D\varphi(x,y)\,e_2,$$

(and similarly for $\varphi(x,z)$ and $\varphi(y,z)$)

then $X_1(p'), X_2(p')$ span $T_p\Sigma \perp N(p)$. Hence $\langle (N\circ\varphi), X_1 \rangle \equiv 0 \equiv \langle (N\circ\varphi), X_2 \rangle$ and

$$0 = \frac{\partial}{\partial x} \langle (N\circ\varphi), X_2 \rangle \Big|_{p'} = \left\langle \frac{\partial}{\partial x}(N\circ\varphi)(p'), \frac{\partial \varphi}{\partial y}(p') \right\rangle + \left\langle (N\circ\varphi)(p'), \frac{\partial^2 \varphi}{\partial x \partial y}(p') \right\rangle$$

$$= \underbrace{\langle DN(p)(X_1(p')), X_2(p') \rangle}_{= -\alpha_p(X_1(p'), X_2(p'))} + \left\langle (N\circ\varphi)(p'), \frac{\partial^2 \varphi}{\partial x \partial y}(p') \right\rangle$$

Similarly,

$$0 = -\alpha_p(X_2(p'), X_1(p')) + \left\langle (N\circ\varphi)(p'), \frac{\partial^2 \varphi}{\partial y \partial x}(p') \right\rangle$$

Since $\frac{\partial^2 \varphi}{\partial x \partial y} = \frac{\partial^2 \varphi}{\partial y \partial x}$ (φ is smooth), we conclude that $\alpha_p(X_1(p'), X_2(p')) = \alpha_p(X_2(p'), X_1(p'))$. $\blacksquare$

Consequently, $A_p = -DN(p) : T_p\Sigma \to T_p\Sigma$ is a symmetric linear map. Its eigenvalues $k_1(p), k_2(p) \in \mathbb{R}$ are called the principal curvatures of Σ at p.

DEFINITION: The mean curvature and the Gaussian curvature of Σ at p are defined by

$$H_\Sigma(p) := \operatorname{tr}(A_p) = \underset{-k_1}{k_1(p)} + \underset{-k_2}{k_2(p)} \quad \text{and} \quad K_\Sigma(p) := \underset{(-k_1)(-k_2)}{k_1(p)k_2(p)} = \det(A_p)$$

RMK: H_Σ is defined everywhere up to sign only (corresponding to the choices N and $-N$), but K_Σ is always defined everywhere.

EXAMPLE: If $\Sigma \subset \mathbb{R}^3$ is a plane, then $N : \Sigma \to \mathbb{R}^3$ is constant, so $DN(p) \equiv 0$, so $K_\Sigma \equiv 0$ and $H_\Sigma \equiv 0$.

EXERCISE: Show that for all bases $\{v, w\}$ for $T_p\Sigma$, it holds

$$v \wedge A_p(w) - w \wedge A_p(v) = H_\Sigma \cdot v \wedge w$$

$v \wedge k_2 w - w \wedge k_1 v = (k_1 + k_2)\, v \wedge w$

$v \times w$

Hint: prove it for a o.n. basis $\{e_1, e_2\}$ made of eigenvectors for A_p; now write v and w as linear combinations of e_1 and e_2.

EXERCISE: Let $\Sigma = \partial B_r(p_0) \subset \mathbb{R}^3$, w/ the global unit normal $N(p) = -\frac{1}{r}(p-p_0)$. Show that

$$DN(p)(v) = -\frac{1}{r}v, \quad \forall v \in T_p\Sigma.$$

$$[DN(p)] = \begin{pmatrix} -1/r & \\ & -1/r \end{pmatrix}$$

Conclude that

$$K_\Sigma \equiv \frac{1}{r^2} \qquad \text{and} \qquad H_\Sigma \equiv \frac{2}{r}$$

$\square$

EXAMPLE: Let $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ be a smooth function, and let $\Sigma = \text{Graph}_u$. From

$$N(p', u(p')) = N_u(p') = \frac{\left(-\frac{\partial u}{\partial x}(p'), -\frac{\partial u}{\partial y}(p'), 1\right)}{\left|\left(-\frac{\partial u}{\partial x}(p'), -\frac{\partial u}{\partial y}(p'), 1\right)\right|}, \quad p' \in \Omega$$

we get

$$DN(p', u(p'))\left(1, 0, \frac{\partial u}{\partial x}(p')\right) = \frac{\partial N_u}{\partial x}(p') \quad \text{and} \quad DN(p', u(p'))\left(0, 1, \frac{\partial u}{\partial y}(p')\right) = \frac{\partial N_u}{\partial y}(p'), \quad \forall p' \in \Omega$$

From that, one computes $(DN(p', u(p')))$ to obtain

$$K_\Sigma = \frac{\frac{\partial^2 u}{\partial x^2}\cdot\frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial y \partial x}\right)^2}{(1+|\nabla u|^2)^2} \qquad \text{and} \qquad H_\Sigma = \frac{\left(1+\left(\frac{\partial u}{\partial x}\right)^2\right)\frac{\partial^2 u}{\partial y^2} + \left(1+\left(\frac{\partial u}{\partial y}\right)^2\right)\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y \partial x}}{(1+|\nabla u|^2)^{3/2}}$$

DEFINITION: We say that a surface $\Sigma \subset \mathbb{R}^3$ is a minimal surface if $H_\Sigma \equiv 0$.

Q: What is the relation between the mean curvature H_Σ of a surface $\Sigma \subset \mathbb{R}^3$ and the area functional?

3. EXAMPLES OF MINIMAL SURFACES (PRESENTATION) AND 1ST VARIATION FORMULA

We will suppose hereafter that $\overline{\Sigma}$ is compact, and Σ is orientable.

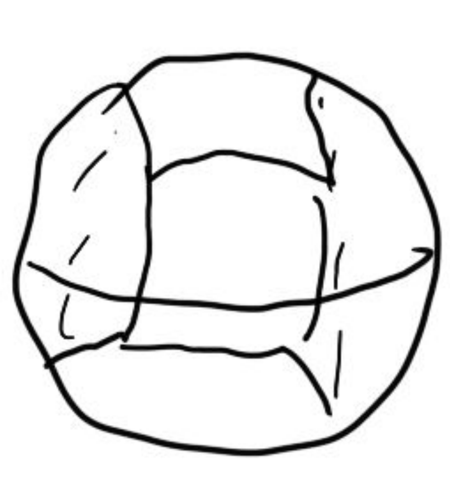
$$\int_\Omega \sqrt{1+|\nabla u|^2} = \int_{\Omega'} \sqrt{1+|\nabla \tilde{u}|^2}$$

We have already defined $\text{Area}(\Sigma)$ when $\Sigma = \text{Graph}_u$ (and this can be naturally extended to graphs over the (x,z) and (y,z) planes). Using the change of variables for integrals in $\mathbb{R}^2$, one checks that if $\Sigma \cap U$ is contained in the image of two distinct parametrizations, then the corresp. local definitions of $\text{Area}(\Sigma \cap U)$ agree. Hence,

$\text{Area}(\Sigma)$ is well-defined, and it can be computed by covering Σ with small domains.

More generally, for a continuous function $f : \Sigma \to \mathbb{R}$, we can make sense of $\int_\Sigma f$ (locally) as

$$\int_\Sigma f = \int_\Omega f \underbrace{\left|\frac{\partial \varphi}{\partial x} \wedge \frac{\partial \varphi}{\partial y}\right|}_{\text{independent of } \varphi} dx\,dy = \int_\Omega \underset{\underset{1}{\shortparallel}}{f}(x,y,u(x,y))\sqrt{1+|\nabla u(x,y)|^2}\,dx\,dy.$$



for a local parametrization $\varphi : \Omega \to \Sigma$, and patch these expressions over Σ.

EXERCISE: Show that $\text{Area}(\partial B_r(p_0)) = 4\pi r^2$.

$X:\Sigma\to\mathbb{R}^3$ be a smooth vector field s.t. $\underbrace{\overline{\{p\in\Sigma \mid X(p)\neq 0\}}}_{=\operatorname{supp}X}$ is compact. $\qquad\overline{\Sigma}$ compact

One checks (see exercise below :)) that $\exists\,\varepsilon>0$ s.t. for all $t\in(-\varepsilon,\varepsilon)$ the set

$$\Sigma_t=\{p+tX(p)\mid p\in\Sigma\}\qquad \Sigma_0=\Sigma$$

is a smooth surface (note that $\Sigma_0=\Sigma$). We say that $\{\Sigma_t\}_{t\in(-\varepsilon,\varepsilon)}$ is a variation of Σ.
It is said to be a normal variation if X is a normal vector field, that is $X(p)\perp T_p\Sigma$, $\forall p\in\Sigma$.

EXERCISE: Show that if $\varphi:\Omega\to\Sigma$ is a parametrization for Σ, then $\varphi_t=\varphi+t(X\circ\varphi)$ is a parametrization for Σ_t. In particular,

$$\operatorname{Area}(\varphi_t(\Omega)\cap\Sigma)=\int_\Omega \boxed{\Big|\underbrace{\Big(\frac{\partial\varphi}{\partial x}+t\,\frac{\partial X}{\partial x}\Big)}_{\frac{\partial\varphi_t}{\partial x}}\wedge\underbrace{\Big(\frac{\partial\varphi}{\partial y}+t\,\frac{\partial X}{\partial y}\Big)}_{\frac{\partial\varphi_t}{\partial y}}\Big|}$$

$t\in(-\varepsilon,\varepsilon)\mapsto \operatorname{Area}(\Sigma_t)$

$d\operatorname{Area}(\Sigma)(X)$

THM (1st variation formula):
$$\frac{d}{dt}\Big|_{t=0}\operatorname{Area}(\Sigma_t)=-\int_\Sigma H_\Sigma\langle X,N_\Sigma\rangle$$

In particular, for a normal variation of the form $X=f\cdot N$, for some function f on Σ, we have $\qquad f=\langle X,N\rangle$

$$\frac{d}{dt}\Big|_{t=0}\operatorname{Area}(\Sigma_t)=-\int_\Sigma f\cdot H_\Sigma$$

Consequently,

Σ is minimal iff $\frac{d}{dt}\big|_{t=0}\operatorname{Area}(\Sigma_t)=0$ for all (compactly supported) normal variations of Σ

Sketch of pf (for normal variations): With the notation from the previous exercise,

$$\Big|\frac{\partial\varphi_t}{\partial x}\wedge\frac{\partial\varphi_t}{\partial y}\Big|=\Big|\underbrace{\frac{\partial\varphi}{\partial x}\wedge\frac{\partial\varphi}{\partial y}}+t\underbrace{\Big(\frac{\partial\varphi}{\partial x}\wedge\frac{\partial X}{\partial y}-\frac{\partial\varphi}{\partial y}\wedge\frac{\partial X}{\partial y}\Big)}+t^2\underbrace{\frac{\partial X}{\partial x}\wedge\frac{\partial X}{\partial y}}\Big|$$

Hence

$$\frac{d}{dt}\Big|_{t=0}\Big|\frac{\partial\varphi_t}{\partial x}\wedge\frac{\partial\varphi_t}{\partial y}\Big|=\frac{\Big\langle\frac{\partial\varphi}{\partial x}\wedge\frac{\partial\varphi}{\partial y},\ \frac{\partial\varphi}{\partial x}\wedge\frac{\partial X}{\partial y}-\frac{\partial\varphi}{\partial y}\wedge\frac{\partial X}{\partial y}\Big\rangle}{\Big|\frac{\partial\varphi}{\partial x}\wedge\frac{\partial\varphi}{\partial y}\Big|}$$

But

$$N=\frac{\frac{\partial\varphi}{\partial x}\wedge\frac{\partial\varphi}{\partial y}}{\Big|\frac{\partial\varphi}{\partial x}\wedge\frac{\partial\varphi}{\partial y}\Big|},\qquad \frac{\partial X}{\partial x}=\frac{\partial}{\partial x}(f\cdot N)=\frac{\partial f}{\partial x}N+f\cdot\frac{\partial N}{\partial x}=\frac{\partial f}{\partial x}N-f\,A\Big(\frac{\partial\varphi}{\partial x}\Big)$$

$$\frac{\partial X}{\partial y}=\frac{\partial f}{\partial y}N-f\cdot A\Big(\frac{\partial\varphi}{\partial y}\Big)$$

So, using another previous exercise

leads to vector $\perp N$

$$\frac{d}{dt}\Big|_{t=0}\Big|\frac{\partial\varphi_t}{\partial x}\wedge\frac{\partial\varphi_t}{\partial y}\Big|=\Big\langle N,\frac{\partial\varphi}{\partial x}\wedge\Big(\cancel{\frac{\partial f}{\partial y}N}-f\,A\Big(\frac{\partial\varphi}{\partial y}\Big)\Big)-\frac{\partial\varphi}{\partial y}\wedge\Big(\cancel{\frac{\partial f}{\partial x}N}-f\,A\Big(\frac{\partial\varphi}{\partial x}\Big)\Big)\Big\rangle$$

$$=-f\Big\langle N,\frac{\partial\varphi}{\partial x}\wedge A\Big(\frac{\partial\varphi}{\partial y}\Big)-\frac{\partial\varphi}{\partial y}\wedge A\Big(\frac{\partial\varphi}{\partial x}\Big)\Big\rangle$$

$$=-f\Big\langle N,H_\Sigma\underbrace{\Big(\frac{\partial\varphi}{\partial x}\wedge\frac{\partial\varphi}{\partial y}\Big)}_{N\cdot\left|\frac{\partial\varphi}{\partial x}\wedge\frac{\partial\varphi}{\partial y}\right|}\Big\rangle=-\,\underbrace{f\cdot H_\Sigma\Big|\frac{\partial\varphi}{\partial x}\wedge\frac{\partial\varphi}{\partial y}\Big|}_{\text{appears in the def. of }\int_\Sigma f\cdot H_\Sigma}$$

Now integrate over the surface. $\square$