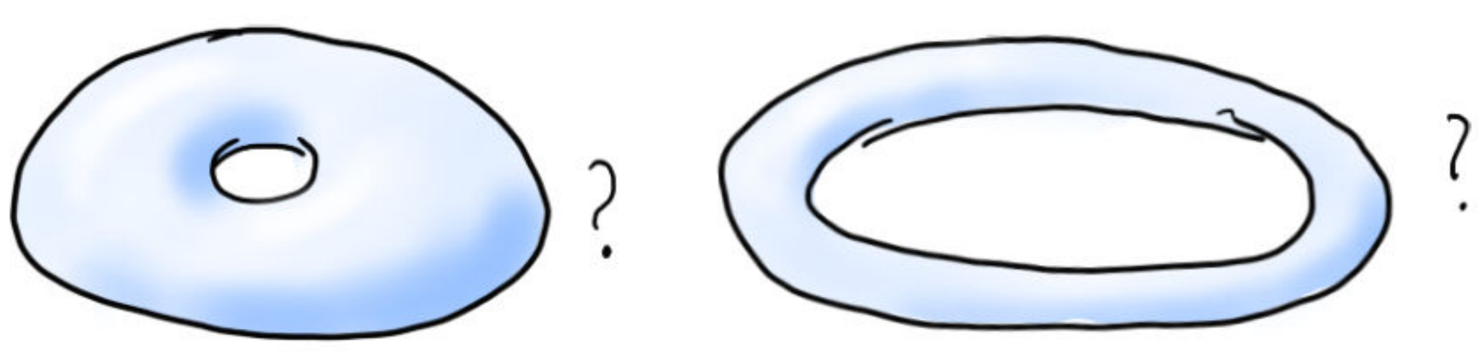
Minimal surfaces in $\mathbb{R}^3$ and Bernstein's Thm (U of Chicago REU 2020)

"A Course in Minimal Surfaces", T. Colding and W. Minicozzi
"Differential Geometry of Curves and Surfaces", M.P. do Carmo

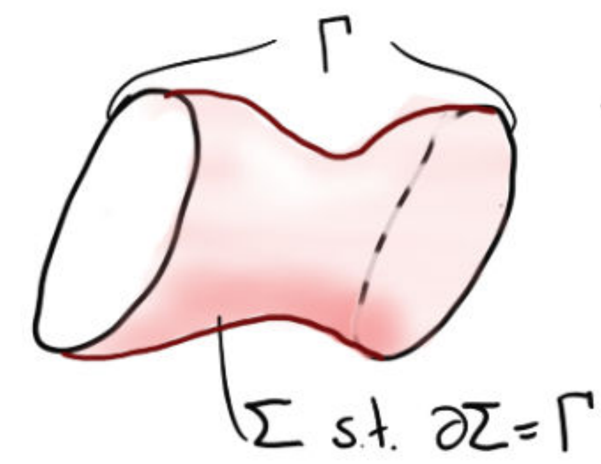
Introduction

Core problem: Finding the best geometric objects.

Q1: What is the best torus (doughnut) in $\mathbb{R}^3$?



Q2: What is the best surface spanning a given contour $\Gamma \subset \mathbb{R}^3$?



Plateau's problem
- Lagrange, 1760
- J. Douglas, T. Radó, '30s

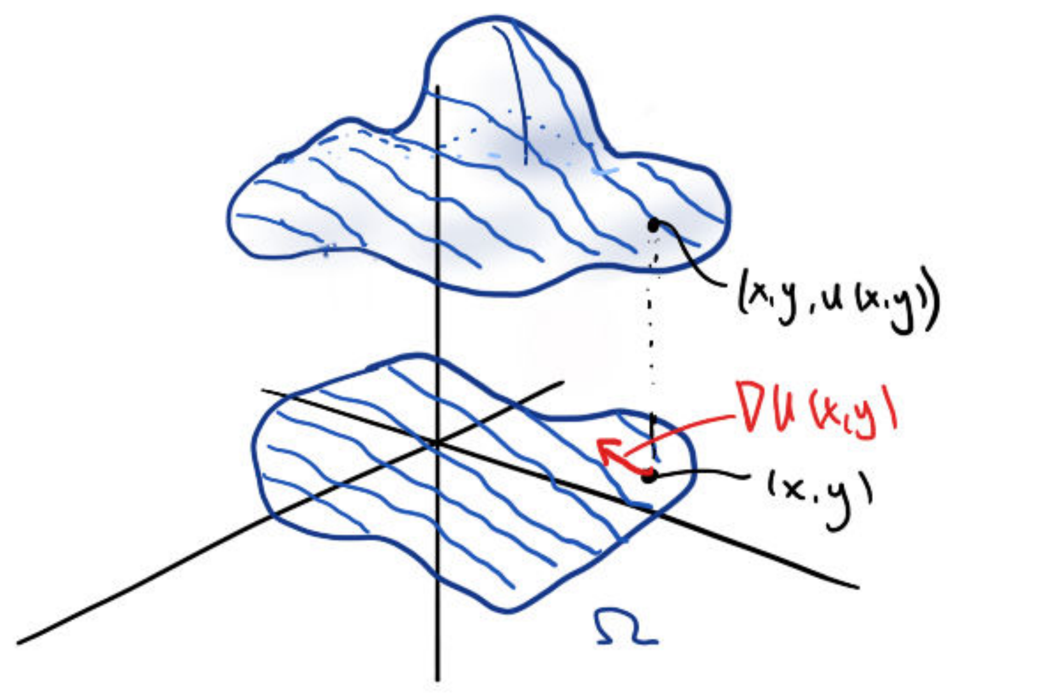
- Geometric quantities: area, volume, curvature
- Candidates: minima/maxima or critical points of certain functionals.

1. Minimal Graphs

$\Omega \subset \mathbb{R}^2$ connected, $\bar{\Omega}$ compact, $\partial\Omega$ nice regular curve

The graph of a function $u : \Omega \to \mathbb{R}^2$ is the set

$$\mathrm{Graph}_u = \{(x, y, u(x,y)) : (x,y) \in \Omega\} \subset \mathbb{R}^3$$



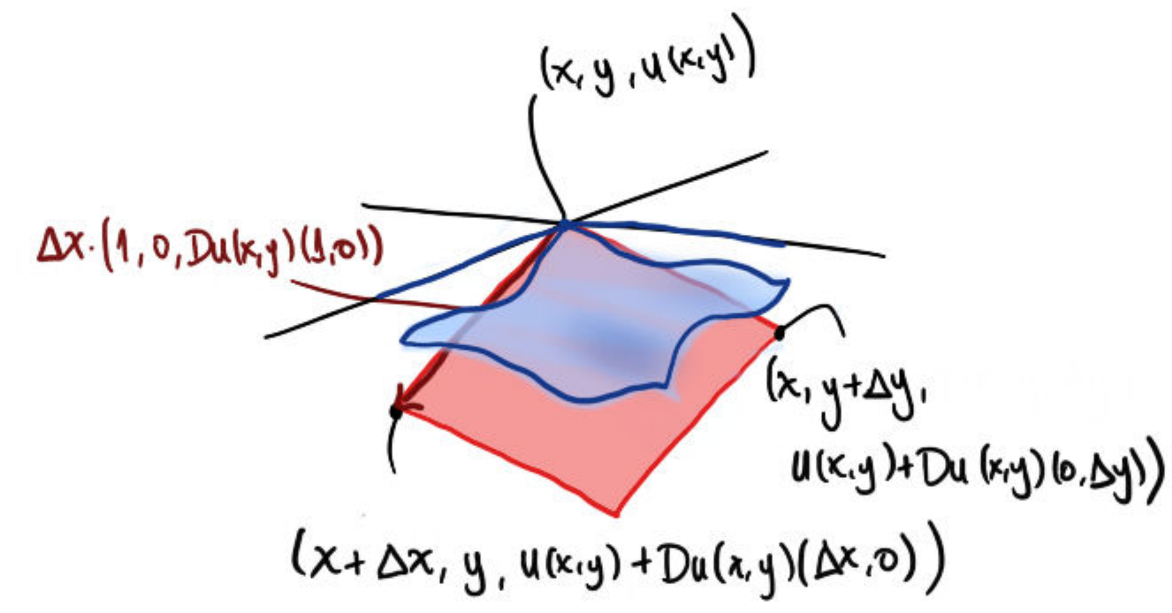
If u is differentiable, we denote by $\nabla u : \Omega \to \mathbb{R}^2$ its gradient is

$$\nabla u(x,y) = \left(\frac{\partial u}{\partial x}(x,y), \frac{\partial u}{\partial y}(x,y)\right)$$

The area of Graph_u is the nonnegative number

$$\mathrm{Area}(\mathrm{Graph}_u) = \int_\Omega \sqrt{1 + |\nabla u(x,y)|^2}\, dx\, dy$$

(if the integral exists, and it is finite)



Example: If $u : \Omega = [a,b] \times [c,d] \to \mathbb{R}$ is constant, w/ $u(x,y) = c$, then $\nabla u \equiv \vec{0}$ and

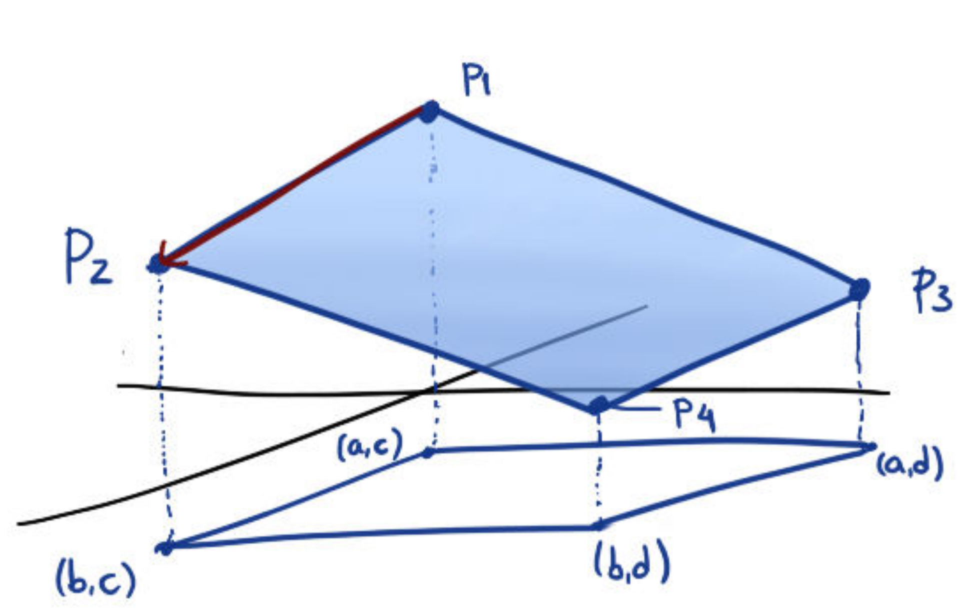
$$\mathrm{Area}(\mathrm{Graph}_u) = \int_\Omega 1\, dx\, dy = \mathrm{Area}(\Omega) = (b-a)(d-c).$$

Example: If $u : \Omega = [a,b] \times [c,d] \to \mathbb{R}$ is $u(x,y) = \alpha x + \beta y + \gamma$, then $\nabla u \equiv (\alpha, \beta)$ and

$$\begin{aligned}\mathrm{Area}(\mathrm{Graph}_u) &= \int_\Omega \sqrt{1+\alpha^2+\beta^2}\, dx\, dy = \sqrt{1+\alpha^2+\beta^2}\,(b-a)(d-c) \\ &= (b-a)(d-c)\,|(-\alpha,-\beta,1)| \overset{\text{Check}}{=} |(b-a, 0, \alpha(b-a)) \wedge (0, d-c, \beta(d-c))| \\ &= \text{Area of parallelogram of vertices } p_1, p_2, p_3, p_4,\end{aligned}$$

where

$$\begin{aligned} p_1 &= (a, c, u(a,c)), \\ p_2 &= (b, c, u(b,c)) = p_1 + (b-a, 0, \underbrace{\alpha}_{=\frac{\partial u}{\partial x}(a,c)}(b-a)) \\ p_3 &= (a, d, u(a,d)) = p_1 + (0, d-c, \beta(d-c)) \\ p_4 &= (b, d, u(b,d)) = p_1 + (b-a, 0, \alpha(b-a)) + (0, d-c, \beta(d-c)) \end{aligned}$$



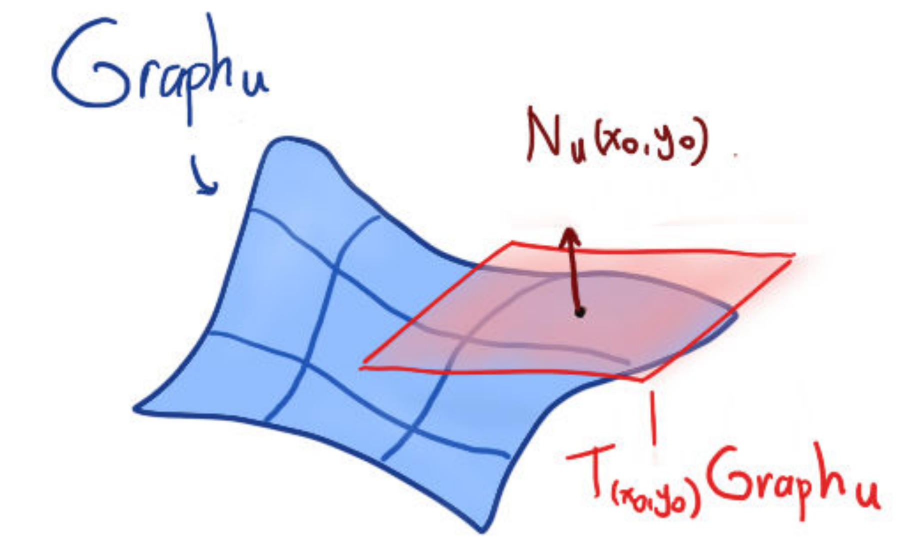
EXERCISE: $B_1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\}$, $u: B_1 \to \mathbb{R}$, $u(x,y) = \sqrt{1-x^2-y^2}$

(a) Draw the graph of u. (b) Show that $\text{Area}(\text{Graph}_u) = 2\pi$ $\square$

Recall that the tangent space to Graph_u at $(x_0, y_0, u(x_0,y_0))$ is

$$T_{(x_0,y_0)}\text{Graph}_u = \{v \in \mathbb{R}^3 \mid \underbrace{\langle v, (-\tfrac{\partial u}{\partial x}(x_0,y_0), -\tfrac{\partial u}{\partial y}(x_0,y_0), 1)\rangle}_{= -v_1 \frac{\partial u}{\partial x}(x_0,y_0) - v_2 \frac{\partial u}{\partial y}(x_0,y_0) + v_3} = 0\}$$

$Du(x,y): \mathbb{R}^2 \to \mathbb{R}$



and the (upward pointing) unit normal is

$$N_u(x_0,y_0) = \frac{(-\frac{\partial u}{\partial x}(x_0,y_0), -\frac{\partial u}{\partial y}(x_0,y_0), 1)}{\left|(-\frac{\partial u}{\partial x}(x_0,y_0), -\frac{\partial u}{\partial y}(x_0,y_0), 1)\right|} = \frac{(-\frac{\partial u}{\partial x}(x_0,y_0), -\frac{\partial u}{\partial y}(x_0,y_0), 1)}{\sqrt{1+|\nabla u|^2}}$$

Q: What are the critical points of $(u: \Omega \to \mathbb{R}) \mapsto \text{Area}(\text{Graph}_u)$?

$f: \bar{\Omega} \subset \mathbb{R}^2 \to \mathbb{R}$ differentiable function w/ $f|_{\partial\Omega} \equiv 0$.

We know

$$\text{Area}(\text{Graph}_{u+tf}) = \int_\Omega \sqrt{1 + |\underbrace{\nabla(u+tf)}_{=\nabla u + t\nabla f}|^2}\, dx\,dy = \int_\Omega \sqrt{1+|\nabla u|^2 + 2t\langle \nabla u, \nabla f\rangle + t^2|\nabla f|^2}\, dx\,dy$$

so

$$\frac{d}{dt}\Big|_{t=0} \text{Area}(\text{Graph}_{u+tf}) = \int_\Omega \frac{d}{dt}\Big|_{t=0} \sqrt{1+|\nabla u|^2 + 2t\langle \nabla u, \nabla f\rangle + t^2|\nabla f|^2}\, dx\,dy$$

$$\underset{\text{check}}{=} \int_\Omega \frac{\langle \nabla u, \nabla f\rangle + t|\nabla f|^2}{\sqrt{1+|\nabla u + t\nabla f|^2}}\Bigg|_{t=0} dx\,dy \overset{(*)}{=} \int_\Omega \left\langle \nabla f, \frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right\rangle dx\,dy$$

(*): inner product is linear

$\langle \nabla f, \vec{F}_u \rangle$

RECALL: $\vec{F}: \Omega \to \mathbb{R}^2$ is a differentiable vector field, $\vec{F} = (F_1, F_2)$

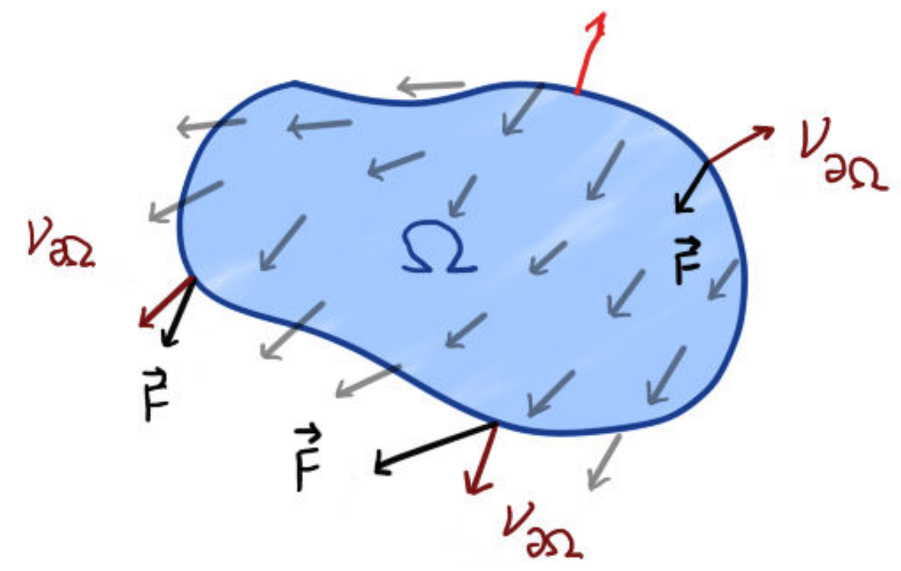
The divergence of $\vec{F}$ is the function $\text{div}\,\vec{F}: \Omega \to \mathbb{R}$ given by

$$\text{div}\,\vec{F}(x,y) = \frac{\partial F_1}{\partial x}(x,y) + \frac{\partial F_2}{\partial y}(x,y) \quad \left(= \text{tr}(D\vec{F}(x,y))\right)$$

The Divergence (or Gauß) Theorem says

$$\int_\Omega \text{div}\,\vec{F}\, dx\,dy = \int_{\partial\Omega} \langle \nu_{\partial\Omega}, \vec{F}\rangle\, ds$$

$\nu_{\partial\Omega}$: outward pointing normal vector



EXERCISE: Show that if $f: \Omega \to \mathbb{R}$ and $\vec{F}: \Omega \to \mathbb{R}$ are differentiable, then

$$\text{div}(f\vec{F}) = \langle \nabla f, \vec{F}\rangle + f \cdot \text{div}\,\vec{F}$$ $\square$

Thus,

$$\frac{d}{dt}\Big|_{t=0} \text{Area}(\text{Graph}_{u+tf}) = \int_\Omega \left[\text{div}\left(f \frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) - f \cdot \text{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)\right] dx\,dy$$

$$= \int_{\partial\Omega} \left\langle \nu_{\partial\Omega}, \underset{0 \text{ along } \partial\Omega}{\cancel{f}} \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right\rangle ds - \int_\Omega f \cdot \text{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) dx\,dy$$

and Graph u is a critical point of Area if u satisfies the minimal surface equation

$$\text{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \frac{\partial}{\partial x}\left(\frac{\frac{\partial u}{\partial x}}{\sqrt{1+|\nabla u|^2}}\right) + \frac{\partial}{\partial y}\left(\frac{\frac{\partial u}{\partial y}}{\sqrt{1+|\nabla u|^2}}\right) = 0 \quad \text{in } \Omega$$

Exercise: Show that this equation is equivalent to

$$\left(1+\left(\frac{\partial u}{\partial x}\right)^2\right)\frac{\partial^2 u}{\partial x^2}+\left(1+\left(\frac{\partial u}{\partial y}\right)^2\right)\frac{\partial^2 u}{\partial y^2}-2\,\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y}\cdot\frac{\partial^2 u}{\partial x\partial y}=0 \qquad \left(\begin{array}{l}\text{quasilinear elliptic}\\ 2^{\text{nd}}\text{ order PDE}\end{array}\right) \qquad \square$$

Such graphs are actually points of minima for the area, among graphs w/ the same values along $\partial\Omega$:

Lemma: If $u:\Omega\to\mathbb{R}$ satisfies the minimal surface equation, then

$$\text{Area}(\text{Graph}_u)\leq\text{Area}(\text{Graph}_v)$$

for any differentiable function $v:\Omega\to\mathbb{R}$ s.t. $v|_{\partial\Omega}=u|_{\partial\Omega}$.

Pf: $\text{Area}(\text{Graph}_v)=\int_\Omega\sqrt{1+|\nabla v|^2}=\int_\Omega\left|\left(-\frac{\partial v}{\partial x},-\frac{\partial v}{\partial y},1\right)\right|\cdot\overbrace{|N_u|}^{=1}\,dx\,dy$

$$\geq\int_\Omega\left\langle\left(-\frac{\partial v}{\partial x},-\frac{\partial v}{\partial y},1\right),\frac{\left(-\frac{\partial u}{\partial x},-\frac{\partial u}{\partial y},1\right)}{\sqrt{1+|\nabla u|^2}}\right\rangle dx\,dy \qquad \text{(by Cauchy-Schwarz)}$$

$$=\int_\Omega\Big\langle\nabla v,\underbrace{\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}}_{=:\vec{X}_u}\Big\rangle dx\,dy+\int_\Omega\frac{1}{\sqrt{1+|\nabla u|^2}}\,dx\,dy$$

But

$$\int_\Omega\langle\nabla v,\vec{X}_u\rangle\,dx\,dy=\int_\Omega \text{div}(v\vec{X}_u)-v\cdot\overbrace{\cancel{\text{div}(\vec{X}_u)}}^{=0\text{, by minimal surface eq.}}\,dx\,dy$$

$$=\int_{\partial\Omega}\langle\vec{\nu}_{\partial\Omega},\underbrace{v}_{=u\text{ along }\partial\Omega}\cdot\vec{X}_u\rangle\,ds \qquad \text{by Divergence Thm}$$

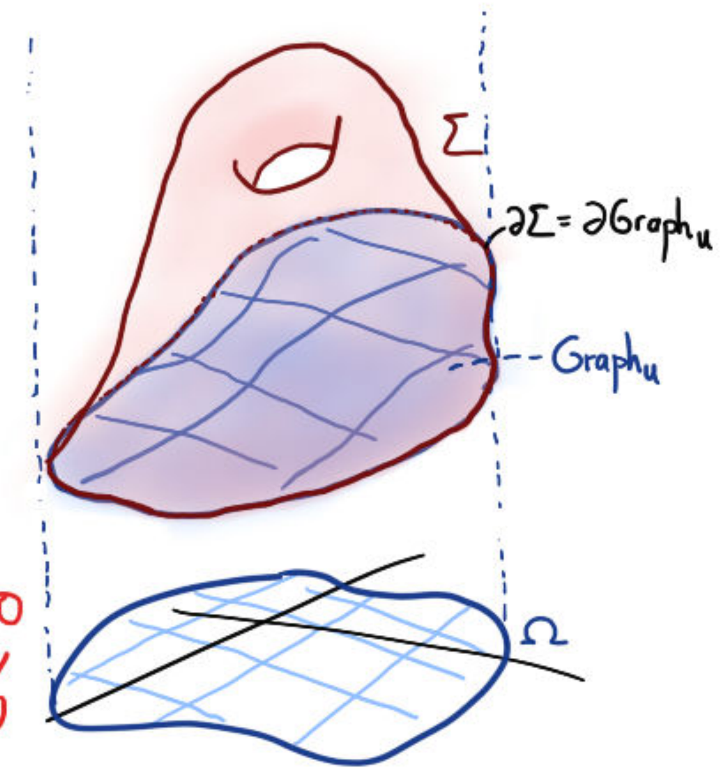
$$=\int_{\partial\Omega}\langle\vec{\nu}_{\partial\Omega},u\vec{X}_u\rangle\,ds=\int_\Omega\langle\nabla u,\vec{X}_u\rangle\,dx\,dy=\int_\Omega\frac{|\nabla u|^2}{\sqrt{1+|\nabla u|^2}}\,dx\,dy$$

Therefore,

$$\text{Area}(\text{Graph}_v)\geq\int_\Omega\frac{|\nabla u|^2}{\sqrt{1+|\nabla u|^2}}\,dx\,dy+\int_\Omega\frac{1}{\sqrt{1+|\nabla u|^2}}\,dx\,dy=\text{Area}(\text{Graph}_u). \qquad \blacksquare$$

RMK: ⚠ The same conclusion holds true if $\Sigma\subset\Omega\times\mathbb{R}$ is any surface s.t. $\partial\Sigma=\partial\,\text{Graph}_u$. We replace the integrand $\langle(-\frac{\partial v}{\partial x},-\frac{\partial v}{\partial y},1),N_u\rangle$ in the proof above by the 2-form ω on $\Omega\times\mathbb{R}$ given by

$$\omega_{(x,y,z)}(X,Y)=\langle X\wedge Y,N_u(x,y)\rangle=\det(X,Y,N_u(x,y))$$



One shows that

$$\omega_{(x,y,z)}=\frac{dx\wedge dy-\frac{\partial u}{\partial x}\,dy\wedge dz-\left(\frac{\partial u}{\partial y}\right)dz\wedge dx}{\sqrt{1+|\nabla u(x,y)|^2}}$$

$$\int_{\text{Graph}_u}\omega-\int_\Sigma\omega=\int_R d\omega=0$$

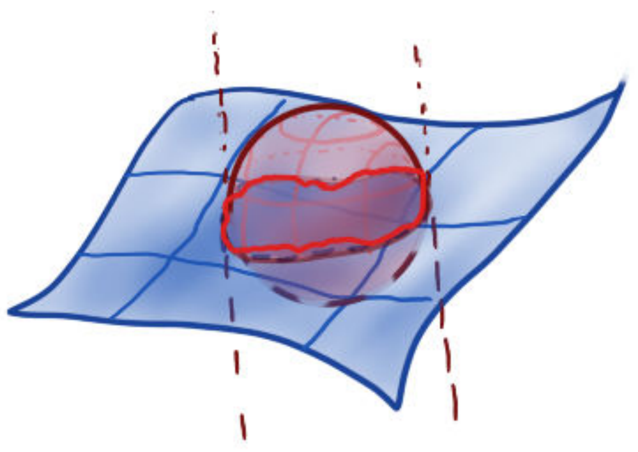
By the minimal surface equation, we get $d\omega=0$ on $\Omega\times\mathbb{R}$. Furthermore, $|\omega(X,Y)|\leq|X\wedge Y|$ (as $|N_u|=1$)

and $\omega\left((1,0,\frac{\partial u}{\partial x}),(0,1,\frac{\partial u}{\partial y})\right)=\sqrt{1+|\nabla u|^2}$, so

$$\text{Area}(\text{Graph}_u)=\int_\Omega\omega\left((1,0,\tfrac{\partial u}{\partial x}),(0,1,\tfrac{\partial u}{\partial y})\right)=\int_{\text{Graph}_u}\omega\underbrace{=}_{\text{Stokes Thm on the region between graphs}}\int_\Sigma\omega\leq\int_\Sigma 1=\text{Area}(\Sigma).$$

We say ω is a calibration.

Exercise: Using this generalized version of the Lemma, show that if $B_r(p_0)=\{p\in\mathbb{R}^2 \mid |p-p_0|<r\}\subset\Omega$, then



$$\text{Area}\left((B_r(p_0)\times\mathbb{R})\cap\text{Graph}_u\right)\leq\tfrac{1}{2}\text{Area}(S^2)\,r^2=2\pi r^2.$$

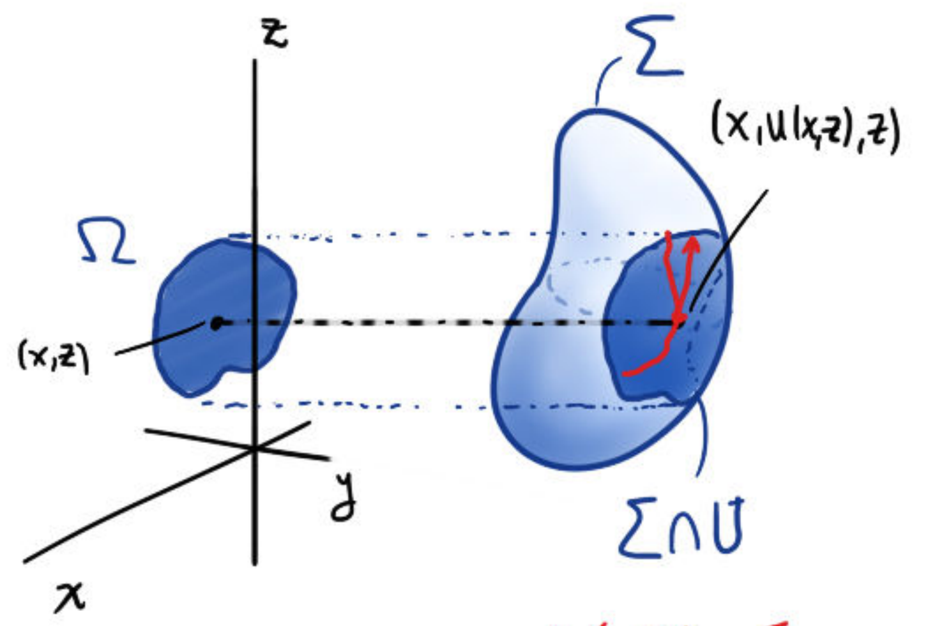
2. Geometry of Surfaces in $\mathbb{R}^3$.

Definition: $\Sigma \subset \mathbb{R}^3$ is a (smooth) surface if it is locally a graph over one of the coord. planes. This means $\forall p \in \Sigma$, $\exists$ open sets $U \subset \mathbb{R}^3$ and $\Omega \subset \mathbb{R}^2$, and a smooth function $u : \Omega \to \mathbb{R}$ s.t. $p \in U$ and $\Sigma \cap U$ is the image of a map $\varphi : \Omega \to \mathbb{R}^3$ (called a parametrization) of the form

(a) $\varphi(x,y) = (x, y, u(x,y))$, (so $\Sigma \cap U = \{(x,y,u(x,y)) \mid (x,y) \in \Omega\} = \mathrm{Graph}\, u$)

(b) $\varphi(x,z) = (x, u(x,z), z)$, or $\{(x,u(x,z),z) \mid (x,z) \in \Omega\}$

(c) $\varphi(y,z) = (u(y,z), y, z)$.



In the cases (b) and (c), we also have a notion of tangent plane, namely

$$\{v \in \mathbb{R}^3 \mid \langle v, (-\tfrac{\partial u}{\partial x}(x_0,z_0), 1, -\tfrac{\partial u}{\partial z}(x_0,z_0))\rangle = 0\} \quad \text{and} \quad \{v \in \mathbb{R}^3 \mid \langle v, (1, -\tfrac{\partial u}{\partial y}(y_0,z_0), -\tfrac{\partial u}{\partial z}(y_0,z_0))\rangle = 0\}$$

Furthermore, if $p_0 = (x_0,y_0,z_0) \in \Sigma$ admits more than one such local description, e.g. $p_0 \in U \subset \mathbb{R}^3$ and

$$\Sigma \cap U = \{(x,y,u(x,y)) \mid (x,y) \in \Omega\} = \{(x, u'(x,z), z) \mid (x,z) \in \Omega'\}$$

for certain functions $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ and $u' : \Omega' \subset \mathbb{R}^2 \to \mathbb{R}$, then these notions of tangent planes agree.

We denote the tangent plane to Σ at p_0 by $T_{p_0}\Sigma$.

Exercise. Prove the statement above by showing that in any of the cases (a), (b) or (c), and for any $v \in \mathbb{R}^3$, the following statements are equivalent:

(i) v lies in the tangent plane to Σ at p_0;

(ii) $v \in D\varphi(p_0')(\mathbb{R}^2)$, where p_0' is either (x_0,y_0), (x_0,z_0) or (y_0,z_0); ($D\varphi(p_0') : \mathbb{R}^2 \to \mathbb{R}^3$ is the total derivative of φ)

(iii) $\exists$ differentiable $\gamma : (-\delta,\delta) \to \mathbb{R}^3$ s.t. $\gamma(t) \in \Sigma$, $\forall t \in (-\delta,\delta)$, $\gamma(0) = p$ and $\gamma'(0) = v$. $\square$

Exercise: Show that for any $p_0 \in \mathbb{R}^3$ and any $r > 0$, the set $\partial B_r(p_0) = \{p \in \mathbb{R}^3 \mid |p - p_0| = r\}$ is a smooth surface, and

$$T_p \partial B_r(p_0) = \{v \in \mathbb{R}^3 \mid \langle v, p - p_0\rangle = 0\}$$

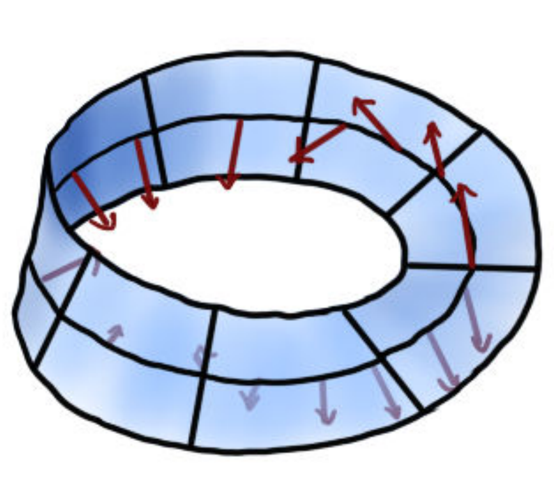
(How many parametrizations do we need to cover $\partial B_r(p_0)$?)

In particular, the unit sphere $S^2 = \{p \in \mathbb{R}^3 \mid |p| = 1\}$ is a smooth surface.

RMK: Similarly, there's a notion of unit normal vector in (b) and (c), namely

$$(-\tfrac{\partial u}{\partial x}, 1, -\tfrac{\partial u}{\partial z}) \quad \text{and} \quad (1, -\tfrac{\partial u}{\partial y}, -\tfrac{\partial u}{\partial z}),$$

but they only agree up to sign (consider the Möbius strip)!



If there is a globally defined continuous unit normal $N : \Sigma \to \mathbb{R}^3$, we'll say Σ is orientable, and such choice of N is an orientation for Σ.

Definition: $f : \Sigma \to \mathbb{R}^k$ is differentiable at $p_0 \in \Sigma$ if $f \circ \varphi$ $: \Omega \to \mathbb{R}^k$ is differentiable for some parametrization $\varphi : \Omega \to \mathbb{R}^3$ s.t. $p_0 \in \varphi(\Omega)$. One checks that this holds iff this is true for any such parametrization. In this case, the derivative of f at p_0 is the linear map $Df(p_0) : T_{p_0}\Sigma \to \mathbb{R}^k$

$v \in T_{p_0}\Sigma$

$$Df(p_0)(v) = D(f \circ \varphi)(p_0')(v_0), \quad \text{where } p_0 = \varphi(p_0') \text{ and } v = D\varphi(p_0')v_0 \quad (T_{p_0}\Sigma = D\varphi(\mathbb{R}^2))$$

Exercise: Show that $Df(p_0)(v)$ does not depend on the choice of φ (so $Df(p_0)$ is well-defined)

Hint: $Df(p_0)(v) = (f \circ \gamma)'(0)$, where $\gamma : (-\delta,\delta) \to \Sigma$ is s.t. $\gamma(0) = p_0$, $\gamma'(0) = v$.