Operads in Cat

I Intro
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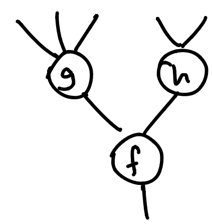
I Intro

In multivariable calculus we consider functions

$$f : \mathbb{R}^n \to \mathbb{R} \qquad f(x_1, \dots, x_n)$$

e.g. $f(x,y) = e^x \cos y$

Compose? $g(a,b,c) = a + 2bc^2$; $h(s,t) = st$



$$f(g(a,b,c), h(s,t)) = e^{a+2bc^2} \cos(st)$$

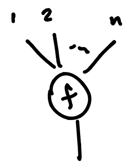
Def An *operad* $\mathcal{O}$ (in Set) consists of

- sets $\mathcal{O}(n)$ for $n \geq 0$
- a (right) Σ_n-action on $\mathcal{O}(n)$
- a special element $\mathbb{1} \in \mathcal{O}(1)$
- a collection of composition maps

$$\gamma : \mathcal{O}(n) \times \mathcal{O}(k_1) \times \dots \times \mathcal{O}(k_n) \longrightarrow \mathcal{O}(k_1 + \dots + k_n)$$

satisfying some conditions.

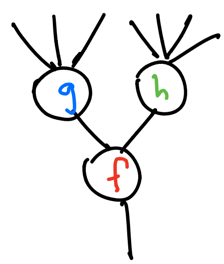
We can depict elements in $\mathcal{O}(n)$ as trees



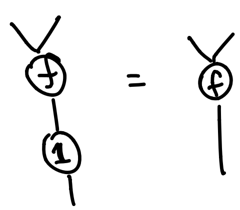
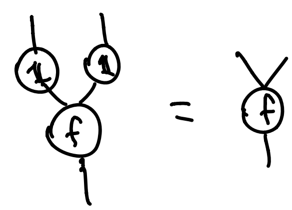
We draw  as 

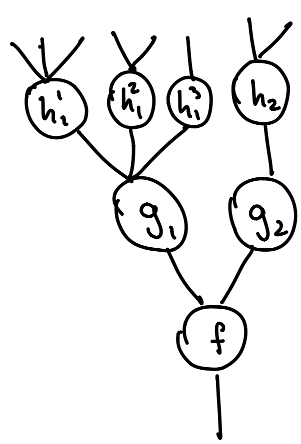
And $\gamma : \underset{f}{\mathcal{O}(2)} \times \underset{g}{\mathcal{O}(3)} \times \underset{h}{\mathcal{O}(4)} \to \mathcal{O}(7)$ (and in general γ)

as



In this graphical language, the conditions become

 ; 



has a unique interpretation

Ex • Let X be a set. $\mathrm{End}(X)$ is the operad with

$$\mathrm{End}(X)(n) = \{f : X^n \to X\}$$

$\mathbb{1} = \mathrm{id}_X : X \to X$ (identity function)
γ = function composition
Σ_n-action : permuting inputs

• Comm with $\mathrm{Comm}(n) = *$

• Assoc with $\mathrm{Assoc}(n) = \Sigma_n$ (action by right mult)

Def Let $\mathcal{O}$ be an operad (in Set). An *$\mathcal{O}$-algebra* consists of

- a set X
- for $n \geq 0$, a map
$$\theta_n : \mathcal{O}(n) \times X^n \longrightarrow X$$

satisfying conditions (wrt Σ_n-action, $\mathbb{1}$, γ)

Note that for each $f \in \mathcal{O}(n)$, we get $\theta(f) : X^n \to X$, ie, an actual n-ary operation on X.

$$\theta(f)(x_1, \ldots, x_n) = \theta_n(f, x_1, \ldots, x_n)$$

Ex Comm-algebra

$$\theta_n : * \times X^n \cong X^n \longrightarrow X$$

including $\theta_0 : * \times X^0 \cong * \longrightarrow X$ (choice of element)
$* \longmapsto e$

Denote $\theta_2(x,y) = x \oplus y$.
The conditions imply: $\theta_1(x) = x$

$$x \oplus (y \oplus z) = \theta_3(x,y,z) = (x \oplus y) \oplus z$$

$$x \oplus e = x = e \oplus x$$

$$x \oplus y = y \oplus x$$

Thm There is a one-to-one correspondence

$$\{\text{Comm-alg}\} \longleftrightarrow \{\text{commutative monoids}\}$$

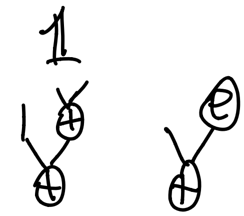
Rmk To go back one needs the following (seemingly trivial) observation:

if $(X, \oplus, e)$ is a commutative monoid, then any n-ary operation constructed by iterating $\oplus$ is equal to

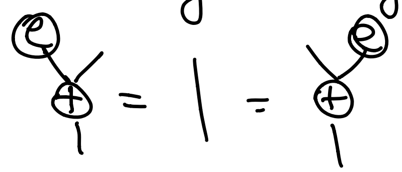
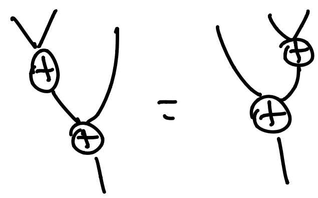
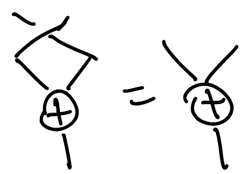
$$(((x_1 \oplus x_2) \oplus x_3) \oplus \cdots) \oplus x_n$$

Reinterpretation Comm is the operad in Set generated by

- a 0-ary operation $e \in \theta(0)$
- a 2-ary operation $\oplus \in \theta(2)$



subject to the relations generated by

- 
- 
- 

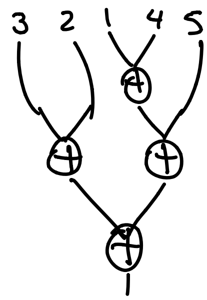
Ex $\{\text{Assoc-alg}\} \longleftrightarrow \{\text{Associative monoids}\}$

$$\theta_n : \underset{\substack{\Downarrow \\ 1_n}}{\Sigma_n} \times X^n \longrightarrow X$$

$\theta_n(1_n) : X^n \longrightarrow X$ gives an n-ary operation,
equal to $(((x_1 \oplus x_2) \oplus x_3) \oplus \cdots) \oplus x_n$, where $\oplus = \theta_2(1_2)$.

Equivariance condition just gives $\theta_n(\sigma)$ is θ_n with inputs reordered, so no new data.

Ex $\mathbb{F}(\oplus)$ = operad freely generated by a 2-ary operation, ie, the elements of $\mathbb{F}(\oplus)(n)$ are given by full binary rooted trees with n leaves, together with a labeling of the leaves



$$\{\mathbb{F}(\oplus)\text{-algebras}\} = \{(X, \oplus)\}$$

X together with a binary operator, no conditions.

II Operads in Cat

Cat = category of small categories and functors.

Def An *operad* $\mathcal{O}$ in Cat consists of

- categories $\mathcal{O}(n)$ for $n \geq 0$
- a (right) Σ_n-action on $\mathcal{O}(n)$
- a special object $\mathbb{1} \in \mathcal{O}(1)$
- a collection of composition functors

$$\gamma : \mathcal{O}(n) \times \mathcal{O}(k_1) \times \cdots \times \mathcal{O}(k_n) \longrightarrow \mathcal{O}(k_1 + \cdots + k_n)$$

Ex Comm, where $\mathrm{Comm}(n) = *$ (category with one obj and only identity morphism)

Assoc, where $\mathrm{Assoc}(n) = \Sigma_n$ (obj given by Σ_n, only identity morphisms)

Q Algebras?

Def Let $\mathcal{O}$ be an operad in Cat. An <u>$\mathcal{O}$-algebra</u> consists of

- a category X
- for $n \geq 0$, a functor

$$\theta_n : \mathcal{O}(n) \times X^n \longrightarrow X$$

satisfying conditions.

Rmk This means that for every $f \in \mathrm{ob}\, \mathcal{O}(n)$, we have a functor

$$\theta_n(f) : X^n \longrightarrow X$$

and for every morphism $\alpha : f \to g$ in $\mathcal{O}(n)$, we get a natural transformation $\theta_n(\alpha) : \theta_n(f) \Rightarrow \theta_n(g)$.

Ex $\{\text{Comm-alg}\} \leftrightarrow \{\text{commutative monoids in Cat}\}$

$\{\text{Asso-alg}\} \leftrightarrow \{\text{associative monoids in Cat}\}$

Ex P Barratt-Eccles operad

$P(n) = \widetilde{\Sigma}_n$ - objects Σ_n
- exactly one morphism between any pair of objects.

Algebras? Since $\operatorname{Ob} P(n) = \operatorname{ob} \mathrm{Assoc}$, we know a P-algebra consists of
- a category X
- an object $e \in X$
- a binary operation (functor) $\oplus : X^2 \to X$

s.t. $(X, \oplus, e)$ is an associative monoid.

The morphisms in P:

In $P(2)$ $\quad 1_2, \sigma \in \Sigma_2$

$$1_2 \xrightarrow[\beta]{!} \sigma$$

This gives rise to a natural transformation:

$$x \oplus y \xrightarrow{\beta_{x,y}} y \oplus x$$

Note that $\sigma \xrightarrow[\beta\sigma]{\beta^{-1}} 1_2$, so

$$\begin{tikzcd} x \oplus y \arrow[r, "\beta_{x,y}"] \arrow[rr, bend right, "id"'] & y \oplus x \arrow[r, "\beta_{y,x}"] & x \oplus y \end{tikzcd} \quad \text{A}$$

In $P(3)$, $\quad \begin{tikzcd} 1_3 \arrow[r] \arrow[dr] & (12) \arrow[d] \\ & (123) \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} x \oplus y \oplus z \arrow[r, bend left, "\beta \oplus id"] \arrow[dr, "\beta"'] & y \oplus x \oplus z \arrow[d, "id \oplus \beta"] \\ & y \oplus z \oplus x \end{tikzcd} \quad \text{B}$

Def A permutative category consists of $(X, \oplus, e, \beta)$ as above.

Thm P-alg $\leftrightarrow$ Permutative categories

Rmk $\leftarrow$ involves both an idea about generation and a coherence result

- Every nonidentity morphism in $P(n)$ can be obtained from $1_2 \to \sigma$ using the Σ_n-action and γ

(follows from the fact that any permutation is a product of transpositions $i \leftrightarrow i+1$)

- In a permutative category, conditions Ⓐ and Ⓑ imply that any diagram involving instances of β commutes (coherence theorem)

Ex Q given by $Q(n) = \widetilde{\mathbb{F}(\oplus, e)}$ $\quad$ $\oplus$ 2-ary, e 0-ary

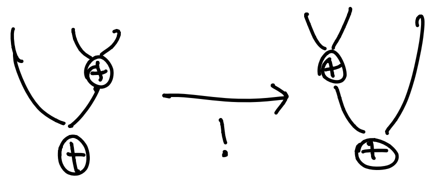
Def A symmetric monoidal category consists of

- a category X
- an object $e \in X$
- a functor $\oplus : X^2 \to X$
- natural isomorphisms

$$\beta: x\oplus y \to y\oplus x \qquad \alpha: x\oplus(y\oplus z) \to (x\oplus y)\oplus z$$
$$\lambda: e\oplus x \to x \qquad \rho: x\oplus e \to x$$

satisfying 5 axioms.

Thm $\{Q\text{-alg}\} \leftrightarrow \{\text{Symm mon cats}\}$



$\rightsquigarrow \alpha$

Upshot: The def of symm mon cat in terms of finite amount of data translates to a question of finitely presenting the operad.

Q: Why do we (algebraic topologists) care?

$B: \mathrm{Cat} \to \mathrm{Top}$ classifying space.

BP and BQ are operads in Top, and are E_∞-operads, so if X is a P-algebra, or a Q-algebra, then BX is an E_∞ space.

$Q \to P$