# COMPLETE MANIFOLDS WITH NONNEGATIVE SECTIONAL CURVATURE

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Abstract. Many major results in Riemannian geometry are of the form curvature implies topology. In this paper, we will specifically study how nonnegative sectional curvature impacts the topology of a manifold. To do this, we first introduce fundamental concepts in Riemannian geometry and briefly discuss fundamental comparison theorems. Finally, we will finish by discussing how complete manifolds with nonnegative sectional curvature can be reduced to compact manifolds.

## Contents



## 1. Introduction

Historically, differential geometry explores the properties of curves and surfaces in $\mathbb{R}^3$. Namely, it explores how curvature can be described and how certain properties of a curve or surface are impacted by its curvature. Many of these ideas originally relied on the presence of a normal vector, which depends on how it is embedded in $\mathbb{R}^n$. We desire to develop a method of studying manifolds which does not rely on how it sits within its ambient space. In doing so, we can also study the geometry of higher dimensional manifolds without relying on a visual intuition of their embeddings. This desire to study manifolds intrinsically and therefore develop an intuition for the geometric properties of arbitrary manifolds will motivate the introduction of Riemannian geometry.

Major results in Riemannian geometry often relate the curvature of a manifold to its topology. The restriction to complete manifolds of nonnegative sectional curvature limits our study to generally well-behaved manifolds and is, therefore, a good first example to the connection between curvature and topology. Some of the more well-known comparison theorems, such as the Rauch Comparison Theorems

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and Toponogov's Theorem, describe how variance in curvature will impact the distance between points and behavior of geodesics. We will see that nonnegative sectional curvature essentially bounds how quickly geodesic curves spread apart. These bounds are what make complete manifolds of nonnegative sectional curvature generally very nice spaces to work with, as we will discuss through the Soul Theorem and Splitting Theorem how the topology of any such manifold can be reduced to that of a compact manifold.

Before moving on to the rest of the paper, we need to establish some notation that will be used throughout the paper. For convenience, we will interchange $\frac{\partial}{\partial x_i} = \partial_i$ as partial derivatives and basis coordinate vectors of the tangent space circumstantially. We will denote the tangent space of a manifold at a point by $T_pM$ and the normal bundle of a subset $S$ of a manifold by $\nu(S)$.

## 2. Riemannian Metric

Smooth manifolds (Definition 2.1) are interesting because they allow us to use techniques of real analysis in non-Euclidean spaces. This is important as we can then define a metric on non-Euclidean spaces. Certain analytic properties, such as the metric, make manifolds normal in some sense.

**Definition 2.1.** A *smooth manifold* is a paracompact Hausdorff topological space $M$ with an atlas $\{(U_\alpha, \phi_\alpha)\}$, where any composition $\phi_\alpha \circ \phi_\beta^{-1}$ restricted to $U_\alpha \cap U_\beta$ is a smooth function from $\mathbb{R}^n$ to $\mathbb{R}^n$.

In Euclidean space, the condition of being locally Euclidean does not restrict our spaces enough to do meaningful analysis. The restriction to paracompact Hausdorff spaces is important. If we want to discuss curvature, then we need a metric. To define a metric on our topological space which induces the topology, a smooth atlas is not sufficient. For example, consider the real line with two origins. We can define open sets of each origin to not contain the other origin. Any two open neighborhoods each including a distinct origin must intersect. Therefore, while this topological space is locally diffeomorphic to $\mathbb{R}$ and paracompact, it is not Hausdorff. See Figure 1.

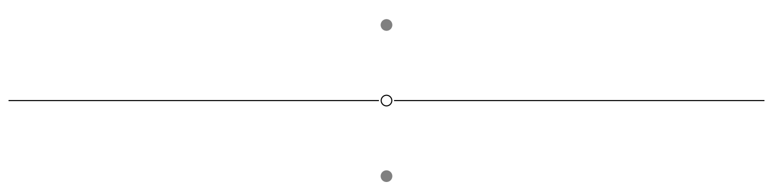


Figure 1. The line with two origins.

Then, we would have two distinct points, but no such distance function which could distinguish these two origins while simultaneously preserving the topology. In short, we require that our spaces are additionally paracompact Hausdorff topological spaces to make our space metrizable. We will also see in the proof of Proposition 2.3 that the existence of a partition of unity subordinate to any open cover is useful. We now define a Riemannian metric on a smooth manifold.

**Definition 2.2.** A *Riemannian metric* on a smooth manifold $M$ is a map $g$ from each point $p \in M$ to inner products on the tangent space $T_pM$, which varies smoothly. We say a smooth manifold paired with a Riemannian metric is a *Riemannian manifold*. For brevity, we may write $M$ is a Riemannian manifold when the metric itself is not needed. Let $\phi\colon \mathbb{R}^n \to M$ be a coordinate chart of $M$. We write $g_{ij} = \langle \partial_i \phi, \partial_j \phi \rangle$, where $\partial_i \phi$ is the partial derivative of $\phi$ and $g^{ij} = g_{ij}^{-1}$.

This definition of a Riemannian metric as an assignment of inner products is stronger than a distance function. The benefit of an inner product is that we can discuss angles between vectors in addition to distances between points. Notice that $g$ depends on the coordinate chart $\phi$. Hence, $g$ is not unique. Though the choice of Riemannian metric is not unique, we want to show that such an inner product exists on any arbitrary smooth manifold.

**Proposition 2.3.** *Any smooth manifold $M$ has a Riemannian metric.*

*Proof.* Since $M$ is smooth, Hausdorff and paracompact, there exists a smooth partition of unity $\{f_\alpha\}$ subordinate to an open covering $\{V_\alpha\}$. We can define a Riemannian metric $\langle \cdot, \cdot \rangle^\alpha$ on each $V_\alpha$ induced by a coordinate chart.

We can then define $\langle u, v \rangle_p = \sum_\alpha f_\alpha(p) \langle u, v \rangle_p^\alpha$ for all $p \in M$ and for all $u, v \in T_pM$. Thus, we only need to show that $\langle u, v \rangle_p$ is indeed a Riemannian metric. As I show below, each condition for an inner product follows naturally since we have a finite sum of inner products.

(1) (Nonnegative length) Suppose $u \neq 0$. Then,
$$\langle u, u \rangle_p = \sum_\alpha f_\alpha(p) \langle u, u \rangle_p^\alpha > \sum_\alpha f_\alpha(p) \cdot 0 = 0.$$
(2) (Linearity)
$$\begin{aligned} \langle au, v \rangle_p &= \sum_\alpha f_\alpha(p) \langle au, v \rangle_p^\alpha = \sum_\alpha f_\alpha(p) \cdot a \langle u, v \rangle_p^\alpha \\ &= a \sum_\alpha f_\alpha(p) \langle u, v \rangle_p^\alpha = a \langle u, v \rangle_p, \end{aligned}$$
(3) (Commutativity over $\mathbb{R}$)
$$\langle u, v \rangle_p = \sum_\alpha f_\alpha(p) \langle u, v \rangle_p^\alpha = \sum_\alpha f_\alpha(p) \langle v, u \rangle_p^\alpha = \langle v, u \rangle_p.$$
□

The proof of the existence of a Riemannian metric here relies on the fact that a partition of unity exists. Partitions of unity are essentially used to glue together open sets smoothly. This proof fails if we did not require manifolds to be paracompact. If we could not find a partition of unity, then any resulting infinite sum doesn't necessarily converge.

We now define an isometry, which is briefly a map that preserves angles and distances of points.

**Definition 2.4.** Let $M, N$ be Riemannian manifolds and $f\colon M \to N$ be a smooth map. We say that $f$ is an *isometry* if $\langle u, v \rangle_p = \langle df_p(u), df_p(v) \rangle_{f(p)}$ for every point $p \in M$ and every $u, v \in T_pM$. We say that $f$ is a *local isometry* at $p \in M$ if there is a neighborhood $p \in U \subset M$ such that $f|_U$ is an isometry. If there exists such an $f$ for all $p \in M$, we say that $M$ and $N$ are *locally isometric*.

What we will show in the next section is how to take a Riemannian metric and produce a corresponding intrinsic first-order derivative theory.

## 3. Levi-Civita Connection

In the standard $\mathbb{R}^n$, it is easy to compare tangent spaces of distinct points. This is because, in $\mathbb{R}^n$, our tangent space is always isomorphic to the entire manifold. This means we can make sense of subtracting two distinct points and take our usual derivative. But what does it mean to compare tangent spaces of two distinct points on some arbitrary manifold? On any arbitrary Riemannian manifold, we cannot necessarily make sense of the difference between two distinct points. To resolve this, we define the Levi-Civita connection, which will be a way of connecting tangent spaces at distinct points within some neighborhood.

The Levi-Civita connection is a way of associating tangent spaces which satisfies properties of our classical derivative, namely linearity and the product rule. This gives an abstraction of the derivative restricted to the tangent space of our manifold.

**Definition 3.1.** Let $\mathcal{T}(M)$ be the set of tangent vector fields on $M$. A *linear connection* is a map $\nabla\colon \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M)$, written $\nabla(X,Y) = \nabla_X Y$, which satisfies the following properties:

(1) $\nabla_{fX_1+gX_2}Y = f\nabla_{X_1}Y + g\nabla_{X_2}Y$,
(2) $\nabla_X(aY_1 + bY_2) = a\nabla_X Y_1 + b\nabla_X Y_2$,
(3) $\nabla_X(fY) = f\nabla_X Y + X(f)Y$.

Given any vector field $X$, we will write $\nabla_{\dot\gamma}X$ to be the linear connection $\nabla_Y X$ restricted to $\gamma$, where $Y$ is any smooth vector field such that $Y(\gamma(t)) = \dot\gamma(t)$. Given a curve $\gamma$, we say that $\nabla_{\dot\gamma}X$ is the *covariant derivative* of $X$ along $\gamma$.

Given a tensor $T$ of order $n$, we get that the *covariant differential* $\nabla T$ of $T$ is a tensor of order $n+1$, where

$$\begin{aligned}\nabla T(X_1,\dots,X_n,Y) &= \nabla_Y T(X_1,\dots,X_n)\\ &= Y(T(X_1,\dots,X_n)) - \sum_{i=1}^{n} T(X_1,X_2,\dots,\nabla_Y X_i,\dots,X_{n-1},X_n).\end{aligned}$$

Given a linear connection $\nabla$ on a Riemannian manifold, a coordinate system $(U,x)$, and basis vector fields $X_i = \partial_i$, we define the *Christoffel symbols* $\Gamma_{ij}^k$ by

$$\nabla_{X_i}X_j = \sum_k \Gamma_{ij}^k X_k.$$

Provided an orthonormal basis $\{e_i\}$ for each point on a curve $\gamma$, we can compute the covariant derivative of a vector field $V$ along $\gamma$ as

$$\nabla_{\dot\gamma}V = \sum_{i,j}\left(\sum_k \dot\gamma(t)^j V^i \Gamma_{ij}^k e_k + \dot\gamma(t)^j \frac{\partial V^i}{\partial x_j} e_i\right).$$

Then, a vector field $V$ is *parallel* along $\gamma$ with respect to the connection $\nabla$ if $\nabla_{\dot\gamma}V = 0$ for all $t$.

This definition of a linear connection does not give uniqueness and does not necessarily preserve the same structure of the tangent space at each point. What we want is essentially an action where going between tangent spaces should mimic the identity map. That is, we want to define parallel transportation of vectors in a way that creates an isometry between tangent spaces and prevents rotation of tangent spaces.

**Definition 3.2.** Let $X, Y, Z$ be vector fields on a smooth manifold $M$ and let $\nabla$ be a linear connection. We say $\nabla$ is *compatible with the metric* if

$$X(\langle Y, Z\rangle) = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle.$$

This definition of compatibility with the metric is a generalization of the product rule for the inner product of two vector fields. It isn't automatically clear why this implies that parallel transport is an isometry of tangent spaces. Then, we look at the following equivalent condition for compatibility with the metric.

**Proposition 3.3.** *$\nabla$ is compatible with the metric if and only if for any two parallel vector fields $P, P'$ along a curve $\gamma$, we have that $\langle P, P'\rangle$ is constant along $\gamma$.*

*Proof.* By definition, we get compatibility with the metric if and only if

$$\dot\gamma\langle P, P'\rangle = \langle \nabla_{\dot\gamma} P, P'\rangle + \langle P, \nabla_{\dot\gamma} P'\rangle = \langle 0, P'\rangle + \langle P, 0\rangle = 0.$$

By definition of the directional derivative, we have that $\dot\gamma\langle P, P'\rangle = 0$ if and only if $\langle P, P'\rangle$ is constant along $\gamma$. This gives us the desired equivalence. □

This proposition gives us a stronger visual intuition of why compatibility implies that parallel transport using the connection $\nabla$ is an isometry. We have that since the inner product of $P$ and $P'$ is constant, parallel transported vector fields preserve angles. However, a linear isometry does not necessarily give us an "identity map" between tangent spaces. We also require the connection to be symmetric.

**Definition 3.4.** The *torsion tensor* is defined as $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$. We say that $\nabla$ is *symmetric* if the torsion tensor is identically equal to 0 for all vector fields $X, Y$.

Symmetry is not defined here in a classical way. Why define symmetry as $\nabla_X Y - \nabla_Y X = [X, Y]$ instead of $\nabla_X Y - \nabla_Y X = 0$? To answer this, we can compute the Christoffel symbols given a symmetric connection. It turns out that a connection is symmetric if and only if $\Gamma^k_{ij} = \Gamma^k_{ji}$. From an analytic perspective, we notice that $\nabla_X Y - \nabla_Y X$ is not a tensor, and tensors generally behave nicely. Additionally, we can think of the torsion tensor as a measurement of how tangent spaces rotate when connected. Consider the following example.

**Example 3.5.** Let $M = \mathbb{R}^3$. Define a connection $\nabla$ by $\Gamma^3_{12} = \Gamma^1_{23} = \Gamma^2_{31} = 1$ and $\Gamma^3_{21} = \Gamma^1_{32} = \Gamma^2_{13} = 1$. Consider the curve $\gamma(t) = (t, 0, 0)$. We get that $\dot\gamma(t) = (1, 0, 0)$. Then, the following picture illustrates a parallel vector field along $\gamma$.

Let $V$ be a vector field such that $V(\gamma(t)) = (0, \sin(t), \cos(t))$. Then, we have that

$$\begin{aligned}
\nabla_{\dot\gamma} V &= \sum_{i,j=1}^{3} \left( \sum_{k=1}^{3} \dot\gamma(t)^j V^i \Gamma^k_{ij} e_k + \dot\gamma(t)^j \frac{\partial V^i}{\partial x_j} e_i \right) \\
&= \sum_{i=2, k\neq i}^{3} V^i \Gamma^k_{i1} e_k + \frac{\partial V^2}{\partial t} e_2 + \frac{\partial V^3}{\partial t} e_3 \\
&= (\cos t(-1)e_3 + \sin t(1)e_2) + (-\sin t e_2) + (\cos t e_3) = 0.
\end{aligned}$$

Hence, $V$ is parallel along $\gamma$ with respect to the connection $\nabla$. We get a similar construction of a parallel vector field along the other two orthogonal axes. By

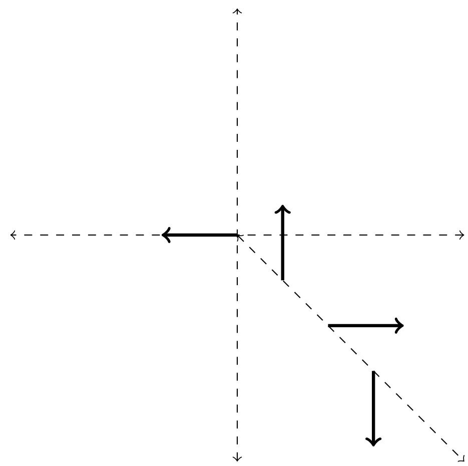


FIGURE 2. Here, we depict how a vector parallel transported along a coordinate axis with respect to $\nabla$ rotates. This goes against our intuition of how a vector should be parallel transported in $\mathbb{R}^3$.

definition of the connection, we have that $\Gamma_{ij}^k = 0$ whenever $i = j$. Therefore, $\nabla_{X_i} X_i = 0$ for all $i$. By linearity over $\mathbb{R}$ in both inputs, we have that every line passing through the origin is a geodesic with respect to the linear connection. Also, since parallel transport of vectors along a geodesic with respect to $\nabla$ is an isometry between tangent spaces with respect to the standard Euclidean inner product, we have that $\nabla$ is compatible with the standard metric.

However, this is not how we want to define parallel transport of vectors in $\mathbb{R}^3$. We notice that $\nabla$ is not symmetric. Computing the torsion tensor, we get

$$\nabla_{X_i} X_j - \nabla_{X_j} X_i - [X_i, X_j] = \sum_k \Gamma_{ij}^k X_k - \sum_k \Gamma_{ji}^k X_k - 0 = 2 \sum_k \Gamma_{ij}^k X_k,$$

which is nonzero for any $i \neq j$.

Now that we have a connection which acts as an identity map between tangent spaces, we want to show that there is exactly one such linear connection. This linear connection will be called the Levi-Civita connection, and it is only unique up to change in the Riemannian metric $g$.

**Theorem 3.6.** *For any smooth manifold $M$ with Riemannian metric $g$, there exists a unique linear connection $\nabla$ such that:*

*(1) $\nabla$ is compatible with the metric,*
*(2) $\nabla$ is symmetric.*

*Proof.* Suppose there exists such a connection $\nabla$. We have that if $\nabla$ is compatible with the metric $g$, then

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$
$$Y\langle Z, X\rangle = \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle$$
$$Z\langle X, Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle$$

Our goal is to isolate the connection and argue that if two such connections existed, they would be equal to each other. Notice that using the symmetry condition can reduce the quantity of distinct connections we deal with. If we apply the symmetry condition to the first connection in each line, we get the following

$$\begin{aligned}
X\langle Y,Z\rangle &= \langle \nabla_Y X + [X,Y], Z\rangle + \langle Y, \nabla_X Z\rangle \\
Y\langle Z,X\rangle &= \langle \nabla_Z Y + [Y,Z], X\rangle + \langle Z, \nabla_Y X\rangle \\
Z\langle X,Y\rangle &= \langle \nabla_X Z + [Z,X], Y\rangle + \langle X, \nabla_Z Y\rangle
\end{aligned}$$

Now we want to cancel some of the terms. Consider the following:

$$\begin{aligned}
X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle &= (\langle \nabla_Y X + [X,Y], Z\rangle + \langle Y, \nabla_X Z\rangle) + \\
&\quad (\langle \nabla_Z Y + [Y,Z], X\rangle + \langle Z, \nabla_Y X\rangle) - \\
&\quad (\langle \nabla_X Z + [Z,X], Y\rangle + \langle X, \nabla_Z Y\rangle) \\
&= (\langle \nabla_Y X, Z\rangle + \langle [X,Y], Z\rangle + \langle \nabla_X Z, Y\rangle) + \\
&\quad (\langle \nabla_Z Y, X\rangle + \langle [Y,Z], X\rangle + \langle \nabla_Y X, Z\rangle) - \\
&\quad (\langle \nabla_X Z, Y\rangle + \langle [Z,X], Y\rangle + \langle \nabla_Z Y, X\rangle) \\
&= 2\langle \nabla_Y X, Z\rangle + \langle [X,Y], Z\rangle + \langle [Y,Z], X\rangle - \\
&\quad \langle [Z,X], Y\rangle.
\end{aligned}$$

This gives us a way to isolate the term with $\nabla_Y X$. If we suppose there are two such connections $\nabla^1$ and $\nabla^2$, we get that $\langle \nabla^1_Y X, Z\rangle = \langle \nabla^2_Y X, Z\rangle$, and therefore $\nabla^1_Y X - \nabla^2_Y X = 0$. Then they must be equal. If we define the connection in this way and follow the same computations, we get that there exists a linear connection $\nabla$ that is symmetric and compatible with the metric. $\square$

Now that we have developed a first-order derivative theory on Riemannian manifolds, we develop a second-order derivative theory.

## 4. Curvature

In a standard calculus class, the Hessian, or total second derivative, is a standard way of discussing curvature of real-valued functions in $\mathbb{R}^n$. We want to discuss curvature intrinsically in a similar way as a total second derivative. Without any embedding in $\mathbb{R}^n$, we need another way to think about curvature. We instead consider curvature as a way to measure how parallel transport commutes. For example, suppose we parallel transport a vector along a sphere in segments of great circles, forming a triangle of all right angles. Then, the resulting vector will rotate by $\frac{\pi}{2}$ upon arrival of the original starting point. The following definition of the curvature operator can then be interpreted as a measurement of the difference in parallel transporting a vector along the flow of two vector fields when their order is commuted.

**Definition 4.1.** The *curvature operator* $R\colon \mathcal{T}(M)\times\mathcal{T}(M)\times\mathcal{T}(M)\to\mathcal{T}(M)$ of a manifold $M$ with Levi-Civita connection $\nabla$ is defined as

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

Then the *curvature tensor* can be defined as $R(X,Y,Z,W) = \langle R(X,Y)Z, W\rangle$

We notice how the curvature operator is defined using second covariant derivatives. This relates to how the Hessian is defined as the second total derivative of a function. As a "second-order derivative theory", we hope for the curvature operator to satisfy linearity like the ordinary derivative does.

**Theorem 4.2.** *The curvature operator is a* $\binom{3}{1}$ *tensor field.*

*Proof.* By the Jacobi identity, we only need to check linearity in one input.

$$\begin{aligned} R(X, fY)Z &= \nabla_X \nabla_{fY} Z - \nabla_{fY} \nabla_X Z - \nabla_{[X,fY]} Z \\ &= \nabla_X (f \nabla_Y Z) - f \nabla_Y (\nabla_X Z) - \nabla_{f[X,Y]+X(f)Y} Z \\ &= f \nabla_X \nabla_Y Z + X(f) \nabla_Y Z - f \nabla_Y \nabla_X Z - f \nabla_{[X,Y]} Z - X(f) \nabla_Y Z \\ &= f(\nabla_X \nabla_{fY} Z - \nabla_{fY} \nabla_X Z - \nabla_{[X,fY]} Z) \\ &= f R(X,Y)Z \end{aligned}$$

□

Since tensors are generally well-behaved, we expect the curvature operator to have nice properties. Below, we list some conveniences which follow from the definition of the curvature operator.

**Proposition 4.3.** *The curvature operator has the following symmetries:*

*(1)* $R(X,Y)Z = -R(Y,X)Z$,
*(2)* $\langle R(X,Y)Z, W\rangle = -\langle R(X,Y)W, Z\rangle$,
*(3)* $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$.

*Proof.* The first symmetry of the curvature operator follows from antisymmetry of the Lie bracket. The second symmetry follows from pulling $\nabla$ out of the inner product. The third symmetry follows from the Jacobi identity of the Lie bracket. This is a quick computation:

$$\begin{aligned} R(X,Y)Z + R(Y,Z)X + R(Z,X)Y =& \nabla_X(\nabla_Y Z - \nabla_Z Y) + \nabla_Y(\nabla_Z X - \nabla_X Z) \\ & + \nabla_Z(\nabla_X Y - \nabla_Y X) - \nabla_{[X,Y]} Z \\ & - \nabla_{[Y,Z]} X - \nabla_{[Z,X]} Y \\ =& \nabla_X [Y,Z] - \nabla_{[Y,Z]} X + \nabla_Y [Z,X] - \nabla_{[Z,X]} Y \\ & + \nabla_Z [X,Y] - \nabla_{[X,Y]} Z \\ =& [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0. \end{aligned}$$

□

Algebraically, 4-tensors can be difficult to work with. We want a simpler object to work with that maintains some of the properties of curvature. One natural thought would be to look at two-dimensional subspaces of the tangent space to help provide a geometric understanding of the curvature tensor. This gives us the sectional curvature.

**Definition 4.4.** The *sectional curvature* of a manifold $M$ at $p \in M$, with $u, v \in T_pM$ is defined as

$$K(u,v) = \frac{R(u,v,u,v)}{\langle u,u\rangle\langle v,v\rangle - \langle u,v\rangle^2}.$$

The sectional curvature of a manifold can be thought of as the Gaussian curvature of an inputted two-dimensional smooth submanifold. We divide by the area to make the curvature independent of choice of basis for a two-dimensional submanifold of the tangent space. That is, if $\text{span}\{u_1, u_2\} = \text{span}\{v_1, v_2\} = S$, then $K(u_1, u_2) = K(v_1, v_2)$. Then, we can write $K(S)$ as the sectional curvature of the submanifold $S$.

Another natural thought would be to take the trace of the Riemannian curvature 4-tensor. If the Riemannian curvature tensor is a generalization of the Hessian, then taking the trace of the Riemannian curvature tensor would give us a generalization for the Laplacian, which we define as the Ricci curvature. This still leaves us with a 2-tensor, so it is possible to simplify further to get the scalar curvature by summing again over pairs of indices to get a scalar-valued function.

**Definitions 4.5.** The *Ricci curvature* $Ric(X, Y)$ is a 2-tensor defined as

$$Ric_{ij} = \sum_{k,m} g^{km} R_{kijm}$$

The *scalar curvature* $S$ is a real valued function defined as

$$S = \sum_{i,j} g^{ij} Ric_{ij}$$

Based on the definition as the trace of the Riemannian curvature, we can also think of the Ricci curvature as an average over each sectional curvature. Similarly, we can view the scalar curvature as an average of the Ricci curvature at a point.

While these definitions of different curvatures on a Riemannian manifold accurately depict our intuition of curvature as an abstraction of the second derivative, they are quite abstract and difficult to visualize. What will follow from studying the generalization of straight lines on Riemannian manifolds is a way to correlate the sectional curvature of a manifold with the general behavior of its geodesics.

## 5. Geodesics and Jacobi Fields

In Euclidean space, we define the distance between two points as the length of the straight line connecting them. Geodesics are the Riemannian equivalent of curves that realize the distance between points. Intrinsically, the defining property of geodesics is that their acceleration is equivalent to zero. The fact that geodesics are distance minimizing is nontrivial locally and not necessarily true globally.

Geodesics are useful for studying properties of path-connected sets. In general, if we have any closed geodesics, then our space should have nontrivial topology. Then, finding subsets that contain all geodesics between points will guarantee that we include any nontrivial loops.

**Definition 5.1.** A curve $\gamma\colon I \to M$ is *geodesic* at $t_0 \in I$ if $\nabla_{\gamma'}\gamma' = 0$. It is called a *geodesic* if it is geodesic at all $t \in I$. We denote the length of a geodesic $\gamma$ by $L[\gamma]$.

**Proposition 5.2.** *Given a Riemannian manifold $(M, g)$ and $p \in M$, there exist a neighborhood $V$ of $p$, some $\varepsilon > 0$, $U = \{(q, w) \in TM : q \in V, w \in T_qM, |w| < \varepsilon\}$ and a smooth mapping $\gamma : (-1, 1) \times U \to M$ such that $t \to \gamma(t, q, w)$ is the unique geodesic of $M$ that at $t = 0$ passes through $q$ with velocity $w$ for every $q \in V$ and for every $w \in T_qM$ such that $|w| < \varepsilon$.*

*Remark* 5.3. The proof of this proposition is essentially a proof of existence and uniqueness of differential equations. We are given an initial condition in each possible direction.

We know that geodesics exist locally for each point $p$ on a manifold. This isn't necessarily true globally. It is possible for geodesics to only be defined on a subset of $\mathbb{R}$. For a simple example, consider the punctured plane $\mathbb{R}^2 - \{0\}$. Then, the unit speed geodesic from $(2, 0)$ passing through $(1, 0)$ is not defined for $t \geq 2$. This motivates a restriction to manifolds where geodesics are defined everywhere.

**Definition 5.4.** We say a Riemannian manifold $(M, g)$ is *(geodesically) complete* if every geodesic $\gamma$ on $M$ is defined over $(-\infty, \infty)$.

With this definition of geodesically complete, we want to find a relation to our manifold being complete as a metric space. We know that it is possible to define a distance function between points on a manifold because, by definition of a manifold, our space must be paracompact, Hausdorff and possess a smooth atlas. We then define the Riemannian distance $d\colon M \times M \to \mathbb{R}$ between two points (or subsets) as the infimum length of a smooth curve connecting them over all possible smooth curves. The following theorem shows the correspondence between geodesic completeness and completeness as a metric space.

**Theorem 5.5.** *[Hopf-Rinow, 1931] A connected Riemannian manifold $(M, g)$ is complete if and only if it is complete as a metric space.*

We will not give a proof of this Theorem. For a complete proof, see (6), page 108, Theorem 6.13. There is also a past University of Chicago REU paper from 2016 by Daniel Spiegel (see (7)) which covers the Hopf-Rinow theorem in greater detail.

**Definition 5.6.** A *ray* is a unit speed geodesic $\gamma\colon [0, \infty) \to M$ such that $\gamma$ restricted to any compact interval is distance realizing. A *line* is a unit speed geodesic $\gamma\colon (-\infty, \infty) \to M$ such that $\gamma$ restricted to any compact interval is distance realizing. Given a ray $\gamma$, we will write $\gamma_a\colon [0, \infty) \to M$ as $\gamma_a(t) = \gamma(a+t)$. Any line $\sigma$ can be decomposed into two rays $\gamma, -\gamma$, where $\gamma(t) = \sigma(t)$ and $-\gamma(t) = \sigma(-t)$ for all $t \geq 0$.

Given the existence and uniqueness of geodesics on a manifold, we want a way to correspond geodesics locally with the initial conditions from the Euclidean tangent space. Studying how a map from the flat tangent space to a manifold behaves can tell us meaningful local information about the manifold.

**Definition 5.7.** For some subset $\Omega \subset T_pM$ containing the origin, the *exponential map* $\exp_p\colon \Omega \to M$ is defined by $\exp_p(v) = \gamma_v(1)$, where $\gamma_v$ is defined as the unique geodesic with initial derivative $\dot{\gamma}(0) = v$ with $v \in \Omega$.

Why do we expect this exponential map to give us a submanifold $\widetilde{M}$ of $M$ such that $p \in \operatorname{int}(\widetilde{M})$? We notice that we can associate each direction of $T_pM$ to a point on the $n$-sphere, where $n$ is the dimension of $M$. Assuming $M$ is finite-dimensional, we have that $S^n$ is compact under the Euclidean topology. Then, we notice that in every direction $v$, there exists a unique geodesic with starting point $p$ and starting velocity $v$ which is a geodesic for at least some length $\alpha_v > 0$. Since the set of all directions, $S^n$, is compact, we know that $\inf_{v \in T_pM} \alpha_v$ is attained and therefore greater than 0. Thus, we have that any submanifold mapped out by the exponential map contains $p$ in its interior.

We desire a way of discussing curvature based on the local behavior of geodesics. Consider parallel transporting a vector $v$ from a point $p$ along a geodesic $\gamma$ to a vector $w$ at a point $q$. Let $\alpha$ and $\beta$ be two geodesics with initial conditions $v, p$ and $w, q$ respectively. Then, in a space of positive sectional curvature such as the sphere or the paraboloid, $d(\alpha(s), \beta(s))$ should be a decreasing function of $s$ on some interval $s \in [0, t]$. Similarly, in a space of negative sectional curvature, $d(\alpha(s), \beta(s))$ should be an increasing function of $s$ and in a space of zero sectional curvature, $d(\alpha(s), \beta(s))$ should be a constant function of $s$ for small $s$. Therefore,

**Definition 5.8.** A *smooth admissable family* is a continuous map $\Gamma\colon (-\varepsilon, \varepsilon) \times [a, b] \to M$ such that for some partition of $[a, b]$, $\Gamma$ restricted to the rectangle $(-\varepsilon, \varepsilon) \times [a_i, a_{i+1}]$ is smooth.

**Definition 5.9.** A *variation* of a geodesic $\gamma\colon [a, b] \to M$ on a manifold is a smooth admissible family $\Gamma\colon (-\varepsilon, \varepsilon) \times [a, b] \to M$ such that $\Gamma_0(t) = \gamma(t)$ for all $t \in [a, b]$ and $\Gamma_s\colon [a, b] \to M$ is a geodesic for each $s \in (-\varepsilon, \varepsilon)$. A *variation field* is a vector field $V$ along $\gamma$ such that $V(t) = \frac{\partial}{\partial s}\Gamma(0, t)$.

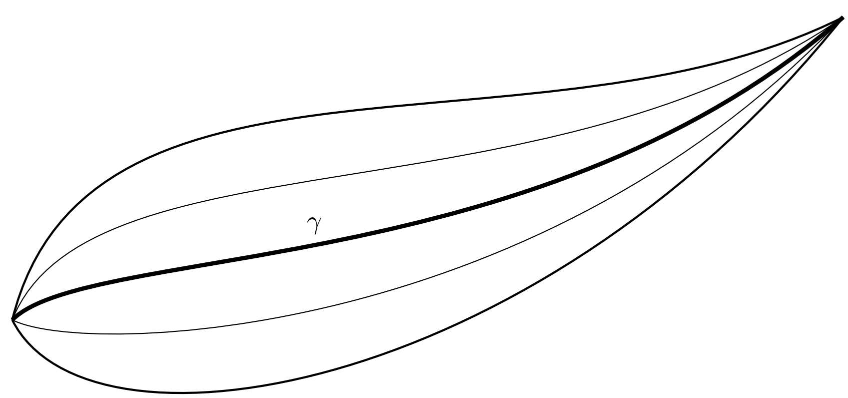



FIGURE 3. We show an example variation of geodesics in some space of positive curvature. The endpoints of this variation are conjugate points because the Jacobi field vanishes.

**Definition 5.10.** Let $p \in M$ and $\gamma_v\colon [0, 1] \to M$ be a geodesic on $M$ such that $\gamma_v(0) = p$ and $\gamma_v'(0) = v$ for each $v \in T_pM$. Then we say a *Jacobi field* is a variation field defined on the variation of geodesic curves.

**Lemma 5.11.** *If $\Gamma$ is a smooth admissible family of curves and $V$ is a smooth vector field along $\Gamma$, then*

$$\nabla_{\partial_s\Gamma}\nabla_{\partial_t\Gamma}V - \nabla_{\partial_t\Gamma}\nabla_{\partial_s\Gamma}V = R(\partial_s\Gamma, \partial_t\Gamma)V$$

We notice that $R(\partial_s\Gamma, \partial_t\Gamma)V$ is equal to $\nabla_{\partial_s\Gamma}\nabla_{\partial_t\Gamma}V - \nabla_{\partial_t\Gamma}\nabla_{\partial_s\Gamma}V - \nabla_{[\partial_s\Gamma, \partial_t\Gamma]}V$. This means the lemma essentially says that the variation field $V$ of a variation of curves is precisely a vector field which satisfies $\nabla_{[\partial_s\Gamma,\partial_t\Gamma]}V = 0$. This is true because the geodesic flow in the direction $\partial_s\Gamma$ and the geodesic flow in the direction $\partial_t\Gamma$ should be independent.

We also notice that there is a direct relation between the second covariant derivatives of our Jacobi field and the curvature of our underlying space. This is precisely what the Jacobi equation expresses.

**Theorem 5.12.** *[Jacobi Equation] Let $\gamma$ be a geodesic and $V$ be a Jacobi field. Then $V$ satisfies*

$$\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}V + R(V,\dot{\gamma})\dot{\gamma} = 0.$$

We know that geodesics are locally minimizing curves. This is not true globally. Considering the standard round 2-sphere embedded in $\mathbb{R}^3$, we have that starting at any point, every geodesic will intersect at the antipodal point. Our next goal is to define the point of intersection of these geodesics.

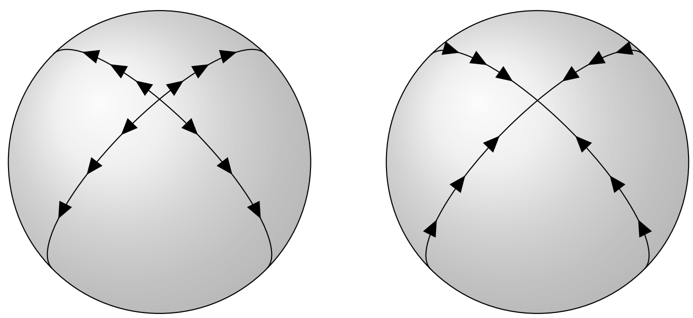


Figure 4. The image on the left represents geodesics extending outward from a base point on $S^2$. The image on the right represents how these geodesics eventually converge to the antipodal point.

**Definition 5.13.** A *conjugate point* of a Jacobi field $V$ is a point $\gamma(t_0) \neq \gamma(0)$ such that $V(\gamma(t_0)) = 0$.

We then have the Jacobi field defined at each point along a geodesic $\gamma$ as $\frac{\partial\Gamma}{\partial s}$, where $\Gamma$ is a smooth admissable family of geodesics such that $\Gamma(0,t) = \gamma(t)$. Then, when a Jacobi field vanishes, we have that arbitrarily close geodesics converge to a point.

Essentially, we want to say that if a Riemannian manifold has conjugate points, then it has positive curvature (somewhere) since for a family of geodesics to converge, it should have positive curvature. This will lead to the Rauch Comparison theorem and more specifically the Cartan-Hadamard theorem.

## 6. First Comparison Theorems

Everything so far has been effective in analyzing the properties of a particular Riemannian manifold. Given a particular manifold, we can discuss its Riemannian metric, compute its Christoffel symbols and curvature tensors. However, our goal is to compare different manifolds given a curvature bound. The Rauch Comparison Theorem will give us a tool to discuss manifolds with different sectional curvatures.

The Rauch comparison theorem will be used to compare different manifolds based on how quickly their geodesics converge. We notice that in Euclidean space, the second derivative of the distance between geodesics is equal to zero. Similarly, in hyperbolic and spherical space, the second derivatives of the distances between geodesics are positive and negative functions respectively. For an example, see Figure 5. This rate of change in the distance function is described for geodesics with arbitrarily close starting conditions through the Jacobi fields. Then, comparing the lengths of Jacobi fields based on curvature will provide a way to compare the convergence of geodesics locally.

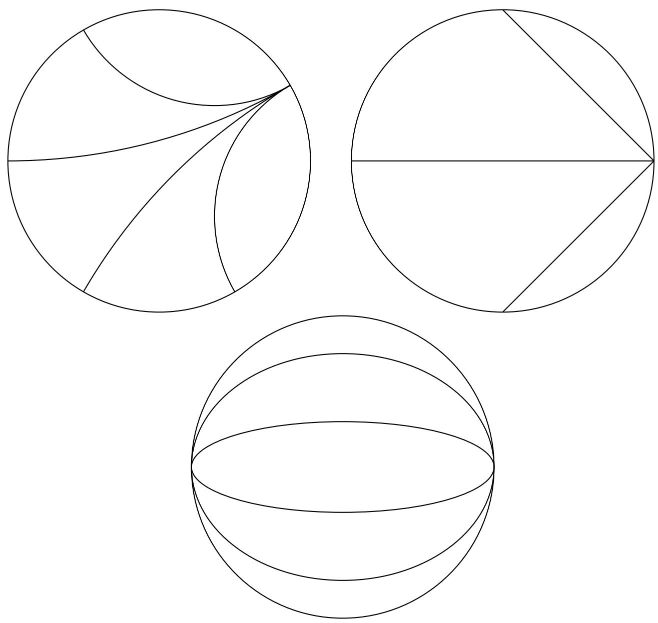


FIGURE 5. In order, we showcase how geodesics behave in hyperbolic space through the Poincaré disk model, the Euclidean plane restricted to the unit disk, and the sphere $S^2$.

**Theorem 6.1.** *[Rauch Comparison I] Let $M^n, \widetilde{M}^{k+n}$ be Riemannian manifolds for $k \geq 0$, $\gamma\colon [0,T] \to M$ and $\tilde{\gamma}\colon [0,T] \to \widetilde{M}$ be geodesic segments such that $\tilde{\gamma}(0)$ has no conjugate points along $\tilde{\gamma}$, and let $J, \widetilde{J}$ be normal Jacobi fields along $\gamma$ and $\tilde{\gamma}$ such that $J(0) = \widetilde{J}(0) = 0$ and $|\nabla_{\dot{\gamma}} J(0)| = |\nabla_{\dot{\tilde{\gamma}}} \widetilde{J}(0)|$. If the sectional curvatures of $M$ and $\widetilde{M}$ satisfy $K(\Pi) \leq \widetilde{K}(\widetilde{\Pi})$ whenever $\Pi \subset T_{\gamma(t)}M$ is a 2-plane containing $\dot{\gamma}(t)$ and $\widetilde{\Pi} \subset T_{\tilde{\gamma}(t)}\widetilde{M}$ is a 2-plane containing $\dot{\tilde{\gamma}}(t)$, then $|J(t)| \geq |\widetilde{J}(t)|$ for all $t \in [0,T]$.*

**Theorem 6.2.** *[Rauch Comparison II (Berger)] Let the notation be as above. Assume further that $\exp_{\gamma(0)} t\dot{\gamma}(0)$ is nonsingular for all $t \in [0,T]$. Let $J, \widetilde{J}$ be Jacobi fields along $\gamma, \tilde{\gamma}$ satisfying $J'(0), = \widetilde{J}'(0)$, $\nabla_{\dot{\gamma}} V = 0$, and $J(0), = \widetilde{J}(0)$. Then for all $t \in [0,T]$, $|J(t)| \geq |\widetilde{J}(t)|$.*

The major difference between the Rauch Comparison Theorems is the initial condition of the Jacobi fields. In the first theorem, we have that the Jacobi field starts as the 0 vector. In the second theorem, we have that the Jacobi field is initially parallel along $\gamma$.

The main idea of proving these theorems is to relate the sectional curvature to the covariant derivatives $\nabla_{\dot{\gamma}} J$ and $\nabla_{\dot{\tilde{\gamma}}} \widetilde{J}$, since we are given a comparison between the sectional curvatures. The goal is to show that the ratio of the derivatives is greater than or equal to 1. Then it shows that one is always larger than the other since they both start at the same magnitude.

Now, we will give some examples of theorems which are direct consequences of the Rauch Comparison Theorem. First is the Cartan-Hadamard theorem, which shows that the exponential map of a manifold with nonpositive sectional curvature is a covering map. This gives us a way to relate the topologies of Euclidean space and spaces of nonpositive sectional curvature.

**Theorem 6.3.** *[Cartan-Hadamard] Let $M$ be a complete Riemannian manifold and let the sectional curvature be bounded above by 0 ($K_M \leq 0$). Then for any $p \in M$, we have that $\exp_p : T_pM \to M$ is a covering map.*

**Corollary 6.4.** *For a manifold $M$ as above, all homotopy groups $\pi_n(M)$ vanish for $n > 1$.*

*Proof.* By 6.1, we have that since the sectional curvature of $M$ is bounded above by zero, then the magnitude of the Jacobi field is nondecreasing along any geodesic from $p$. Therefore, since Euclidean space has no conjugate points, $M$ also has no conjugate points.

Therefore, $\exp_p$ has no critical points. This means the rank of

$$d(\exp_p)_q \colon T_q(T_pM) \to T_qM$$

is equal to the dimension of $M$. That is, the kernel of $d(\exp_p)_q$ is trivial and $d(\exp_p)_q$ is nonsingular. Hence, $\exp_p$ is a covering map. □

The Cartan-Ambrose-Hicks theorem is a global result regarding the nature of geodesics. What we want to say is that given any manifolds $M$ and $N$, with a linear isometry between them, then the crossings of geodesics are preserved.

**Theorem 6.5.** *[Cartan-Ambrose-Hicks] Let $M, N$ be complete manifolds and let $M$ be simply connected. Let $I \colon T_pM \to T_qN$ be a linear isometry. Suppose every piecewise geodesic $\gamma$ satisfies*

$$I_\gamma(R(X,Y,Z)) = \bar{R}(I_\gamma(X), I_\gamma(Y), I_\gamma(Z)).$$

*Then for all piecewise geodesics with common intersections, there is a local isometry $\Phi \colon M \to N$ which preserves intersections. That is, if $\gamma_0, \gamma_1$ are piecewise geodesics where $\gamma_0(t_0) = \gamma_1(t_1)$, then $\Phi(\gamma_0(t_0)) = \Phi(\gamma_1(t_1))$.*

The next theorem is a global result regarding geodesic triangles. The theorem gives bounds on distances and angles of geodesic triangles. This theorem will be particularly useful for studying manifolds of nonnegative sectional curvature because of the initial condition $K \geq H$ which allows us to compare distances and angles on our manifold directly with Euclidean space.

**Theorem 6.6.** *[Toponogov] Let $M$ be a complete manifold with sectional curvature $K \geq H$ for some $H$. The following equivalent statements hold:*

(1) *Let $(\gamma_1, \gamma_2, \gamma_3)$ be a geodesic triangle in $M$, where $\gamma_1$ and $\gamma_3$ are minimal. If $H > 0$, suppose $L[\gamma_2] \leq \frac{\pi}{\sqrt{H}}$. Then in $M^H$, the standard simply connected two-dimensional space of constant sectional curvature $H$, there is some geodesic triangle in $M^H$ such that the geodesics are of equal length and the angles are bounded above by the angles in $M$. That is, $L[\gamma_i] = L[\bar{\gamma}_i]$ and $\bar{\alpha}_1 \leq \alpha_1, \bar{\alpha}_3 \leq \alpha_3$.*

   *The triangle in $M^H$ is unique unless $H > 0$ and $L[\gamma_i] = \frac{\pi}{\sqrt{H}}$ for some $i$.*

(2) *Let $\gamma_1, \gamma_2$ be geodesics such that $\gamma_1(l_1) = \gamma_2(0)$ and $\sphericalangle(\gamma_1(l_1), \gamma_2(0)) = \alpha$. Let $\gamma_1$ be a minimal distance realizing geodesic. If $H > 0$, let $L[\gamma_2] \leq \frac{\pi}{\sqrt{H}}$. Let $\bar{\gamma}_1, \bar{\gamma}_2$ be geodesics in $M^H$ such that $\bar{\gamma}_1(l_1) = \bar{\gamma}_2(0)$, $L[\gamma_i] = L[\bar{\gamma}_i]$ and $\sphericalangle(\bar{\gamma}_1, \bar{\gamma}_2) = \alpha$. Then,*

$$d(\gamma_1(0), \gamma_2(l_2)) \leq d(\bar{\gamma}_1(0), \bar{\gamma}_2(l_2)).$$

Since the proof of this theorem is extensive and mostly involves checking cases, we will guide the reader to page 35 of (2) for a complete proof. Instead, we discuss the importance of the bound $L[\gamma_i] \leq \frac{\pi}{\sqrt{H}}$.

At a first glance, the requirement $L[\gamma_i] \leq \frac{\pi}{\sqrt{H}}$ for $H > 0$ seems random. We will discuss why this is an important condition, which is not discussed in (2). To do so, we look at how geodesics behave in locally symmetric spaces. A locally symmetric space is defined as a space such that for any point $p$, parallel transport in any direction is locally isometric to parallel transport in the opposite direction. We will prove the following proposition, which will show that the requirement $L[\gamma_i] \leq \frac{\pi}{\sqrt{H}}$ is to prevent degenerate geodesic triangles in our model space with constant curvature $H$.

**Proposition 6.7.** *Let $M$ be a smooth manifold with a Levi-Civita connection satisfying $\nabla R \equiv 0$. Let $\gamma\colon [0, \infty) \to M$ be a geodesic in $M$ and let $v = \gamma'(0)$ be its velocity at $p = \gamma(0)$. Let $K_v\colon T_pM \to T_pM$ be a linear transformation defined as $K_v(x) = R(v, x)v$. Then the conjugate points of $p$ along $\gamma$ are given by $\gamma(\pi k/\sqrt{\lambda_i})$, where $k \in \mathbb{Z}^+$ and $\lambda_i$ is a positive eigenvalue of $K_v$.*

*Proof.* We have that $\langle K_v(x), y\rangle = R(v, x, v, y) = R(v, y, v, x) = \langle K_v(y), x\rangle = \langle x, K_v(y)\rangle$. Therefore, $K_v$ is self adjoint. Hence, we can find an orthonormal basis $\{e_i\}$ of $T_pM$ that diagonalizes $K_v$ and let $e_i(t)$ be a vector field of $e_i$ parallel transported along $\gamma$. Then $\lambda_i e_i(t) = K_{\dot{\gamma}}(e_i(t))$. Since $\nabla R \equiv 0$, we have

$$0 = \nabla K_{\dot{\gamma}(t)}(e_i(t)) = \dot{\gamma}(K_{\dot{\gamma}}(e_i(t))) - K_{\nabla_{\dot{\gamma}}\dot{\gamma}}(e_i(t)) - K_{\dot{\gamma}(t)}(\nabla_{\dot{\gamma}} e_i(t)).$$

Since $\{e_i(t)\}$ is an orthonormal basis of $T_{\gamma(t)}M$, we have that $\nabla_{\dot{\gamma}} e_i(t) = 0$ for all $t$. Also, since $\gamma$ is a geodesic, we get that $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$ for all $t$. This gives us

$$\nabla K_{\dot{\gamma}(t)}(e_i(t)) = \dot{\gamma}(K_{\dot{\gamma}}(e_i(t))) = 0.$$

What this means is that $K_{\dot{\gamma}(t)}(e_i(t))$ is parallel transported along $\gamma$. Hence, the inner product $\langle e_i(t), K_{\dot{\gamma}(t)}(e_i(t))\rangle = \lambda_i$ is constant.

Let $J(t) = \sum_i x_i(t) e_i(t)$ be a Jacobi field along $\gamma$. Then, by the Leibniz rule, linearity, and the definition of $e_i(t)$, we get

$$\nabla_{\dot{\gamma}} J = \sum_i (\dot{x}_i(t) e_i(t) + x_i(t) \nabla_{\dot{\gamma}} e_i(t)) = \sum_i \dot{x}_i(t) e_i(t).$$

With the same logic, we get

$$\nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} J = \sum_i \ddot{x}_i(t) e_i(t).$$

By the Jacobi equation, we get

$$-\sum_i \lambda_i x_i(t) e_i(t) = -K_{\dot{\gamma}(t)}(J) = -R(\dot{\gamma}, J)\dot{\gamma} = \nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} J = \sum_i \ddot{x}_i(t) e_i(t).$$

Since $\{e_i(t)\}$ is an orthonormal basis, we have that $\ddot{x}(t) = -\lambda_i x_i(t)$. Therefore, $x_i(t) = A\sin(\sqrt{\lambda_i}t) + B\cos(\sqrt{\lambda_i}t)$. Solving for when $x_i(t) = 0$, we get that the conjugate points are when $t = \frac{\pi k}{\sqrt{\lambda_i}}$. □

Consider our space of constant curvature $H > 0$. Let $v = \gamma_2'(0)$. We now compute the eigenvalues of $K_v$. Let $x$ be a unit eigenvector of $K_v$ and $\lambda$ its corresponding eigenvalue. Then, we have

$$\begin{aligned}\lambda = \lambda\langle x, x\rangle &= \langle \lambda x, x\rangle = \langle K_v(x), x\rangle \\ &= \langle R(v, x)v, x\rangle = K(v, x) = H.\end{aligned}$$

Therefore, $\lambda = H$ is an eigenvalue of $K_v$. This gives us that $L[\gamma_i] \leq \frac{\pi}{\sqrt{H}}$ is the first conjugate point on the manifold of constant curvature $H$.

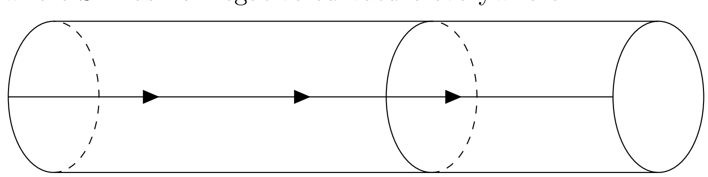
## 7. Complete Manifolds of Nonnegative Sectional Curvature

Our goal, as stated before, is to realize connections between nonnegative sectional curvature and topology. One important characteristic of nonnegative sectional curvature is that, by Rauch Comparison, geodesics cannot diverge. This gives us some "predictable" behavior of geodesics. Our goal is to establish some regulations on how geodesics impact such manifolds. We then want to reduce complete manifolds with nonnegative sectional curvature to compact manifolds. To do so, we will discuss the Splitting Theorem and the Soul Theorem. The Soul Theorem states that we can produce this reduction. Then, the Splitting Theorem will provide an explicit way to collapse manifolds.

**Theorem 7.1.** *[Toponogov Splitting Theorem, 1964] Let $M$ be a complete connected Riemannian manifold with nonnegative sectional curvature. If $M$ has a line, then $M$ is isometric to $\mathbb{R} \times N$, where $N$ is a Riemannian manifold with nonnegative sectional curvature.*

**Theorem 7.2.** *[The Soul Theorem, 1972] Let $M$ be a complete, connected manifold with nonnegative sectional curvature. Then there exists a submanifold $S \subset M$ that is totally convex, totally geodesic and compact without boundary. It follows that $M$ is diffeomorphic to the normal bundle of $S$.*

**Example 7.3.** Consider the cylinder, which has a line shown below. The cylinder is $\mathbb{R} \times S^1$, where $S^1$ has nonnegative curvature everywhere.



We emphasize that the cylinder decomposes into its line, $\mathbb{R}$, and a compact, totally convex submanifold of strictly lower dimension. The circle $S^1$ is the only such one-dimensional submanifold without boundary. Also, notice that the cylinder is diffeomorphic to the normal bundle of the circle. This is a good image to keep in mind for the proof of the splitting theorem.

To prove these theorems, we digress to discussing convex sets. A *convex set* is a set $C \subset M$ such that if $x, y \in C$, then there is some minimal geodesic $\gamma$ connecting $x$ and $y$ that is entirely contained in $C$. Convex sets are nice to work with because they must be path connected by definition and, in some sense, they capture the behavior of geodesics. However, convexity is not a strong enough condition to determine any meaningful information about the topology. For example, any singleton is a convex set because $d(x,x) = 0$. Hence, every geodesic loop on our manifold cannot be distance minimizing. If we want to discuss the topology, and more specifically the homotopy groups of a manifold, entirely based on paths between points, then it is important to restrict to sets that contain all geodesic loops. This motivates the restriction to complete manifolds and the following definition of totally convex sets.

**Definition 7.4.** A *totally convex set* $C \subset M$ is a subset such that for each $x, y \in C$, every geodesic $\gamma$ connecting $x$ and $y$ is contained in $C$. A *soul* $S \subset M$ is a compact, totally convex submanifold without boundary.

This restriction is enough to begin discussing homotopy groups. We can think of the $n$-sphere as a 1-point compactification of $\mathbb{R}^n$. Then, we can relate maps $f\colon S^n \to C$ to geodesic loops in $n$ linearly independent directions. To get a visual, see Figure 4 for the case $n = 2$.

**Theorem 7.5.** *Let $M$ be a complete Riemannian manifold and $C \subset M$ be compact and totally convex such that $\partial C = \varnothing$. Then the inclusion $i\colon C \hookrightarrow M$ is a homotopy equivalence.*

*Proof.* Let $\Omega_C$ be the space of piecewise smooth curves $\gamma\colon [0,1] \to M$ such that $\gamma(0), \gamma(1) \in C$. We define the energy of a curve $\gamma$ to be $\int_0^1 |\dot{\gamma}|^2 dt$. We will denote the space of curves of energy less than or equal to $a$ as $\Omega_C^a \subseteq \Omega_C$. Of particular importance, we have that $\Omega_C^0$, the space of constant geodesics in $C$, is diffeomorphic to $C$ as each point can be associated with the constant geodesic mapping to that particular point. $\Omega_C^0$ is also the set of critical points of the energy functional on $\Omega_C$. This is true because, by completeness, the energy of any piecewise smooth curve $\gamma$ can be decreased locally by following a projection onto the geodesic connecting $\gamma(0)$ and $\gamma(1)$. Then, since $C$ is totally convex, the energy of a geodesic can be decreased while staying in $\Omega_C$ by shortening the geodesic.

Let $\phi \in \Omega_C$ be defined on the unit interval $[0,1]$. Then, we can project $\phi(t)$ onto $\gamma(t)$ with some geodesic $\alpha_t$, where $\alpha_t(0) = \phi(t)$ and $\alpha_t(1) = \gamma(t)$ for each value of $t$. We will define a curve $\beta_t$ as the partial projection of $\phi$ onto $\gamma$, where $\beta_t(s) = \alpha_s(t)$ for all values of $s$ and $t$. From this, we get a continuous function $F\colon \Omega_C \times [0,1] \to \Omega_C$, defined by

$$F(\phi, s)(t) = \begin{cases} \beta_{2s}(t), & 0 \le s < \frac{1}{2}, \\ \gamma\left((2-2s)t\right), & \frac{1}{2} \le s \le 1. \end{cases}$$

We notice, in particular, that $F(x_0, s) = x_0$ for all values of $s$ and all constant geodesics $x_0 \in \Omega_C^0$. Therefore, the constant geodesics $\phi\colon [0,1] \to \{x_0\} \in C$ remain fixed under $F$. It then follows that $\Omega_C^0$, diffeomorphic to $C$, is a strong deformation retract of $\Omega_C$. Hence the relative homotopy groups $\pi_k(\Omega_C, C)$ are trivial for all values of $k$.

Now let $I^k$ denote the $k$-dimensional cube. For any $\overline{\mathbf{x}} = (x_1, \ldots, x_{k-1}) \in I^{k-1}$, we notice that $(\overline{\mathbf{x}}, t) = (x_1, \ldots, x_{k-1}, t)$ is an element of $I^k$. Then for any map $f\colon I^k \to M$ where $f(\partial I^k) \subseteq C$, there exists a corresponding map $g\colon I^{k-1} \to \Omega_C$ with $g(\partial I^{k-1}) \subseteq \Omega_C^0$, defined by

$$g(\overline{\mathbf{x}})(t) = f\left(\overline{\mathbf{x}},\ t(1-t)\cdot\left(\frac{1}{2} - \left\|\left(x_1 - \frac{1}{2}, \ldots, x_{k-1} - \frac{1}{2}\right)\right\|_\infty\right)\right)$$

for any choice of $\overline{\mathbf{x}} = (x_1, \ldots, x_k)$. We notice that if $\overline{\mathbf{x}} \in \partial I^{k-1}$, then there is some $x_i$ which is equal to either 0 or 1. Hence, $|x_i - \frac{1}{2}| = \frac{1}{2}$, which is the maximum distance possible from $\frac{1}{2}$. Then, we have that $g(\overline{\mathbf{x}})$ is a constant geodesic when $\overline{\mathbf{x}} \in \partial I^{k-1}$ as $\frac{1}{2} - \left\|\left(x_1 - \frac{1}{2}, \ldots, x_{k-1} - \frac{1}{2}\right)\right\|_\infty = 0$. Also, if $t$ is equal to zero or one, then $g(\overline{\mathbf{x}})(t) = f(\overline{\mathbf{x}}, 0) \in C$ for any $\overline{\mathbf{x}} \in I^{k-1}$. By this construction of $g$ for a corresponding $f$, we have that the relative homotopy groups of the pair $(\Omega_C, \Omega_C^0)$ are isomorphic to the relative homotopy groups of the pair $(M, C)$.

Therefore $\pi_k(M, C)$ is trivial for all $k \geq 1$, since $\pi_k(\Omega_C, C)$ is trivial for all $k$. By the Whitehead Theorem, we notice that since $M$ and $\Omega_C$ are smooth manifolds, if $h\colon (M, C) \to (\Omega_C, \Omega_C^0)$ is a weak homotopy equivalence of pairs, then $h$ is also a homotopy equivalence of pairs. Hence, $C$ is a deformation retract of $M$ and it follows that the inclusion map $i\colon C \hookrightarrow M$ is a homotopy equivalence. $\square$

*Remark* 7.6. In fact, the inclusion $i\colon C \hookrightarrow D$ is a homotopy equivalence for any locally convex set $D$. That is, for any $p \in D$, there exists an open set $U$ containing $p$ such that $U \cap D$ is convex. The proof is essentially the same, and is shown in a slightly different manner in (1).

*Remark* 7.7. The restriction of $C$ to a totally convex set without boundary is needed in the construction of the deformation retraction $F$. Indeed, if $C$ were convex but not totally convex, then we may not be able to compress the geodesic $\gamma$ while maintaining both endpoints in $C$. As an example, let $M = \mathbb{R} \times S^1$ and let $C$ be three quarters of a circle. Then, $C$ is convex and the shortest length geodesic containing the boudary points of $C$ cannot be compressed to a constant geodesic.

We begin the construction of totally convex sets. From now on, $M$ will be a noncompact manifold since the soul of a compact manifold is itself. As an example, notice in $\mathbb{R}^n$ that every closed, totally convex subset is an intersection of half-spaces. This is because, from total convexity, we can separate any point $x$ in the complement of the set from the set itself with some line. Then, take the half-space containing the totally convex set. We look to generalize the idea of a half-space to get an explicit construction of a totally convex set.

We notice there is a way to associate rays in $\mathbb{R}^n$ to half-spaces. What we can do is take the hyperplane orthogonal to the ray at the base point and the half-space will include the side of the hyperplane which does not contain any points on the ray except the base point. On any noncompact manifold, there must exist a ray $\gamma$. We use this ray to define a half-space on our manifold $M$, then show the half-space is totally convex.

**Definition 7.8.** Given a ray $\gamma$, we define $B_\gamma = \bigcup_{t \geq 0} B(\gamma(t), t)$, where $B(\gamma(t), t)$ is the geodesic ball at $\gamma(t)$ of radius $t$. We define the half-space as $H_\gamma = B_\gamma^c$.

**Proposition 7.9.** *Let $M$ be a noncompact Riemannian manifold with nonnegative sectional curvature. Then, $H_\gamma$ is totally convex.*

*Proof.* Suppose, for some ray $\gamma$, we have that $H_\gamma$ is not totally convex. Then we have two points $x, y \in H_\gamma$ and a geodesic $\phi\colon [0, l] \to M$ connecting them which is not entirely contained in $H_\gamma$. That is, there is some point $z$ along $\phi$ such that $z \notin H_\gamma$. Then there is some $t_0 > 0$ such that $z \in B(\gamma(t), t)$ for all $t \geq t_0$. Since $z \in B(\gamma(t), t)$, we have that $d(z, \gamma(t)) < t$.

For each $t \geq t_0$, there is some $s_t \in (0, 1)$ such that $d(\phi(s_t), \gamma(t)) \leq d(\phi(s), \gamma(t))$ for all $s \in [0, 1]$. Let $\phi_0^t = \phi|_{[0, s_t]}$. Then, let $\phi_1^t, \phi_2^t$ be minimal geodesics connecting $\phi(s_t)$ and $\phi(0)$ to $\gamma(t)$, respectively. We notice that $\phi_0^t, \phi_1^t, \phi_2^t$ form a geodesic triangle. By Theorem 6.6, there is a triangle in Euclidean space ($H = 0$) with equal side lengths such that $\bar{\alpha}_2^t \leq \alpha_2^t$, where $\alpha_2^t$ is the angle of the geodesic triangle at the point $\phi(s_t)$.

For convenience, let $L_i = L[\phi_i^t]$. Then, by the triangle inequality, we have that $L_2 + L_1 > L_0$. Also, we have that $L_2 > L_1$ since $\phi(s_t)$ is the closest point to $\gamma(t)$.

We use the Law of Cosines to get the following:

$$\cos \bar{\alpha}_2^t = \frac{(L_0)^2 + (L_1)^2 - (L_2)^2}{2 L_0 L_1} = \frac{(L_1 + L_2)(L_1 - L_2)}{2 L_0 L_1} + \frac{L_0}{2 L_1}.$$

We notice the following:

(1) $L_1 - L_2 < 0$ and $L_1 + L_2 > 0$. Hence, $(L_1 + L_2)(L_1 - L_2) < 0$.
(2) $L_0$ is bounded above by $L[\phi]$, which is fixed.
(3) $L_2 \geq t$ for all $t$ large enough.

It follows that, for $t$ large, $\cos \bar{\alpha}_2^t < 0$. Therefore, $\alpha_2^t \geq \bar{\alpha}_2^t > \frac{\pi}{2}$, which is a contradiction because $s_t \in (0, 1)$, which is an open interval. Therefore, there would be some $s \in (0, 1)$ either greater than or less than $s_t$ such that the angle formed would be closer to right. $\square$

*Remark* 7.10. Note that the proof presented in (2) has a minor mistake in the constant factor of $\cos \bar{\alpha}_2^t$. The constant does not actually impact the growth of $\cos \bar{\alpha}_2^t$ as $t \to \infty$.

This gives us a natural construction of a totally convex set. We want to reduce $H_\gamma$ to a compact set. This can be done by taking the intersection over all rays $\gamma$ from a point $\gamma(0)$. What we get from this construction is that we can exhaust $M$ with increasing compact, totally convex sets. Having an exhaustion like this will be useful for expanding on local information to get global properties of our manifold. For example, the exhaustion will be used in the proof of the splitting theorem, where we want that the submanifold $\exp_{\sigma(0)} \nu(\sigma)$ normal to a line $\sigma$ is at a fixed distance from each set of our exhaustion.

**Proposition 7.11.** *Let $M$ be a noncompact manifold of everywhere nonnegative sectional curvature. Then for $p \in M$ and rays $\gamma$ such that $\gamma(0) = p$, we have that $C_t = \cap_\gamma H_{\gamma_t}, t \geq 0$ is a family of compact, totally convex subsets of $M$ that exhausts $M$ where $p \in \partial C_0$.*

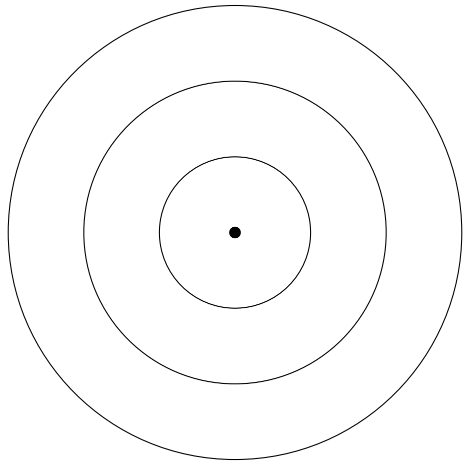


FIGURE 6. $\mathbb{R}^2$ can be exhausted by closed, totally convex balls of increasing radius.

*Proof.* By construction, we have that $p \in \partial C_0$ and that each $C_t$ is totally convex and closed. We also have that, for any $t \geq d(p,q)$, $q \in C_t$. What remains is to show that $C_t$ is compact.

Suppose $C_t$ is not compact for some $t$. Then, since $C_t$ is closed and $M$ is a metric space, $C_t$ contains all of its limit points. If $C_t$ is not compact, there is a convergent sequence of points $p_i \in C_t$ such that $d(p_i, p) \to \infty$. Since $C_t$ is totally convex, each geodesic $\psi_i$ connecting $p, p_i$ is entirely contained in $C_t$. Then, $\psi_i$ converges uniformly to some ray $\psi$. This is a contradiction since, by construction, for any $t_0 > t$, we have that $\psi(t_0) \in H_\psi^c$. $\square$

The following lemma can be thought of as an extension of the idea that the shortest path from an interior point to the boundary of a closed and convex set will form a right angle. We will state the lemma without proof and briefly discuss why this should be true visually.

**Lemma 7.12.** *Let $C \subset M$ be closed and convex. If there exist $p \in \partial C, q \in intC$ and a minimal normal geodesic $\gamma : [0,d] \to C$ such that $L[\gamma] = d(q, \partial C)$, then $T_pC - \{0\}$ is the open half-space, where $T_pC$ is the set of vectors $v$ such that any geodesic with initial velocity $v$ is contained in $C$ for some interval $[0,t]$.*

Essentially, if $T_pC\ \{0\}$ was not the open half space, then there would be some direction $v$ such that the geodesic $\gamma$ with initial velocity $v$ does not intersect $T_pC\ \{0\}$ at $\gamma(\varepsilon)$ for any $\varepsilon > 0$ small. If there exists a point $q$ such that the geodesic connecting $p$ and $q$ realizes the distance from $q$ to $\partial C$, then the distance from $q$ to any other point on $\partial C$ must be greater than $d(p,q)$.

The following theorem will be useful in determining when a subset of a manifold is flat.

**Theorem 7.13.** *Let $M$ be a Riemannian manifold with nonnegative sectional curvature. Let $C \subset M$ be closed and convex such that $\partial C \neq \varnothing$. Then the function $\psi \colon C \to \mathbb{R}$ defined by $\psi(x) = d(x, \partial C)$ is such that for any geodesic segment $\gamma$ in*

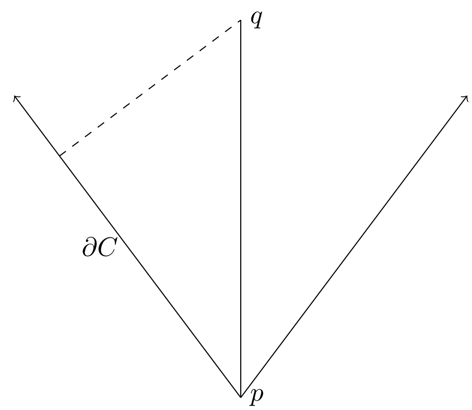



FIGURE 7. Here, we illustrate the failure of a two-dimensional counterexample to Lemma 7.12.

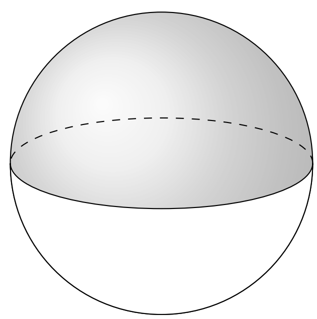


FIGURE 8. The tangent space of the sphere at a point on the boundary of the upper hemisphere is restricted to the hemisphere is the open half-space

*$C$, we have*

$$\psi(\gamma(\alpha t_1 + \beta t_2)) \geq \alpha\psi(\gamma(t_1)) + \beta\psi(\gamma(t_2))$$

*with $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$. Further, if $V$ is a parallel vector field along $\gamma$ such that $V(\gamma(s)) = \dot{\gamma}(s)$ for all $s$ and $\psi \circ \gamma \equiv c \in \mathbb{R}$ is constant on some interval $[t_1, t_2]$, then $\exp_{\gamma(s)} tV(s)$ is a minimal length geodesic for all $t \in [0, d]$ and the rectangle defined by the union of $\exp_{\gamma(s)} tV(s)$ over all $s \in [t_1, t_2]$ and $t \in [0, d]$ is flat and totally geodesic.*

The proof of this theorem is essentially a quick application of Rauch Comparison. For a full proof, see (3) or (2). Instead, we give some reasoning behind why this result should intuitively be true.

We notice that this function $\psi$ is only weakly convex. This is important since it allows for linear functions rather than strictly convex functions. When restricting $\psi$ to be identically constant, we guarantee that it is linear. This allows for the claim that our rectangle defined by $\phi$ is flat. The restriction to nonnegative sectional curvature is also what gives us weak convexity. Consider triangles in spaces of constant sectional curvature.

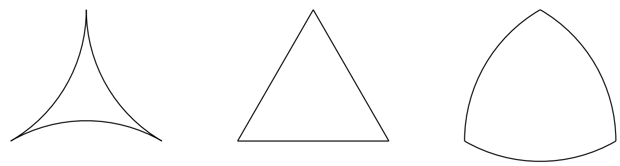


FIGURE 9. From left to right, we depict geodesic triangles in spaces of constant negative curvature, zero curvature and positive curvature. The only geodesic triangle with a geodesic contained in the interior that is at a constant distance from the boundary is the Euclidean triangle.

Taking a geodesic segment contained in the hyperbolic triangle, for example, the vertical geodesic from the top vertex will show that $\psi \circ \gamma$ as defined above is not necessarily weakly convex. We can similarly see that there is no such $\gamma$ contained in the spherical triangle. This is because, in spaces of positive sectional curvature, geodesics converge. Therefore, since the boundary of a geodesic triangle is by definition a union of geodesics, any geodesic defined on a compact interval which is contained inside a triangle of positive sectional curvature will attain a maximum distance from the boundary. It is possible to find a curve that is a fixed distance from the boundary of $C$. However, this curve is not a geodesic.

Now that we can exhaust our manifold with totally convex sets and determine when a geodesic rectangle is flat, we are ready to prove the Splitting Theorem. The proof we will give is a slightly modified version of Cheeger and Gromoll's proof which puts a greater emphasis on the flat rectangles exhausting $M$ and the fact that every such flat rectangle shares the same line. The basic idea of this proof is that we can take the exponential map restricted to any direction perpendicular to the line $\sigma$. We want this set to be entirely contained in an intersection of half-spaces, and we want it to represent the points farthest from the boundary of our intersection from half-spaces. As $a$ approaches infinity, we get that this set of points farthest from the boundary stays constant and the distance from any point in this set to the boundary is strictly equal to $a$.

If the distance from the set of maximal distance to the boundary was greater than $a$, we would have a geodesic along a negatively curved manifold. If the distance from the set of maximal distance to the boundary was less than $a$, then we would have a geodesic along a positively curved manifold. The fact that the distance is exactly $a$ gives us the desired decomposition of the line. Since our sectional curvature is zero, there is a principal curvature in the two-dimensional submanifold that is equal to zero. We want this to be the principal curvature of the line.

It turns out that every two-dimensional submanifold of $M$ containing a line is either diffeomorphic to a cylinder or $R^2$, entirely depending on whether the exponential map restricted to the other principal direction is compact. Notice that, in either case, the image of the exponential map restricted to one orthogonal direction has no boundary.

*Proof of Splitting Theorem (7.1).* Let $\sigma$ be a line in $M$. We notice that the family $\left\{H_{\sigma_{t_1}} \cap H_{-\sigma_{t_2}}\right\}$ of closed, totally convex sets exhausts $M$. Then, consider the closed, totally convex sets $D_t = H_{\sigma_t} \cap H_{-\sigma_0}$. Then, $\sigma(0) \in \partial D_t$ for all $t \in \mathbb{R}$. Also, $\sigma(\frac{t}{2}) \in \operatorname{int}(D_t)$. Since $\sigma$ is a line, the minimal geodesic connecting $\sigma(0)$ and $\sigma(\frac{t}{2})$ is $\sigma$. Then $\sigma(0)$ and $\sigma(\frac{t}{2})$ satisfy the conditions for Lemma 7.12. We get that the set of all directions from $x \in H_{\sigma_0} \cap H_{-\sigma_0}$ where exp maps into $\operatorname{int}(D_t)$ is precisely the open Euclidean half-space. By Proposition 7.11, we can exhaust $H_{\sigma_0}$ with the closed convex sets $D_t$. We get a similar claim for $-\sigma$. Therefore, every orthogonal geodesic to $\sigma$ is contained in $H_{\sigma_0} \cap H_{-\sigma_0}$.

We claim that the set of points in $D_t$ farthest from $B_{\sigma_t}$ is precisely $H_{\sigma_0} \cap H_{-\sigma_0}$. We also claim that $H_{\sigma_0} \cap H_{-\sigma_0} = \exp_{\sigma(0)} \nu(\sigma)$. We have shown above that $H_{\sigma_0} \cap H_{-\sigma_0} \subset \exp_{\sigma(0)} \nu(\sigma)$.

Let $\tau$ be a geodesic such that $\tau(0) = \sigma(0)$ and $\dot{\tau} \perp \dot{\sigma}$. Then as above, we have that $\tau(t) \in H_{\sigma_0} \cap H_{-\sigma_0}$ for all $t \in \mathbb{R}$. By Theorem 7.13, we have that $d(\tau(t), H_{\sigma_s}) = d(\tau(t), H_{-\sigma_s}) = s$ for all $s, t \in \mathbb{R}$.

Suppose there is some $p \in H_{\sigma_s} \cap H_{-\sigma_s}$ such that $d(p, \partial(H_{\sigma_s} \cap H_{-\sigma_s})) \geq s$. Let $\gamma$ be a geodesic from $\sigma(0)$ to $p$. Then, since $H_{\sigma_s} \cap H_{-\sigma_s}$ is totally convex, we have that $\gamma$ is entirely contained in $H_{\sigma_s} \cap H_{-\sigma_s}$. Since $\gamma$ never crosses the boundary of $H_{\sigma_s} \cap H_{-\sigma_s}$, we have by Lemma 7.12 that $\dot{\gamma}(0) = \dot{\sigma}(0)$. Then, as above, $d(p, \partial(H_{\sigma_s} \cap H_{-\sigma_s})) = s$. Therefore, we have that $\exp_{\sigma(0)} \nu(\sigma) \subset H_{\sigma_0} \cap H_{-\sigma_0}$, and hence $H_{\sigma_0} \cap H_{-\sigma_0} = \exp_{\sigma(0)} \nu(\sigma)$ is the set of points in $H_{\sigma_s} \cap H_{-\sigma_s}$ farthest from the boundary.

For every $\tau$ orthogonal to $\sigma$ at $\sigma(0)$, the distance $d(\tau(t), \partial(H_{\sigma_s} \cap H_{-\sigma_s})$ is a constant $s$. Then by Theorem 7.13, we get that the rectangle formed by $\tau|_{[-t,t]}$ and $\sigma|_{[-s,s]}$ is flat for all $s, t \in \mathbb{R}$. Since this rectangle is flat for any $\tau$ orthogonal to a fixed line $\sigma$, the two-dimensional submanifold spanned by $\tau$ and $\sigma$ is locally isometric to $\mathbb{R}^2$.

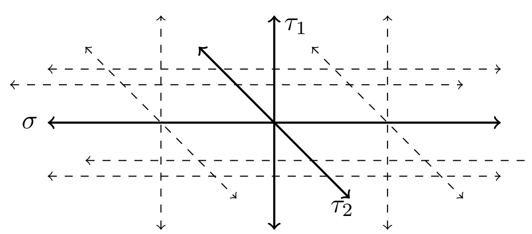



FIGURE 10. Here, we depict a decomposition of $M = \mathbb{R}^3$ into a line $\sigma$ and a submanifold spanned by $\tau_1$ and $\tau_2$. Notice that if $\sigma$ and $\tau_i$ both had nonzero principal curvature for any $i$, then by Theorem 7.13, we would not have a flat geodesic rectangle.

Let $S = H_{\sigma_0} \cap H_{-\sigma_0}$. Then, we have the normal bundle $\nu(p)$ for each $p \in S$ is a line parallel to $\sigma$ because $p = \tau(t)$ for some $t$ and the rectangle formed by $\tau$ and $\sigma$ is flat. Since *every* rectangle spanned by $\sigma$ is locally isometric to $\mathbb{R}^2$, we have that each rectangle has at least one principal curvature equal to zero. Then either the principal curvature of $\sigma$ is equal to zero or the principal curvature of every such $\tau$ orthogonal to $\sigma$ at $\sigma(0)$ is equal to zero for all $t$.

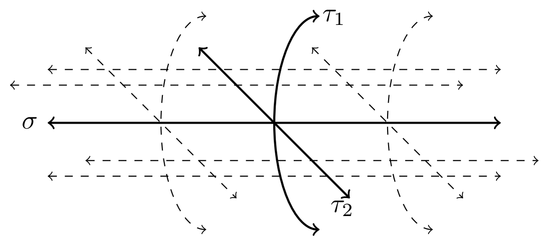



FIGURE 11. As long as the principal curvature of $\sigma$ is identically zero, the specific behavior of $\tau_i$ does not impact the existence of a decomposition. Similarly, if the principal curvature of every possible $\tau$ is identically zero, then the principal curvature of $\sigma$ does not affect the decomposition of $M$.

If $\sigma$ has zero principal curvature for all $t$, then since every geodesic rectangle defined by $\sigma$ and $\tau$ is flat, $M$ decomposes into $\mathbb{R} \times S$. If $\sigma$ has nonzero principal curvature, then since the principal curvature of every such $\tau$ orthogonal to $\sigma$ at $\sigma(0)$ is equal to zero for all $t$, $S$ must be isometric to $R^{n-1}$. We then get that since $\sigma$ is a line, $M$ is isometric to $R^n$. In either case, we can split $M$ isometrically into $\mathbb{R} \times N$, for some manifold $N$.

$\square$

*Remark* 7.14. By induction, we get that $M$ is isometric to a product $N \times \mathbb{R}^k$, where $N$ contains no lines.

In proving the Splitting Theorem, we have also shown a construction of a soul. On any noncompact such manifold, we can find at least two rays. There exists some point $p \in M$ such that $C_0$ is a set of maximum distance from the set $C_t$. For example, $C_t$ in Euclidean space should be a closed disk of radius $t$ and hence $C_0$ is just a point. Similarly, on the paraboloid, if we pick the vertex as the base point, then each $C_t$ is the set of points of distance at most $t$ from the vertex. By symmetry, we get that the geodesic in every direction emanating from the vertex is a ray. Then, we construct $C_0$ to be the vertex. On the cylinder, there are only two rays emanating from any fixed point $p$, and these rays form the line. Therefore, $C_0$ is the intersection of two half-spaces, which is a circle.

Remarkably, the Splitting Theorem for sectional curvature can be generalized to manifolds of nonnegative Ricci curvature. This is not true for the Soul Theorem. Changing the condition to nonnegative Ricci curvature is a weaker condition since the Ricci curvature is an average of the sectional curvatures. Intuitively, in two-dimensional subspaces where the sectional curvature is negative, we should get that geodesics diverge. Therefore, a totally convex, compact submanifold is unreasonable.

**Theorem 7.15.** *[Cheeger-Gromoll Splitting Theorem, 1972] Let $M$ be a complete connected Riemannian manifold with nonnegative Ricci curvature. If $M$ has a line, then $M$ is isometric to $\mathbb{R} \times N$, where $N$ is a Riemannian manifold with nonnegative Ricci curvature.*

The proof of this theorem is beyond the scope of this paper and requires a background in analysis. For a complete proof of the theorem, see (4) or (5). In summary, the theorem is proved by the superharmonicity of the Busemann functions, which essentially measure how much geodesics deviate from Euclidean straight lines. Proving superharmonicity is, roughly speaking, proving a less restrictive notion of convexity.

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