A GLIMPSE OF DIMENSION THEORY

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Abstract. This expository paper gives a brief introduction of dimension theory used in commutative algebra, focusing on Noetherian rings. We begin with introducing the motivation and basic properties of the Krull dimension. First we prove Krull's height theorem. Then we define the Hilbert polynomial of the associated graded ring of a Noetherian local ring, and give three characterizations of the dimension. After that we move to regular local rings and see its correspondence to nonsingularity. We end this exposition by mentioning how homological algbera can be used in the study of regular local rings.

Contents

All rings are assumed to be commutative and unital. In most cases, they are assumed to be Noetherian. For the ease of notation, a local ring is written as $(R, \mathfrak{m})$, with $\mathfrak{m}$ being the unique maximal ideal. For an arbitrary ring R, $\mathfrak{p}$ any prime ideal, write $R_\mathfrak{p}$ as the localization of R at $\mathfrak{p}$, i.e., $R_\mathfrak{p} = R[(R\backslash\mathfrak{p})^{-1}]$.

1. Introduction

We have encountered the idea of *dimension* in linear algebra, and we say that the dimension of a vector space is the number of elements in its basis. Alternatively, we can say that the dimension is the maximum length of descending chains of subspaces. Such definition is a geometric notion: a dimension 1 subspace looks like a line, a dimension 2 subspace looks like a sheet of paper, etc.

However, this becomes more complicated once we move to algebraic geometry, where we study algebraic varieties, the zero set of polynomials in affine space. What is the dimension of an algebraic variety? If the polynomials are nice enough, then we can still use our intuition from linear algebra, but that is not always the case.

Consider the following example: Let k be an algebraically closed field, and $\mathbb{A}^3$ be the affine space over k of dimension 3, with axis labeled x, y, z. Let $M = \{(0, y, z) \mid y, z \in k\}$ be the yz-plane (the zero-set of the polynomial x), $L = \{(x, 0, 0) \mid x \in k\}$ be the x-axis (the zero-set of polynomials y, z), and let $N = (1, 0, 0)$ be a point on L (the zero-set of polynomials $x - 1, y, z$).

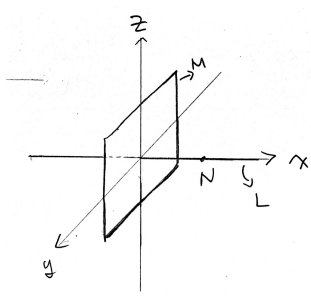

Figure 1. A sheet with a line

But what is the dimension of the space $M \cup L \cup N$, the zero set of polynomials xy, xz? The plane M has dimension 2, the line L has dimension 1, and the point N has dimension 0.

Thus we need to study dimension more or less locally. We are mostly interested in the dimension of an algebraic variety (zero-set of polynomials), and to study varieties locally is to look at their irreducible subsets. We put Zariski topology on the affine space, and for some algebraic variety in the affine space, we give it the subspace topology.

For a topological space X, we say X is *irreducible* if it is non-empty, and whenever we have $X = Z_1 \cup Z_2$, Z_1, Z_2 closed, then $X = Z_1$ or $X = Z_2$. Topologically, we define the *dimension of* X to be the supremum of lengths of descending chains of irreducible subsets of X.

Suppose $X \subseteq \mathbb{A}^n$ is an algebraic variety. Let $R = k[x_1, \ldots, x_n]/I(X)$ be its coordinate ring, where $I(X) = \{f \in k[x_1, \ldots, x_n] \mid f(x) = 0 \, \forall x \in X\}$ is the ideal of X. Then irreducible subsets in X given the Zariski topology correspond to prime ideals in R, and a descending chain of irreducible subsets in X becomes an ascending chain of prime ideals in R. Such correspondence motivates the definition of Krull dimension of an arbitrary ring.

Definition 1.1 (Krull Dimension)**.** The *Krull dimension* of a ring R, denoted $\dim(R)$, is the supremum of the lengths of strictly ascending chains of prime ideals

in R. Symbolically, it is

$$\dim(R) = \sup_n \{\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \mid \mathfrak{p}_0, \ldots, \mathfrak{p}_n \text{ are prime ideals in } R\}.$$

For $I \subseteq R$ any ideal, define $\dim(I) = \dim(R/I)$.

Krull's definition of dimension using chains of prime ideals gives us some nice properties as shown below, and we will prove them in the exposition.

Theorem 1.2. *Let R be a Noetherian ring. Then:*

(1) (Dimension is a local property) For any prime ideal $\mathfrak{p}$ of R, let $\hat{R}_{\mathfrak{p}}$ be the completion of $R_{\mathfrak{p}}$ according to $\mathfrak{p}$. Then

$$\dim(R) = \sup_{\mathfrak{p} \text{ prime ideal}} \{\dim(R_{\mathfrak{p}})\}, \qquad \dim(R_{\mathfrak{p}}) = \dim(\hat{R}_{\mathfrak{p}}).$$

(2) (Nilpotent elements do not affect dimension) Let $I \subseteq R$ be a nilpotent ideal. Then

$$\dim(R) = \dim(R/I).$$

(3) (Dimension is preserved by a map with finite fibers). If $R \subseteq S$ are rings such that S is a finitely generated R-module, then

$$\dim(R) = \dim(S).$$

(4) Let $x_1, \ldots, x_n$ be independent indeterminants. Then

$$\dim(R[x_1, \ldots, x_n]) = \dim(R[[x_1, \ldots, x_n]]) = \dim(R) + n.$$

On the other hand, if we return to our first example, we would also like to know how "far" is the point N away from the line L, dimension-wise. If $Y \subseteq X$ is an irreducible subvariety, the *codimension of Y in X*, defined as the supremum of lengths of descending chains of irreducible subsets in X ending at Y, measures such difference. Then Y corresponds to some prime ideal $\mathfrak{p}$ in the coordinate ring R, and the chain becomes a strictly ascending chain of prime ideals ending at $\mathfrak{p}$. And in general, we have:

Definition 1.3 (Codimension)**.** The *codimension*, or the *height* of a prime ideal $\mathfrak{p} \subseteq R$, denoted $\operatorname{codim}(\mathfrak{p})$, or $\operatorname{ht}(\mathfrak{p})$, is the supremum of the lengths of strictly ascending chains of prime ideals in R ending at $\mathfrak{p}$. Symbolically, it is

$$\operatorname{ht}(\mathfrak{p}) = \sup_n \{\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p} \mid \mathfrak{p}_0, \mathfrak{p}_1, \cdots, \mathfrak{p}_n \text{ are prime ideals in } R\}.$$

For an arbitrary ideal I, its codimension is the minimum of codimensions of prime ideals containing I.

Then for any prime ideal $\mathfrak{p}$ of R, since prime ideals in the localization of R at $\mathfrak{p}$ are prime ideals contained in $\mathfrak{p}$, one may see that $\operatorname{ht}(\mathfrak{p}) = \dim(R_{\mathfrak{p}})$. For example, in *Dedekind domains*, all non-zero prime ideals are maximal, so Dedekind domains have dimension 1. However, we can also have rings of infinite dimension. Here is an example from Nagata of an infinite dimensional Noetherian ring([8] 104.14):

Example 1.4. Let k be a field, and consider the ring $R = [x_1, x_2, \ldots, x_n, \ldots]$ with infinite indeterminants. Let $\mathfrak{p}_i = (x_{2^{i-1}}, x_{2^{i-1}+1}, \ldots, x_{2^i})$ for $i \geq 1$. Then the $\mathfrak{p}_i$'s are all prime. Now let $U = \bigcap_{i \geq 1}(R \backslash \mathfrak{p}_i)$. This is multiplicatively closed, so we may localize at U and put $S = R[U^{-1}]$. It can be proven that S is a Noetherian ring. If we accept that in S, the maximal ideals are the ideals $\mathfrak{m}_i = \mathfrak{p}_i S$, and

$S_{\mathfrak{m}_i} = R_{\mathfrak{p}_i}$ is a Noetherian local ring of dimension 2^i, then by 1) of Theorem 1.2, $\dim(S) = \sup_{i\geq 1}(\dim(S_{\mathfrak{m}_i})) = \infty$.

One might also hope that if $\dim(R) < \infty$, dimension and codimension of an ideal add up, i.e., for $I \subseteq R$ an ideal, we would like

$$\dim(I) + \operatorname{codim}(I) = \dim(R).$$

But this is not quite the case. Nonetheless, it is true that

$$\dim(I) + \operatorname{codim}(I) \leq \dim(R).$$

Let us return to the example of a sheet and a line in Figure 1, and see how the equality fails:

Let $R = k[x, y, z]/(x)(y, z) = k[x, y, z]/(xy, xz)$ be the coordinate ring of the plane M together with the line L, and put $I = (x - 1, y, z)$ a maximal ideal in R, which corresponds to the point N by Hilbert's Nullstellensatz. Then R is the coordinate ring of the plane M together with the line L.

First note that a longest ascending chain of prime ideals in R corresponds to a chain of decreasing subspaces consisting of M, a line in M, and a point on M on the line as shown in Figure 2. If the chain of subspaces start at L, then the corresponding chain of prime ideals have height at most 1, since there is no non-trivial subspace of a point. Thus $\dim(R) = 2$.

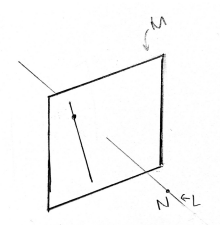

FIGURE 2. Chain of prime ideals correspond to chain of subspaces

However, $\dim(I) = \dim(R/I) = 0$ since I is maximal, and $\operatorname{codim}(I) = 1$ since there is only the line L containing N. Therefore $\dim(I) + \operatorname{codim}(I) = 1 \neq 2$.

Organization:

We will see in Section 2 that the codimension of a prime ideal is bounded by the number of its generators. We will also look at dimension of polynomial extensions of a Noetherian ring. In Section 3, we will introduce the Hilbert function, and focus on Noetherian local rings. We will prove the dimension theorem (Theorem 3.13), establishing the equality of three quantities: the degree of the Hilbert function, the Krull dimension, and the minimal number of the generators of an $\mathfrak{m}$-primary ideal. In Section 4, we will briefly study regular local rings, the case when the number of generators of the maximal ideal is exactly its height, see an example of regular local rings corresponding to non-singular points in an algebraic variety, and end this exposition by briefly discussing how homological algebra is used in the study of regular local rings. Relevant results from commutative algebra will be included in Section 5, though without proof.

2. Krull's Principal Ideal Theorem

2.1. The Theorems.

Krull proved the principal ideal theorem in 1928. The theorem describes height of prime ideals in Noetherian rings. We may obtain the height theorem through induction. The theorem has a converse, which we will also include in this section.

Theorem 2.1 (Krull's Principal Ideal Theorem)**.** *Let R be a Noetherian ring, and pick $x \in R$. Let $\mathfrak{p}$ be minimal amongst all prime ideals in R containing x. Then* $\operatorname{codim}(\mathfrak{p}) \leq 1$.

Proof. We shall show that if $\mathfrak{q}$ is any prime ideal with $\mathfrak{q} \subsetneq \mathfrak{p}$, then $\dim(R_\mathfrak{q}) = \operatorname{codim}(\mathfrak{q}) = 0$. And it follows that $\operatorname{codim}(\mathfrak{p}) \leq 1$.

If x is a unit, then there is no prime ideal containing x, so we may as well suppose x is not a unit.

First we replace R by $R_\mathfrak{p}$ since $\operatorname{ht}\mathfrak{p} = \dim(R_\mathfrak{p})$. In this way $\mathfrak{p}$ is maximal. Define the n-th symbolic power of $\mathfrak{q}$: $\mathfrak{q}^{(n)} = \{r \in R \mid rs \in \mathfrak{q}^n, s \in R\backslash\mathfrak{q}\}$. One may check that in the localization $R_\mathfrak{q}$, $(\mathfrak{q}^{(n)})_\mathfrak{q} = (\mathfrak{q}_\mathfrak{q})^n$, and $\mathfrak{q}^{(n+1)} \subseteq \mathfrak{q}^{(n)}$.

Since $\mathfrak{p}$ is the minimal prime containing x, i.e., minimal over (x), we have $R/(x)$ as an Artinian ring by Theorem 5.11. So in the quotient ring, the descending chain of ideals

$$\mathfrak{q}^{(1)} \supseteq \mathfrak{q}^{(2)} \supseteq \cdots \supseteq \mathfrak{q}^{(n)} \supseteq \cdots$$

stabilizes at some N, and in R, we have a corresponding descending chain of ideals

$$(x) + \mathfrak{q}^{(1)} \supseteq (x) + \mathfrak{q}^{(2)} \supseteq \cdots \supseteq (x) + \mathfrak{q}^{(n)} \supseteq \cdots,$$

where $(x) + \mathfrak{q}^{(n)} = (x) + \mathfrak{q}^{(n+1)}$ for all $n \geq N$, and $\mathfrak{q}^{(n)} \subseteq (x) + \mathfrak{q}^{(n+1)}$.

Using this inclusion of ideals, for any $f \in \mathfrak{q}^{(n)}$, we have $f = ax + r$ for some $a \in R, r \in \mathfrak{q}^{(n+1)}$. At the same time, $\mathfrak{q}^{(n+1)} \subseteq \mathfrak{q}^{(n)}$ gives us $ax \in \mathfrak{q}^{(n)}$. Since $\mathfrak{p}$ is minimal amongst all prime ideals containing x, and $\mathfrak{q} \subsetneq \mathfrak{p}$ is prime, it follows that $x \notin \mathfrak{q}$. Therefore, in order for ax to be in $\mathfrak{q}^{(n)}$, we need to have $a \in \mathfrak{q}^{(n)}$, and thus

$$\mathfrak{q}^{(n)} = (x)\mathfrak{q}^{(n)} + \mathfrak{q}^{(n+1)}.$$

Now we mod out both sides by $\mathfrak{q}^{(n+1)}$:

$$\mathfrak{q}^{(n)}/\mathfrak{q}^{(n+1)} = ((x)\mathfrak{q}^{(n)} + \mathfrak{q}^{(n+1)})/\mathfrak{q}^{(n+1)} = ((x)\mathfrak{q}^{(n)})/\mathfrak{q}^{(n+1)} = (x)(\mathfrak{q}^{(n)}/\mathfrak{q}^{(n+1)}).$$

Nakayama's Lemma 5.2 tells us that $\mathfrak{q}^{(n)}/\mathfrak{q}^{(n+1)} = 0$. Therefore $\mathfrak{q}^{(n)} = \mathfrak{q}^{(n+1)}$ for $n \geq N$.

Then in the local ring $R_\mathfrak{q}$, we have $(\mathfrak{q}_\mathfrak{q})^n = (\mathfrak{q}_\mathfrak{q})^{n+1}$ for $n \geq N$. Again by Nakayama, $(\mathfrak{q}_\mathfrak{q})^n = 0$. Hence $\bigcap_{n\in\mathbb{N}}(\mathfrak{q}_\mathfrak{q})^n = 0$. By Proposition 5.13, $R_\mathfrak{q}$ is an Artinian local ring, and by Theorem 5.12, $\dim(R_\mathfrak{q}) = 0 = \operatorname{codim}(\mathfrak{q})$. □

Theorem 2.2 (Krull's Height Theorem)**.** *Let R be a Noetherian ring, $\mathfrak{p}$ be a prime ideal of R. If there are $x_1, \ldots, x_n \in R$ such that $\mathfrak{p}$ is a minimal prime over the ideal $I = (x_1, \ldots, x_n)$, then* $\operatorname{ht}(\mathfrak{p}) \leq n$.

Proof. We have already shown the case when $n = 1$ in the proof of Theorem 2.1. Now we shall proceed by induction to show the rest of the theorem.

Suppose the theorem is true for the ideal generated by $x_1, \ldots, x_{n-1}$, and we want to look at minimal prime ideals over the ideal $(x_1, \ldots, x_n)$. Suppose $\mathfrak{p}$ is a minimal prime ideal over $(x_1, \ldots, x_n)$, and suppose for contradiction that there is a chain of prime ideals in R:

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \subsetneq \mathfrak{p}_{n+1} = \mathfrak{p}.$$

If $x_1 \in \mathfrak{p}_1$, then $\mathfrak{p}$ is a minimal prime ideal over $\mathfrak{p}_1 + (x_2, \ldots, x_n)$, and $\mathfrak{p}/\mathfrak{p}_1$ is the minimal prime over $(x_2, \ldots, x_n)$ in the quotient ring $R/\mathfrak{p}_1$, in which the chain becomes

$$0 = \mathfrak{p}_1/\mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n/\mathfrak{p}_1 \subsetneq \mathfrak{p}_{n+1}/\mathfrak{p}_1 = \mathfrak{p}/\mathfrak{p}_1.$$

This is a chain of prime ideals in $R/\mathfrak{p}_1$ of length n, which contradicts with the inductive hypothesis.

Now it remains to show that by replacing certain prime ideals in the chain, we may achieve $x_1 \in \mathfrak{p}_1$, i.e., if $x_1 = x \in \mathfrak{p}_k$, $x \notin \mathfrak{p}_{k-1}$ for $k \geq 2$, then there is a prime ideal strictly between $\mathfrak{p}_k$ and $\mathfrak{p}_{k-2}$ that contains x.

Consider the quotient ring after localization at $\mathfrak{p}_k$: $R_{\mathfrak{p}_k}/(\mathfrak{p}_{k-2})_{\mathfrak{p}_k}$. It is an integral domain since $(\mathfrak{p}_{k-2})_{\mathfrak{p}_k}$ is still prime. The image $\bar{x}$ of x in this quotient ring is neither zero nor a unit, so there is a minimal prime ideal in $R_{\mathfrak{p}_k}/(\mathfrak{p}_{k-2})_{\mathfrak{p}_k}$ containing $\bar{x}$, call it $\mathfrak{p}'$.

Note that the image of $(\mathfrak{p}_k)_{\mathfrak{p}_k}$ cannot be $\mathfrak{p}'$ since its height is at least 2 in the quotient ring (with the image of $(\mathfrak{p}_{k-1})_{\mathfrak{p}_k}$ in between itself and (0)), while $\mathfrak{p}'$ has height at most 1 by Theorem 2.1. The bijection of prime ideals containing $(\mathfrak{p}_{k-2})_{\mathfrak{p}_k}$ and prime ideals in the quotient ring gives us a prime ideal in the $R_{\mathfrak{p}_k}$localization corresponding to $\mathfrak{p}'$. The bijection of prime ideals in $R_{\mathfrak{p}_k}$ and prime ideals in R contained in $\mathfrak{p}_k$ gives us a prime ideal strictly between $\mathfrak{p}_k$ and $\mathfrak{p}_{k-2}$, and we are done. □

Observe that for a Noetherian local ring $(R, \mathfrak{m})$, $\dim(R) = \mathrm{ht}(\mathfrak{m})$. The height theorem tells us that the latter is bounded by the number of its generators. By the following corollary, $\dim(R) < \infty$.

Corollary 2.3. *The prime ideals in a Noetherian ring satisfy the descending chain condition, with the length of a chain of prime ideals descending from a prime $\mathfrak{p}$ bounded by the number of generators of $\mathfrak{p}$.*

There is also a converse to the height theorem:

Theorem 2.4. *Let R be a Noetherian ring, and $\mathfrak{p}$ a prime ideal with height $\leq n$. Then there are $x_1, \ldots, x_n \in \mathfrak{p}$ such that $\mathfrak{p}$ is minimal over $(x_1, \ldots, x_n)$.*

The proof, however, follows from Proposition 3.17, which works with a Noetherian local ring. We will prove this converse to height theorem after proving that. For now we may assume it is true, and use it to observe some consequences of the height theorem, especially the dimension of a polynomial ring.

2.2. Consequence of Krull's Theorems: Polynomial Extensions.

One may see that the principal ideal theorem somehow corresponds to the situation when looking for solutions to polynomial, which says that if we add one more equation, the dimension of the solution set decreases by one. We will be able to prove this after the dimension Theorem 3.13, and it is stated as Corollary 3.21.

Now let us look at polynomial extensions of a Noetherian ring, and prove the third part of Theorem 1.2. Recall that the statement goes:

Let $x_1, \ldots, x_n$ be independent indeterminants. Then

$$\dim(R[x_1, \ldots, x_n]) = \dim(R[[x_1, \ldots, x_n]]) = \dim(R) + n.$$

Using induction, it suffices to show the cases of $R[x]$ and $R[[x]]$. Before proving this, we need to build some results.

Proposition 2.5. *Let R be any ring, $S = R[x]$, $\mathfrak{p}$ a prime ideal in R, $\mathfrak{q} \subsetneq \mathfrak{q}'$ two prime ideals in S lying over $\mathfrak{p}$ (i.e., $\mathfrak{q} \cap R = \mathfrak{q}' \cap R = \mathfrak{p}$). Then $\mathfrak{q} = \mathfrak{p}S$.*

Proof. If R is a field, then $S = R[x]$ is a principal ideal domain. Then every non-zero prime ideal is maximal, and in particular $\mathfrak{q}, \mathfrak{q}'$ are maximal. But by assumption $\mathfrak{q} \subsetneq \mathfrak{q}'$, so $\mathfrak{q} = (0)$. At the same time, R being a field gives us $\mathfrak{p} = (0)$, and one may see that $\mathfrak{p}S = \mathfrak{q}$.

It remains to show that we may turn R into a field without changing the relationship between $\mathfrak{p}$ and $\mathfrak{q}$. First observe that $R/\mathfrak{p}$ can be considered as a subring of $S/\mathfrak{q}$, thus we may as well replace R by $R/\mathfrak{p}$, where $\mathfrak{p}$ becomes the zero ideal. Now we localize the domain $R/\mathfrak{p}$ at the zero ideal, and the local ring obtained is the fraction field of $R/\mathfrak{p}$ with the zero ideal being the unique maximal ideal. □

Corollary 2.6. *Let R be a Noetherian ring, $I \subsetneq R$ a proper ideal. Let $\mathfrak{p}$ be a prime ideal containing I. Let $S = R[x]$, put $J = IS, \mathfrak{q} = \mathfrak{p}S$. If $\mathfrak{p}$ is minimal over I, then $\mathfrak{q}$ is minimal over J.*

Proof. First note that

$$S/\mathfrak{q} = R[x]/\mathfrak{p}R[x] \cong R[x]/\mathfrak{p}[x] \cong (R/\mathfrak{p})[x].$$

Then since $R/\mathfrak{p}$ is an integral domain, $S/\mathfrak{q}$ is also an integral domain, and $\mathfrak{q}$ is a prime ideal in S.

It remains to show that $\mathfrak{q}$ is minimal over J. Since there is a bijection between prime ideals containing I and prime ideals in R/I, we may replace R by R/I. Thus $I = (0)$, and therefore $J = IS = (0)$. Suppose $\mathfrak{q}' \subsetneq S$ is a minimal over J such that $\mathfrak{q}' \subsetneq \mathfrak{q}$. Then

$$\mathfrak{q}' \cap R \subseteq \mathfrak{q} \cap R = \mathfrak{p}S \cap R = \mathfrak{p}.$$

For the last equality, in general we have $\mathfrak{p} \subseteq \mathfrak{p}S \cap R$. But in this case, one may see that, polynomials in the intersection $\mathfrak{p}S \cap R$ are the constant polynomials in $\mathfrak{p}S$. Since $\mathfrak{p}$ is prime, it follows that $\mathfrak{p}S \cap R$ is exactly $\mathfrak{p}$. Also, note that $\mathfrak{q}' \cap R$ is a prime ideal, since $R/(\mathfrak{q}' \cap R)$ is a subring of $S/\mathfrak{q}'$, and the latter is an integral domain.

But by assumption, $\mathfrak{q} \cap R$ is prime lying over I, so $\mathfrak{p} \subseteq \mathfrak{q}' \cap R$, and we have the equality $\mathfrak{p} = \mathfrak{q}' \cap R$. By Proposition 2.5, $\mathfrak{q}' = \mathfrak{p}S = \mathfrak{q}$, and we have reached a contradiction. Hence $\mathfrak{q}$ is a minimal prime over J. □

Proposition 2.7. *Let R be a Noetherian ring, $S = R[x]$, $\mathfrak{p}$ a prime ideal in R, and $\mathfrak{q} = \mathfrak{p}S$. Then $\mathrm{ht}(\mathfrak{q}) = \mathrm{ht}(\mathfrak{p})$.*

Proof. Let $\mathrm{ht}(\mathfrak{p}) = n$. By Theorem 2.4, there is $I \subsetneq R$ generated by n elements such that $\mathfrak{p}$ is minimal over I. By Corollary 2.6, $\mathfrak{q} = \mathfrak{p}S$ is a minimal prime over $J = IS$, where J is also generated by n elements in S. So by Theorem 2.2, $\mathrm{ht}(\mathfrak{q}) \leq n = \mathrm{ht}(\mathfrak{p})$.

On the other hand, suppose there is a chain of prime ideals in R,

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}.$$

Put $\mathfrak{q}_i = \mathfrak{p}_i[x] \subseteq S$ for $0 \leq i \leq n$. It is a quick check that each $\mathfrak{q}_i$ is a prime ideal, and we have a chain of prime ideals in S:

$$\mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_n = \mathfrak{q}.$$

Thus $\mathrm{ht}(\mathfrak{q}) \geq n = \mathrm{ht}(\mathfrak{p})$, and equality is obtained. □

Now we are ready to prove the theorem about the dimension of a polynomial ring.

Theorem 2.8. *Let R be a Noetherian ring, $S = R[x]$. Then* $\dim(S) = \dim(R) + 1$.

Proof. If $\dim(R) = \infty$, then there is nothing to show. Suppose $\dim(R) = n < \infty$, and there is a chain of prime ideals in R:

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n.$$

If we put $\mathfrak{q}_i = \mathfrak{p}_i S$ for $0 \leq i \leq n$, then by Proposition 2.7, $\mathrm{ht}(\mathfrak{q}_n) = \mathrm{ht}(\mathfrak{p}_n)$. However, note that $\mathfrak{q}_{n+1} = \mathfrak{p}_n + (x) \supsetneq \mathfrak{q}_n$ is prime: For $f(x), g(x) \in S$ such that $f(x)g(x) \in \mathfrak{q}_{n+1}$, the product of two constant terms in $f(x), g(x)$ is in $\mathfrak{p}_n$, so one of $f(x), g(x)$ has constant term in $\mathfrak{p}_n$, and hence itself in $\mathfrak{p}_n + (x) = \mathfrak{q}_{n+1}$. Then $\dim(S) \geq n + 1 = \dim(R) + 1$.

It remains to show the other inequality. Suppose $\dim(S) = m$, and we have a chain of prime ideals in S:

$$\mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_m.$$

Let $\mathfrak{p}_i = \mathfrak{q}_i \cap R$ for $0 \leq i \leq m$. The $\mathfrak{p}_i$'s are prime using argument in the proof of Corollary 2.6. If all the $\mathfrak{p}_i$'s are distinct, then $\dim(R) \geq \dim(S) > \dim(S) - 1$ and we have a contradiction. Suppose not, and let j be the largest index such that $\mathfrak{p}_j = \mathfrak{p}_{j+1}$.

Then by Proposition 2.5, $\mathfrak{q}_j = \mathfrak{p}_j S$. By Proposition 2.7, $\mathrm{ht}(\mathfrak{p}_j) = \mathrm{ht}(\mathfrak{q}_j) \geq j$. By the choice of j, we still have a strictly ascending chain of prime ideals:

$$\mathfrak{p}_{j+1} \subsetneq \mathfrak{p}_{j+2} \subsetneq \cdots \subsetneq \mathfrak{p}_m.$$

So $\mathrm{ht}(\mathfrak{p}_j) + (m - (j+1)) \leq \dim(R)$, and

$$m - 1 = j + (m - (j+1)) \leq \mathrm{ht}(\mathfrak{p}_j) + (m - (j+1)) \leq \dim(R).$$

This gives us $\dim(S) = m \leq \dim(R) + 1$, and we are home. □

The case of $R[[x]]$ is similar, as we may follow the same procedure as the case of $R[x]$: if R is a field, then $R[[x]]$ is a Euclidean domain (with norm measuring the degree of the first non-zero term), and hence a principal ideal domain. Since $R[x, y] = (R[x])[y], R[[x, y]] = (R[[x]])[[y]]$, using induction on the number of variables, we have shown the desired result.

3. The Hilbert Function and Krull Dimension

3.1. The Length Function of Finitely Generated Modules.

Before introducing the Hilbert function, we need to know the length function of finitely generated modules of a ring R.

Definition 3.1 (Composition Series)**.** Suppose we have a chain of submodules of M of length n:

$$M = M_n \supsetneq M_1 \supsetneq \cdots \supsetneq M_n = 0.$$

We call this chain a *composition series of M* if it is a maximal chain, i.e., no extra submodules can be inserted in the chain.

Proposition 3.2 ([1] Proposition 6.7)**.** *Suppose that M has a composition series of length n. Then every composition series of M has length n, and every chain in M can be extended to a composition series.*

Proposition 3.3 ([1] Proposition 6.8)**.** *M has a composition series if and only if M satisfies both descending chain condition and ascending chain condition.*

We call a module M satisfying both the descending chain condition and the ascending chain condition a *module of finite length*, and denote the length of its composition series as $l(M)$. This is a function on modules over R, and has value in $\mathbb{Z}$. For example, if R is a field, then finitely generated moduels over R are finite dimensional vector spaces, and their lengths are exactly their dimensions.

Definition 3.4 (Additive Function)**.** An *additive function* λ is defined on a collection of modules, $\mathcal{C}$, takes its value in $\mathbb{Z}$, and satisfies the following condition: If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of R-modules, and $M', M, M'' \in \mathcal{C}$, then $\lambda(M') - \lambda(M) + \lambda(M'') = 0$. It follows that if we have an exact sequence $0 \to M_0 \to M_1 \to \cdots \to M_n \to 0$ with all terms and kernels in $\mathcal{C}$, then $\sum_{i=0}^{n} \lambda(M_n) = 0$ ([1] Proposition 2.11).

Proposition 3.5 ([1] Proposition 6.9)**.** *The length function l is additive on the class of all R-modules of finite length.*

3.2. **The Hilbert Function.**

Let $R = \bigoplus_{n=0}^{\infty} R_n$ be a Noetherian graded ring, and $M = \bigoplus_{n=0}^{\infty} M_n$ be a finitely generated graded R-module. By Proposition 5.19, both R and M are finitely generated R_0 modules. Suppose R has generators $x_1, \ldots, x_s$ with corresponding degrees $k_1, \ldots, k_s$. Suppose M has homogeneous elements $m_1, \ldots, m_t$ as generators over R, with corresponding degrees $r_1, \ldots, r_t$. Then for each M_n, as an R_0-module, it has generators $\{g_j(x)m_j\}_{j=1}^{t}$, where $g_j(x)$ is a monomial in x_i's $(1 \leq i \leq s)$ with total degree $n - r_j$.

Let λ be an integer-valued additive function on finitely generated R_0-modules. Define the *Poincare series of M* with respect to λ to be the generating function of λ:

$$P(M,t) = \sum_{n=0}^{\infty} \lambda(M_n)t^n \in \mathbb{Z}[[t]].$$

We have a theorem from Hilbert and Serre saying that the Poincare series can be written as a rational function dependent on the number of generators of R as an R_0-module:

Theorem 3.6. *The Poincare series of M with respect to λ, $P(M,t)$ is a rational function in t of the form*

$$P(M,t) = \frac{f(t)}{\prod_{i=1}^{s}(1 - t^{k_i})},$$

where $f(t)$ is a polynomial with integer coefficients.

Proof. We shall use induction on s, the number of generators of R as an R_0-module.

When $s = 0$, $R = R_0$, and $R_n = 0$ for all n. Then M is a finitely generated, graded R_0 module, and $M_n = 0$ for n large enough. Hence $P(M,t)$ is a polynomial.

Now let $s > 0$, and suppose the statement is true for $s - 1$. Note that multiplication by x_s gives an R-module homomorphism on M, i.e., $M_n \to M_{n+k_s}$. Let $K_n \subseteq M_n$ denote the kernel, and $L_{n+k_s} = M_{n+k_s}/(x_s M_n)$ denote the cokernel. Then we have the following exact sequence:

$$0 \longrightarrow K_n \longrightarrow M_n \xrightarrow{\cdot x_s} M_{n+k_s} \longrightarrow L_{n+k_s}.$$

Since λ is additive, we have

$$\lambda(K_n) - \lambda(M_n) + \lambda(M_{n+k_s}) - \lambda(L_{n+k_s}) = 0,$$
$$\lambda(M_{n+k_s}) - \lambda(M_n) = \lambda(L_{n+k_s}) - \lambda(K_n).$$

Let $K = \bigoplus_{n=0}^{\infty} K_n$, $L = \bigoplus_{n=0}^{\infty} L_n$. Multiply both sides by t^{n+ks} and take sum according to n gives us:

$$(\sum_{n=0}^{\infty} \lambda(K_n)t^n - \sum_{i=0}^{k_s-1} \lambda(M_i)t^i) - t^{k_s} \sum_{n=0}^{\infty} \lambda(M_n)t^n = \sum_{n=0}^{\infty} \lambda(L_n)t^n - t^{k_s} \sum_{n=0}^{\infty} \lambda(K_n)t^n,$$

$$(1 - t^{k_s})P(M,t) = P(L,t) - t^{k_s}P(K,t) + \sum_{i=0}^{k_s-1} \lambda(M_i)t^i$$

Note that for both K, L, since each component is annihilated by x_s, we may say that both are $R_0[x_1, \ldots, x_{s-1}]$-modules. Then using the hypothesis, we reach the desired outcome. □

When $k_i = 1$ for $1 \leq i \leq s$, we have $P(M,t) = \frac{f(t)}{(1-t)^s}$, and we use $d(M)$ to denote the order of the pole of $P(M,t)$ at $t = 1$.

Corollary 3.7. *When $k_i = 1$ for $1 \leq i \leq s$, for sufficiently large n, $\lambda(M_n)$ is a polynomial in n (with rational coeffients) of degree $d-1$, where $d = d(M)$.*

Proof. When $k_i = 1$ for all i, we have $P(M,t) = \frac{f(t)}{(1-t)^s}$, and $\lambda(M_n)$ is the coefficient of t^n. In this case, d is simply the degree of $(1-t)$ in the denominator after simplifying the fraction. We may as well suppose that $s = d$, $f(1) \neq 0$, and $f(t) = \sum_{k=0}^{N} a_k t^k$.

Using the negative binomial series, we get

$$(1-t)^{-d} = \sum_{k=0}^{\infty} \binom{d+k-1}{d-1} t^l.$$

Then to find the coefficient of t^n in $P(M,t)$, we pair $a_k t^k$ in $f(t)$ with the t^{n-k} term in the expansion of $(1-t)^{-d}$, and we get

$$\lambda(M_n) = \sum_{k=0}^{N} a_k \binom{d+(n-k)-1}{d-1}.$$

The claim is that if we expand the terms, then the leading term ordered according to the degree of n, is

$$(\sum_{k=0}^{N} a_k) \frac{n^{d-1}}{(d-1)!}.$$

It suffices to show that for each $0 \leq k \leq N$, the leading term of $a_k \binom{d+(n-k)-1}{d-1}$ is $a_k \frac{n^{d-1}}{(d-1)!}$. Note that

$$\binom{d+(n-k)-1}{d-1} = \frac{(d+(n-k)-1)!}{(d-1)!(n-k)!} = \frac{(d+(n-k)-1)(d+(n-k)-2)\cdots(n-k+1)}{(d-1)!}.$$

Since n, d, k are known, each factor in the numerator can be written in the form $(n+m)$ for some $m \in \mathbb{Z}$. There are in total $(d+(n-k)-1)-(n-k+1)+1 = d-1$ factors, so the leading term according to n is of degree $d-1$, and the leading term is indeed $a_k \frac{n^{d-1}}{(d-1)!}$. □

Corollary 3.8. *If $x \in R_k$ is not a zero-divisor of M, then $d(M/xM) = d(M) - 1$.*

Proof. Since x is not a zero-divisor of M, multiplication by x is an injective map $M \xrightarrow{\cdot x} M$. As in the proof of Theorem 3.6, let K denote the kernel of this map, and set $L = M/xM$. Then $K = 0$, and in terms of the Poincare series, we have

$$(1 - t^k)P(M, t) = P(L, t) + g(t),$$

where $g(t)$ is some polynomial in t. Thus the degrees of the pole at $t = 1$ on both sides are the same, and we have the desired result. □

Usually, the polynomial in Corollary 3.7 is called the *Hilbert function with respect to λ*. In this exposition, we will focus on the case when λ is the length function l over finitely generated modules of M, as defined in the previous subsection. Then by Corollary 3.7, $l(M_n)$ is a polynomial in n, and we call it the *Hilbert polynomial*. Note that if R_0 is a field, and we put $M = R$, then $l(R_n)$ measures the dimension of R_n as a vector space over R_0, and Corollary 3.7 tells us how fast the dimension grows as n increases.

Example 3.9. Let $R = k[x_1, \ldots, x_d]$, where $x_1, \ldots, x_d$ are independent indeterminants. Note that the polynomial ring R comes with a natural grading given by the degree of each monomial, assuming that $x_1, \ldots, x_d$ all have degree 1. Then the degree-n part R_n is generated freely as a k-module by monomials in $x_1, \ldots, x_d$ of degree n. There are in total $\binom{d+n-1}{d-1}$ of these monomials, all independent over k, so $l(R_n) = \binom{d+n-1}{d-1}$, and the Poincare series is $P(R, t) = \sum_{n=0}^{\infty} l(R_n)t^n = (1 - t)^{-d}$.

The following theorem discusses the relationship between the length of a finitely generated module, and the minimal number of generators of an $\mathfrak{m}$ primary ideal.

Proposition 3.10. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, $\mathfrak{q}$ be an $\mathfrak{m}$-primary ideal, M a finitely generated R module. Suppose (M_n) is a stable $\mathfrak{q}$-filtration of M. Then:*

(1) M/M_n is of finite length for each $n \geq 0$.
(2) For all sufficiently large n, this length is a polynomial $g(n)$ of degree $\leq s$, where s is the least number of generators of $\mathfrak{q}$.
(3) The degree and coefficient of $g(n)$ depend only on $M, \mathfrak{q}$, i.e., they are independent of the filtration chosen.

Proof. Let $G(R) = \bigoplus_{n=0}^{\infty} \mathfrak{q}^n/\mathfrak{q}^{n+1}$, $G(M) = \bigoplus_{n=0}^{\infty} M_n/M_{n+1}$ (see Definition 5.21 on associated graded rings).

First note that $G_0(R) = R/\mathfrak{q}$ is Artinian: R being Noethrian implies $R/\mathfrak{q}$ being Noethrian. By Proposition 5.4, the radical of $\mathfrak{q}$, $r(\mathfrak{q}) = \mathfrak{m}$ is the smallest prime ideal containing $\mathfrak{q}$, and the zero ideal in $R/\mathfrak{q}$ is prime if and only if $\mathfrak{q} = \mathfrak{m}$. Therefore $\dim(R) = 0$, and Theorem 5.12 tells us that $G_0(R)$ is Artinian. Also, since R is Noetherian, by Proposition 5.22, $G(R)$ is Noethrian, and $G(M)$ is a finitely generated, graded $G(R)$-module.

Since (M_n) is a $\mathfrak{q}$-filtration, each $G_n(M) = M_n/M_{n+1}$ is annihilated by $\mathfrak{q}$, and hence $G_n(M)$ is a Noetherian $R/\mathfrak{q}$-module. Also, since $R/\mathfrak{q}$ is Artinian, $G_n(M)$ is Artinian. By Proposition 3.3, it has finite length.

Observe that we can form a composition series of M/M_n from composition series of M_{r-1}/M_r when $1 \leq r \leq n$ and $M_0 = M$: A composition series of M_{r-1}/M_r corresponds to a maximal chain of strictly decreasing submodules starting from M_{r-1} and ending at M_r. Then we may concatenate these chains to form a strictly

decreasing chain starting from M and ending at M_n, into which we cannot insert any more submodules. Modding out by M_n gives us a composition series of M/M_n.

Let l be the length function on $R/\mathfrak{q} = G_0(R)$-modules of finite length. Define g to be a function that takes in an integer n and outputs the length of M/M_n. Then

$$g(n) = l(M/M_n) = \sum_{r=1}^{n} l(M_{r-1}/M_r).$$

Since each $l(M_{r-1}/M_r)$ is finite, $g(n) < \infty$ for all n, and M/M_n is of finite length. This finishes the proof of (1).

Suppose $\mathfrak{q} = (x_1, \ldots, x_s)$. Then the images $\bar{x}_i$ in $\mathfrak{q}/\mathfrak{q}^2$ generate $G(R)$ as a $G_0(R)$-module. Each $\bar{x}_i$ has degree 1. So by Corollary 3.7, $l(G_n(M)) = l(M_n/M_{n+1}) = g(n+1) - g(n)$ is some polynomial of degree at most $s-1$ for all large n. Hence g is a polynomial for sufficiently large n of degree not greater than s. This finishes proof of (2).

Now let $(\widetilde{M}_n)$ be another stable $\mathfrak{q}$-filtration, and let $\widetilde{g}(n) = l(M/\widetilde{M}_n)$ be the corresponding polynomial obtained using a similar process as above. By Lemma 5.18, $(M_n), (\widetilde{M}_n)$ have bounded difference: there exists n_0 such that $M_{n+n_0} \subseteq \widetilde{M}_n$, $\widetilde{M}_{n+n_0} \subseteq M_n$. In terms of lengths of composition series, this means that a composition series of M/M_{n+n_0} can include $\widetilde{M}_n/M_{n+n_0}$, and one of $M/\widetilde{M}_{n+n_0}$ can include $M_n/\widetilde{M}_{n+n_0}$. Then $g(n+n_0) \geq \widetilde{g}(n)$, and $\widetilde{g}(n+n_0) \geq g(n)$. As n gets larger, the ratio $\frac{g(n)}{\widetilde{g}(n)}$ approaches 1, suggesting that the two polynomials have the same degree and leading coefficient. □

Thus it suffices to look at a stable $\mathfrak{q}$-filtration $(\mathfrak{q}^n M)$. Let us denote the corresponding length polynomial by $\chi_{\mathfrak{q}}^M(n)$. For sufficiently large n, $\chi_{\mathfrak{q}}^M(n) = l(M/\mathfrak{q}^n M)$. Denote $\chi_{\mathfrak{q}}^R(n)$ by $\chi_{\mathfrak{q}}(n)$, and call it *the characteristic polynomial of the* $\mathfrak{m}$*-primary ideal* $\mathfrak{q}$. Put $M = R$, and we have:

Corollary 3.11. *The length of $R/\mathfrak{q}^n$ for all sufficiently large n is a polynomial $\chi_{\mathfrak{q}}(n)$ of degree is at most s, where s is the least number of generators of $\mathfrak{q}$.*

Proposition 3.12. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, and let $\mathfrak{q}$ be an $\mathfrak{m}$-primary ideal. Then* $\deg(\chi_{\mathfrak{q}}(n)) = \deg(\chi_{\mathfrak{m}}(n))$.

Proof. Since $\mathfrak{q}$ is $\mathfrak{m}$-primary, by Proposition 5.5, $\mathfrak{m}^r \subseteq \mathfrak{q} \subseteq \mathfrak{m}$ for some r. Then $\mathfrak{m}^{rn} \subseteq \mathfrak{q}^n \subseteq \mathfrak{m}^n$, and for large n, $\chi_{\mathfrak{m}}(n) \leq \chi_{\mathfrak{q}}(n) \leq \chi_{\mathfrak{m}}(rn)$ using a similar argument as in the proof of (3) of Proposition 3.10. As n increases, the rate of growth of the polynomial is determined by its term of highest degree, so the two polynomials $\chi_{\mathfrak{q}}(n)$ and $\chi_{\mathfrak{m}}(n)$ have the same degree. □

Now we have seen that the degree of the characteristic polynomial $\chi_{\mathfrak{q}}(n)$ does not change if we choose another $\mathfrak{m}$-primary ideal. We may as well denote this degree as $d(R)$. Note that this number is also $d(G_{\mathfrak{m}}(R))$, the order of the pole of the Poincare series of $G_{\mathfrak{m}}(R)$, the associated graded ring, with respect to the length function l. We shall see in the coming part that $d(R) = \dim(R)$, for R being a Noetherian local ring. This will give us another description of the dimension of a ring.

3.3. **The Dimension Theorem.**

Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let $\delta(R)$ denote the least number of generators of $\mathfrak{m}$-primary ideals of R. We shall prove the following theorem by

showing that

$$\delta(R) \geq d(R) \geq \dim(R) \geq \delta(R).$$

Theorem 3.13. *For any Noetherian local ring $(R, \mathfrak{m})$, the following three integers are equal:*

(1) The maximal length of chains of prime ideals in R, $\dim(R)$;
(2) The degree of the characteristic polynomial $\chi_{\mathfrak{m}}(n) = l(R/\mathfrak{m}^n)$;
(3) The least number of generators of an $\mathfrak{m}$-primary ideal of R.

From Corollary 3.11, we got $\delta(R) \geq d(R)$. It remains to show the other two inequalities. Proposition 3.16 shows that $d(R) \geq \dim(R)$, and Proposition 3.17 shows that $\dim(R) \geq \delta(R)$.

Proposition 3.14. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, $\mathfrak{q}$ an $\mathfrak{m}$-primary ideal. Let M be a finitely generated R-module, and $x \in R$ be a non-zerodivisor of M. Put $M' = M/xM$. Then $\deg(\chi_{\mathfrak{q}}^{M'}) \leq \deg(\chi_{\mathfrak{q}}^{M}) - 1$.*

Proof. Put $N = xM$. Then since x is a non-zerodivisor of M, the kernel of multiplication by x is trivial, and we have a R-module isomorphism $N \cong M$. Let $N_n = N \cap \mathfrak{q}^n M$. Then we have a short exact sequence

$$0 \longrightarrow N/N_n \xrightarrow{\cdot x} M/\mathfrak{q}^n M \longrightarrow M'/\mathfrak{q}^n M' \longrightarrow 0.$$

By Proposition 5.20 (Artin-Rees), (N_n) is a stable $\mathfrak{q}$-filtration. Since $N \cong M$ as R-modules, their composition series are of the same length. It follows that $g(n) = l(N/N_n)$ and $\chi_{\mathfrak{q}}^{M}(n)$ have the same degree and leading coefficient. Recall that length is additive, so $g(n) = \chi_{\mathfrak{q}}^{M}(n) - \chi_{\mathfrak{q}}^{M'}(n)$ for all sufficiently large n, which implies that $\deg(\chi_{\mathfrak{q}}^{M}(n)), \deg(\chi_{\mathfrak{q}}^{M'}(n))$ differ by at least 1. □

If we put $M = R$, then we get the following corollary:

Corollary 3.15. *For R a Noetherian local ring, $x \in R$ not a zerodivisor, we have $d(R/(x)) \leq d(R) - 1$.*

Proposition 3.16. *For $(R, \mathfrak{m})$ a Noetherian ring, $d(R) \geq \dim(R)$.*

Proof. We shall use induction on $d = d(R)$. When $d = 0$, the polynomial $\chi_{\mathfrak{m}}(n)$ is a constant, i.e., $l(R/\mathfrak{m}^n) = l(R/\mathfrak{m}^{n+1})$ for large n. Hence $\mathfrak{m}^n = \mathfrak{m}^{n+1}$. Then by Nakayama's Lemma 5.2, $\mathfrak{m}^n = 0$, and by Proposition 5.13, R is Artinian. By Theorem 5.12, $\dim(R) = 0 = d$.

Now suppose $d > 0$, and the statement holds for $d-1$. Suppose we have a chain of prime ideals

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_r.$$

The goal is to show that $r \leq d$. By the definition of supremum, it follows that $\dim(R) \leq d$. Pick $x \in \mathfrak{p}_1 \backslash \mathfrak{p}_0$. Then in the integral domain $R' = R/\mathfrak{p}_0$, the image of x, x' is not a zero divisor. Hence by Corollary 3.15, $d(R'/(x')) \leq d(R') - 1$.

Let $\mathfrak{m}'$ denote the image of $\mathfrak{m}$ in R'. Then $\mathfrak{m}'$ is also the maximal ideal in R', and we have $R'/\mathfrak{m}'^n$ contained in $R/\mathfrak{m}^n$ as a subring. Therefore $l(R'/\mathfrak{m}'^n) \leq l(R/\mathfrak{m}^n)$ Then: $d(R'/(x')) \leq d(R') - 1 \leq d(R) - 1 = d - 1$.

By the induction hypothesis, $\dim(R'/(x')) \leq d-1$, i.e., length of chains of prime ideals is no greater than $d-1$. The image of the chain $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_r$ in $R'/(x')$ has length $r-1$, since only $\mathfrak{p}_0$ has image the zero ideal. Thus $r - 1 \leq d - 1$. □

Proposition 3.17. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, $\dim(R) = d$. Then there exists an $\mathfrak{m}$-primary ideal in R generated by d elements. Hence $\dim(R) \geq \delta(R)$.*

Proof. We shall use induction on $i \leq d$ to show that for any $x_1, \ldots, x_i \in R$, every prime ideal containing $(x_1, \ldots, x_i)$ has codimension at least i.

Codimension of a prime ideal is always nonnegative, so the case $i = 0$ is trivial. Suppose $i > 0$, and we have already constructed the ideal $(x_1, \cdots, x_{i-1})$ such that all primes containing it have codimension at least $i - 1$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ be minimal prime ideals over $(x_1, \ldots, x_{i-1})$ that have height $i - 1$. If there is no such minimal prime, then if we add another generator x_i, all prime ideals containing $(x_1, \ldots, x_i)$ have height at least i.

Then by the induction hypothesis, $i-1 < d = \dim(R)$, which means that $\mathfrak{m} \neq \mathfrak{p}_j$ for $1 \leq j \leq s$, and $\mathfrak{m} \neq \bigcup_j \mathfrak{p}_j$ by Proposition 5.1. We may take $x \in \mathfrak{m} \backslash \bigcup_j \mathfrak{p}_j$. Let $\mathfrak{q}$ be a minimal prime ideal containing $(x_1, \ldots, x_i)$. Then $(x_1, \ldots, x_{i-1}) \subseteq \mathfrak{q}$ means that $\mathfrak{q}$ contains some minimal prime $\mathfrak{p}$ over $(x_1, \ldots, x_{i-1})$. If $\mathfrak{p} = \mathfrak{p}_j$ for some j, then $\mathrm{ht}(\mathfrak{q}) \geq \mathrm{ht}(\mathfrak{p}) + 1 = \mathrm{ht}(\mathfrak{p}_j) + 1 = i$. Otherwise, $\mathrm{ht}(\mathfrak{q}) \geq \mathrm{ht}(\mathfrak{p}) + 1 > i - 1 + 1 = i$.

It remains to show that when $i = d$, the ideal $(x_1, \ldots, x_d)$ is $\mathfrak{m}$-primary. But for a minimal prime ideal $\mathfrak{q}$ containing $(x_1, \ldots, x_d)$, $\mathrm{ht}(\mathfrak{q}) \geq d = \dim(R) = \mathrm{ht}(\mathfrak{m})$ means $\mathfrak{q} = \mathfrak{m}$, since $\mathfrak{m}$ is the unique maximal ideal of R. Therefore $(x_1, \ldots, x_d)$ is $\mathfrak{m}$-primary. □

Now as promised, we are ready to prove the converse of Krull's height theorem. Recall that the statement goes:

Theorem 3.18 (As Theorem 2.4)**.** *Let R be a Noetherian ring, and $\mathfrak{p}$ a prime ideal with height $\leq n$. Then there are $x_1, \ldots, x_n \in \mathfrak{p}$ such that $\mathfrak{p}$ is minimal over $(x_1, \ldots, x_n)$.*

Proof. Suppose $\mathrm{ht}(\mathfrak{p}) = n$. Then in the local ring $R_\mathfrak{p}$, $\dim(R_\mathfrak{p}) = n$. Hence by Proposition 3.17, there exists a $\mathfrak{p}_\mathfrak{p}$-primary ideal $\mathfrak{q}$ generated by n elements. We may multiply all the denominators of these generators together such that they all have the same denominator, and $\mathfrak{q} = (\frac{x_1}{s}, \ldots, \frac{x_n}{s})$ for some $s \in R \backslash \mathfrak{p}$. Note that all the x_i's are in $\mathfrak{p}$ or otherwise $\mathfrak{q} = R$. Also, by Proposition 5.5, $\mathfrak{q}$ being $\mathfrak{p}_\mathfrak{p}$-primary means that $r(\mathfrak{q}) = \mathfrak{p}_\mathfrak{p}$.

Now let $I = (x_1, \ldots, x_n)$ be an ideal in R, and it follows that $r(I) = \mathfrak{p}$. Thus by Proposition 5.4, $\mathfrak{p}$ is minimal over I, an ideal generated by n elements in $\mathfrak{p}$. □

3.4. Consequences of the Dimension Theorem.

Let us see some immediate corollaries of the dimension theorem, and prove the properties mentioned in Theorem 1.2. Corollary 3.19 bounds the dimension of R as a vector space over its residue field $R/\mathfrak{m}$. Corollary 3.20 says that dimension is a local property, as under the Zariski topology, passing from localization to completion is the same as restricting oneself to even smaller neighorhoods around a point. Finally, Corollary 3.21 has a geometric interpretation: if we add one more restriction to an algebraic variety, then its dimension is reduced by 1.

Corollary 3.19. *Let $(R, \mathfrak{m})$ be a Noetherian local ring. Then $\dim(R) \leq \dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2)$.*

Proof. If $x_1, \ldots, x_d \in \mathfrak{m}$ have images in $\mathfrak{m}/\mathfrak{m}^2$ that form a basis over $R/\mathfrak{m}$, then by Lemma 5.2, $x_1, \ldots, x_n$ generate $\mathfrak{m}$. Now use Theorem 3.13. □

Corollary 3.20. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, $\hat{R}$ be the $\mathfrak{m}$-adic completion of R. Then $\dim(R) = \dim(\hat{R})$.*

Proof. The isomorphism $R/\mathfrak{m}^n \cong \hat{R}/\hat{\mathfrak{m}}^n$ by Proposition 5.17 gives us another isomorphism $G_{\mathfrak{m}}(R) \cong G_{\hat{\mathfrak{m}}}(\hat{R})$. Hence the characteristic polynomials of R and $\hat{R}$ have the same degree. □

Corollary 3.21. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, $x \in \mathfrak{m}$ not a zero-divisor. Then* $\dim(R/(x)) = \dim(R) - 1$.

Proof. Let $d = \dim(R/(x))$. Then by Theorem 3.13 and Corollary 3.15, we have $d \leq \dim(R) - 1$. On the other hand, let $x_1, \ldots, x_d$ be elements of $\mathfrak{m}$ whose images in $R/(x)$ generate an $\mathfrak{m}/(x)$-primary ideal as described in Proposition 3.17. Then $(x, x_1, \ldots, x_d) \in R$ generates a $\mathfrak{m}$-primary ideal. Hence $d + 1 \geq \dim(R)$, and the equality is obtained. □

Now we are ready to prove some properties of dimension mentioned in the introduction, Theorem 1.2. We will state them again:

Theorem 3.22 (As Theorem 1.2)**.** *Let R be a Noetherian ring. Then:*

(1) (Dimension is a local property) For any prime ideal $\mathfrak{p}$ of R, let $\hat{R}_{\mathfrak{p}}$ be the completion of $R_{\mathfrak{p}}$ according to $\mathfrak{p}$. Then

$$\dim(R) = \sup_{\mathfrak{p} \text{ prime ideal}} \{\dim(R_{\mathfrak{p}})\}, \qquad \dim(R_{\mathfrak{p}}) = \dim(\hat{R}_{\mathfrak{p}}).$$

(2) (Nilpotent elements do not affect dimension) Let $I \subseteq R$ be a nilpotent ideal. Then

$$\dim(R) = \dim(R/I).$$

(3) (Dimension is preserved by integral extensions). If $R \subseteq S$ are rings such that S is a finitely generated R-module, then $\dim(R) = \dim(S)$.

(4) Let $x_1, \ldots, x_n$ be independent indeterminants. Then

$$\dim(R[x_1, \ldots, x_n]) = \dim(R[[x_1, \ldots, x_n]]) = \dim(R) + n.$$

Proof. (1) The first part follows from definition, and the second is Corollary 3.20.

(2) Since I is nilpotent, there exists k such that $I^k = 0$. Then for any $r_1, \ldots, r_k \in I$, $r_1 r_2 \cdots r_k = 0$, and in particular, $r^k = 0$ for all $r \in I$. Hence elements in I are all contained in the nilradical of R. Since the nilradical is the intersection of all prime ideals of R, all prime ideals in R contain I. Hence taking the quotient does not affect the lengths of chains of prime ideals.

(3) Note that if S is a finitely generated R-module, then S is integral over R. We will prove instead that if S is integral over R, then $\dim(R) = \dim(S)$.

Since integral extensions satisfy lying-over and going-up, by Theorem 5.7 and Theorem 5.8, given a chain of strictly increasing prime ideals in R:

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n,$$

we may find a chain of prime ideals of the same length in S:

$$\mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_n,$$

such that $\mathfrak{q}_i \cap R = \mathfrak{p}_i$ for $0 \leq i \leq n$. Thus $\dim(R) \leq \dim(S)$.

On the other hand, suppose we have a chain of prime ideals in S:

$$\mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_n.$$

Then there is a corresponding chain of prime ideals in R:

$$\mathfrak{p}_0 \subseteq \mathfrak{p}_1 \subseteq \cdots \subseteq \mathfrak{p}_n,$$

such that $\mathfrak{p}_i = \mathfrak{q}_i \cap R$ for $0 \leq i \leq n$. If there are $0 \leq i \neq j \leq n$ such that $\mathfrak{p}_i = \mathfrak{p}_j$, then by incomparability (Theorem 5.9), $\mathfrak{q}_i \nsubseteq \mathfrak{q}_j, \mathfrak{q}_j \nsubseteq \mathfrak{q}_i$, and we have a contradiction. Thus all prime ideals in the corresponding chain are distinct, and $\dim(R) \geq \dim(S)$.

(4) This is proved in Section 2.2.

□

4. Regular Local Rings

4.1. Regular Local Rings and Nonsingularity.

The dimension Theorem 3.13 gives a lower bound of generators of the maximal ideal. We are interested in the case when equality is achieved. We say a Noetherian local ring $(R, \mathfrak{m})$ of dimension d is a *regular local ring* if $\mathfrak{m}$ is generated by exactly d elements. Regularity comes from the geometric notion of nonsingularity, as we shall see in an example later.

First let us see two easy examples of regular local rings:

(1) If R is a field, its only proper ideal is (0), generated by the empty set, and $\dim(R) = 0$. So any field is a regular local ring.

(2) A *discrete valuation ring* R is an integral domain with a unique non-zero prime ideal $\mathfrak{m}$ that is maximal. Since discrete valuation rings are principal ideal domains (see [1] Proposition 9.2), $\mathfrak{m}$ is generated by exactly one element. There is only one chain of prime ideals, namely $(0) \subsetneq \mathfrak{m}$. So $\dim(R) = 1$, and R is a regular local ring. It turns out that regular local rings of dimension 1 are all discrete valuation rings.

There is a characterization of regular local rings using the associated graded rings, and the dimension of $\mathfrak{m}/\mathfrak{m}^2$ as a $R/\mathfrak{m}$-vector space.

Proposition 4.1. *Let $(R, \mathfrak{m})$ be a Noetherian local ring with residue field $k = R/\mathfrak{m}$. Suppose $\dim(R) = d$. Then the following are equivalent:*

(1) $G_{\mathfrak{m}}(R) \cong k[t_1, \ldots, t_d]$, where $t_1, \ldots, t_d$ are indeterminants over k.
(2) $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = d$.
(3) $\mathfrak{m}$ can be generated by exactly d elements, i.e., R is a regular local ring.

Proof. i) $\implies$ ii) If $G_{\mathfrak{m}}(R) = \bigoplus_{n=0}^{\infty} \mathfrak{m}^n/\mathfrak{m}^{n+1} \cong (R/\mathfrak{m})[t_1, \ldots, t_d]$, then the degree 1 part of $G_{\mathfrak{m}}(R)$ may be generated by $t_1, \ldots, t_d$, while $t_1, \ldots, t_d$ are all independent indeterminants. This is exactly what $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = d$ means.

ii) $\implies$ iii) This is the second part of Lemma 5.2.

iii) $\implies$ i) Suppose $\mathfrak{m} = (x_1, \ldots, x_d)$. Consider the map

$$\alpha : k[t_1, \ldots, t_d] \longrightarrow G_{\mathfrak{m}}(R) = \bigoplus_{n=0}^{\infty} \mathfrak{m}^n/\mathfrak{m}^{n+1}$$

that sends t_i to the image of x_i in $R/\mathfrak{m} = k$. It is surjective by definition. It remains to show that α is injective. We will do so by proving the following claim:

Claim: Let $f \in R[t_1, \ldots, t_d]$ be homogeneous of degree n. Suppose $f(x_1, \ldots, x_d) \in \mathfrak{m}^{n+1}$, i.e., f is in the kernel of α once we mod out the coefficients by $\mathfrak{m}$. Then the coefficients of f lie in $\mathfrak{m}$.

Proof of claim: Let us denote the image of f in $k[t_1, \ldots, t_d]$ as $\bar{f}$, and call it the reduction of f. Suppose for contradiction that there are some coefficients of f that are not in $\mathfrak{m}$, i.e., they are units in the local ring R. Then $\bar{f}$ is not a zero divisor in $k[t_1, \ldots, t_d]$. Since $f \in \ker(\alpha)$, we have $d(G_{\mathfrak{m}}(R)) \leq d(k[x_1, \ldots, x_d]/(\bar{f}))$. Now we may apply Corollary 3.8 and the example following it to see that

$$d(k[x_1, \ldots, x_d]/(\bar{f})) = d(k[x_1, \ldots, x_d]) - 1 = d - 1.$$

However, by Theorem 3.13, $d(G_{\mathfrak{m}}(R)) = d$, and we have a contradiction. □

Therefore, the kernel of α, i.e., any $f \in R[x_1, \ldots, x_d]$ whose reduction $\bar{f}$ is mapped into $\bigcap_n \mathfrak{m}^n$, is exactly those $f \in R[x_1, \ldots, x_d]$ with coefficients in $\mathfrak{m}$. Thus α is a bijection and hence an isomorphism. □

Example 4.2. Let k be a field, $R = k[x_1, \ldots, x_d]$, $\mathfrak{m} = (x_1, \ldots, x_d)$ a maximal ideal. Let $S = R_{\mathfrak{m}}$ be the localization of R at $\mathfrak{m}$. Its associated graded ring is a polynomial ring over the residue field $R_{\mathfrak{m}}/\mathfrak{m}_{\mathfrak{m}}$ in d indeterminants. Thus S is a regular local ring.

Using this characterization of regular local rings, and properties of completion, we get the following corollary:

Corollary 4.3. *Let R be a Noetherian local ring. Then R is regular if and only if the completion with respect to $\mathfrak{m}$, $\hat{R}$ is regular.*

Proof. Observe that we have an isomorphism $G_{\mathfrak{m}}(R) \cong G_{\hat{\mathfrak{m}}}(\hat{R})$. □

Proposition 4.4. *A regular local ring is an integral domain.*

Proof. Let $(R, \mathfrak{m})$ be a regular local ring. By Krull's intersection theorem (Theorem 5.16), if we put $M = R$, then $\bigcap_{n \geq 0} \mathfrak{m}^n = 0$. We want to show that $G_{\mathfrak{m}}(R)$ being an integral domain implies that R is an integral domain. Using the characterization $G_{\mathfrak{m}}(R) \cong (R/\mathfrak{m})[x_1, \ldots, x_d]$, $G_{\mathfrak{m}}(R)$ is a unique factorization domain and hence an integral domain.

Take $x, y \in R$ such that $x, y \neq 0$. Then since $\bigcap_{n \geq 0} \mathfrak{m}^n = 0$, there exist integers r, s such that $x \in \mathfrak{m}^r, x \notin \mathfrak{m}^{r+1}$, $y \in \mathfrak{m}^s, y \notin \mathfrak{m}^{s+1}$. Let $\overline{x}, \overline{y}$ be images of x, y in $\mathfrak{m}^r/\mathfrak{m}^{r+1}$ and $\mathfrak{m}^s/\mathfrak{m}^{s+1}$ respectively. Then $\overline{x}, \overline{y} \neq 0$ by assumption, and in $G_{\mathfrak{m}}(R)$, $\overline{xy} = \bar{x} \cdot \bar{y} \neq 0$, which in turn means $xy \neq 0$. □

Let us see an example using this characterization.

Example 4.5. Let $R = k[x]/(x^2)$. It is a Noetherian local ring with the unique maximal ideal being the image of (x) in R. One may see that R is not an integral domain since x is a zero-divisor. Therefore R is not a regular local ring.

Now let us see an example of how regular local rings correspond to nonsingular points in algebraic geometry.

Let k be an algebraically closed field, $\mathbb{A}^n$ the affine n-space over k. Let $Y \subseteq \mathbb{A}$ be an algebraic variety. Suppose the ideal of Y, $I(Y) \subseteq k[x_1, \ldots, x_n]$ is generated by $f_1, \ldots, f_m \in k[x_1, \ldots, x_n]$. For a point $p \in Y$, we say Y *is nonsingular at* p if the Jacobian matrix of $(f_1, \ldots, f_m)$ is zero at p, i.e., $(\frac{\partial f_i}{\partial x_j}(p))_{1 \leq i \leq m, 1 \leq j \leq n} = 0$.

If $p = (a_1, \ldots, a_n)$, then by Hilbert's Nullstellensatz, there is a corresponding maximal ideal in $k[x_1, \ldots, x_n]$, which is $\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n)$. By [4] theorem 5.1, Y is non-singular at p if and only if the stalk at p,

$$\mathcal{O}_{Y,p} = (k[x_1, \ldots, x_n]/I(Y))_{\mathfrak{m}}$$

is a regular local ring. The proof requires more technique in algebraic geometry, so we will point the reader to the reference. Here let us see an example of the cusp, which has a singular point corresponding to a local ring that is not regular.

Example 4.6. Let k be an algebraically closed field, say $\mathbb{C}$, $f = x^3 - y^2 \in k[x, y]$, and put $R = k[x, y]/(f)$. Then $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})(0, 0) = (3x^2, -2y)(0, 0) = (0, 0)$, and $Y = V(f)$ has a singular point at $p = (0, 0)$. The stalk at p, $\mathcal{O}_{Y,p}$, is the localization of R at the maximal ideal in $k[x, y]$ corresponding to $(0, 0)$, i.e., $\mathcal{O}_{Y,p} = R_{(x,y)}$. Now we show that $\mathcal{O}_{Y,p}$ is not a regular local ring.

Since the polynomial $y^2 - x^3$ is irreducible, R is an integral domain, and furthermore, the maximal ideal in $R_{(x,y)}$, $(x, y)_{(x,y)} \overset{\text{call}}{=} \mathfrak{m}$, is generated by at least two elements. But $\dim(R_{(x,y)}) = 1$ as the curve looks like a line.

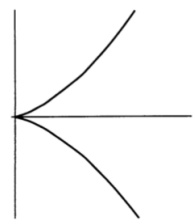

FIGURE 3. A Cusp, [3] Figure 11.1

4.2. **Global Dimension and Regular Local Rings.**

In the last part of this paper, we would like to briefly discuss how homological algebra is used in the study of regular local rings as a rather powerful tool. We will not include proofs, and the reader may consult [3] Chapter 17, 18, and [6] Chapter 6, 7 for more details.

First we introduce the notion of depth via regular sequences.

Let R be a ring, M an R-module. A sequence of elements $x_1, \ldots, x_n \in R$ is called a *M-regular sequence* if it satisfies the following two conditions:

(1) f_i is not a zero divisor of $M/(x_1, \ldots, x_{i-1})M$ for $1 \leq i \leq n$;
(2) $M/(x_1, \ldots, x_n) \neq 0$.

If $x_1, \ldots, x_n$ are contained in some ideal I, then we say $x_1, \ldots, x_n$ is a *regular M-sequence in I*.

If furthermore, $IM \neq M$, then we may define the *I-depth of M*, $\operatorname{depth}_I(M)$, to be the supremum of regular M-sequences in I. If $IM = M$, then we define $\operatorname{depth}_I(M) = \infty$. If R is a local ring with unique maximal ideal $\mathfrak{m}$, then we call $\operatorname{depth}_{\mathfrak{m}}(M)$ simply the *depth of M*.

Note that in general, whether a sequence of elements is a regular sequence depends on the ordering of these elements. For example, let $R = k[x, y, z]$, where k is a field. Then x, $y(1 - x)$, $z(1 - x)$ is a regular sequence, since

$$k[x, y, z]/(x) \cong k[y, z],$$
$$k[x, y, z]/(x, y(1 - x)) \cong k[x, y, z]/(x, y) \cong k[z],$$
$$k[x, y, z]/(x, y(1 - x), z(1 - x)) \cong k.$$

However, $y(1-x), z(1-x), x$ is not a regular sequence, one may check that $z(1-x)$ is a zero-divisor in $k[x, y, z]/(y(1 - x))$.

If, instead, we are working with regular sequences in a Noetherian local ring, then any permutation of these elements still will form a regular sequence.

Next we introduce projective resolution and global dimension of a module.

Definition 4.7. Let R be a ring. An R-module P is *projective* if for every epimorphism of modules $\alpha : M \twoheadrightarrow N$, and every map $\beta : P \to N$, there exists a map $\gamma : P \to M$ such that $\beta = \alpha\gamma$. For any R-module M, a *projective resolution of* M $\mathcal{P}$ is a sequence of projective modules P_i and maps $\phi_i : P_i \to P_{i-1}$ for $i > 0$, and $\phi_0 : P_0 \to M$ such that we have an exact sequence

$$\mathcal{P} : \cdots \longrightarrow P_{i+1} \xrightarrow{\phi_{i+1}} P_i \xrightarrow{\phi_i} P_{i-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0.$$

If for some n, $P_{n+1} = 0$ but $P_i \neq 0$ for $0 \leq i \leq n$, then we say $\mathcal{P}$ is a *finite projective resolution of length* n.

Note that a free module is always projective, but a projective module is not necessarily free. If we replace projective modules with free modules, we get a *free resolution of* M. Geometrically, projective modules correspond to algebraic vector bundles, whereas projective resolution is helpful as it tells us how "far" a module is from being projective. Every module has a free resolution, and hence a projective resolution. But these resolutions are not necessarily of finite lengths.

Definition 4.8. Let R be a ring, M an R-module. The *projective dimension* of M, $\mathrm{pd}_R(M)$, is the minimal length of projective resolutions of M. The *global dimension* of R, $\mathrm{gd}_R(M)$, is the supremum of projective dimension of all R-modules. For a local ring $(R, \mathfrak{m})$, we say a chain complex $\mathcal{P}$ of R-modules is *minimal* if the induced maps in the complex $\mathcal{P} \otimes (R/\mathfrak{m})$ are all zero, i.e., $\phi_n(P_n) \subseteq \mathfrak{m}P_{n-1}$.

The following result states that to calculate the global dimension of a local ring, it suffices to focus on the projective dimension of the residue field $R/\mathfrak{m}$.

Proposition 4.9 ([3] Corollary 19.5)**.** *Let $(R, \mathfrak{m})$ be a local ring, $k = R/\mathfrak{m}$, and $M \neq 0$ a finitely generated R-module. Then $\mathrm{pd}_R(M)$ is the length of every minimal free resolution of M, and $\mathrm{gd}(R) = \mathrm{pd}_R(k)$.*

Projective resolution also provides a tool to understand the generators of a finitely generated module, and we have the Hilbert Syzygy theorem:

Theorem 4.10 (Hilbert Syzygy Theorem, [3] Corollary 19.7)**.** *Let k be a field. Then every finitely generated graded module over $k[x_1, \ldots, x_n]$ has graded free resolution of length at most n.*

There is also a relation between projective dimension and depth, discovered by Auslander and Buchsbaum, stated as follows:

Theorem 4.11 (Auslander-Buchsbaum Formula, [3] Theorem 19.9)**.** *Let $(R, \mathfrak{m})$ be a local ring, M finitely generated R-module with finite projective dimension. Then*

$$\mathrm{pd}_R(M) = \mathrm{depth}(\mathfrak{m}, R) - \mathrm{depth}(\mathfrak{m}, M).$$

And finally, Serre gave his characerization of regular local rings using global dimension:

Theorem 4.12 (Serre's Characterization of Regular Local Rings, [3] Theorem 19.12, [7] Theorem 9.2)**.** *A local ring has finite global dimension if and only if it is a regular local ring. If so, its global dimension is its Krull dimension.*

This characterization of regular local rings gives us that regularity is a local property:

Corollary 4.13 ([3] Corollary 19.14)**.** *Every localization of a regular local ring is again regular. Every localization of a polynomial ring over a field is regular.*

5. Appendix

This section includes necessary background in commutative algebra. Only the results are stated here, and the reader may consult the references for proofs and thourough discussions.

5.1. Miscellaneous Results.

Proposition 5.1 ([1] Proposition 1.11)**.** *Let R be any ring.*

(1) Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be prime ideals, I be an ideal contained in $\bigcup_{i=1}^n \mathfrak{p}_i$. Then $I \subseteq \mathfrak{p}_i$ for some i.
(2) Let $I_1, \ldots, I_n$ be ideals, $\mathfrak{p}$ be a prime ideal such that $\bigcap_{i=1}^n I_i \subseteq \mathfrak{p}$. Then $I_i \subseteq \mathfrak{p}$ for some i. If $\mathfrak{p} = \bigcap_{i=1}^n I_i$, then $\mathfrak{p} = I_i$ for some i.

Lemma 5.2 (Nakayama's Lemma, [3] Corollary 4.8a)**.** *Let I be an ideal contained in the Jacobson radical of a ring R (intersection of maximal ideals of R), and let M be a finitely generated R-module.*

(1) If $IM = M$, then $M = 0$.
(2) If $m_1, \ldots, m_n \in M$ have images in M/IM that generate it as an R-module, then $m_1, \ldots, m_n$ generate M as an R-module.

5.2. Primary ideals.

Definition 5.3 (Primary ideal)**.** Let R be a ring, $\mathfrak{q} \subseteq R$ a proper ideal. We say $\mathfrak{q}$ is a *primary ideal* if $xy \in \mathfrak{q}$ implies $x \in \mathfrak{q}$ or $y^n \in \mathfrak{q}$ for some $n \geq 0$. For a prime ideal $\mathfrak{p}$ in R, if $\mathfrak{p} = r(\mathfrak{q})$, the radical of $\mathfrak{q}$, then we say $\mathfrak{q}$ is $\mathfrak{p}$-primary.

Proposition 5.4 ([1] Proposition 4.1)**.** *Let $\mathfrak{q}$ be a primary ideal in a ring R. Then its radical $r(\mathfrak{q})$ is the smallest prime ideal containing $\mathfrak{q}$.*

Proposition 5.5 ([1] Corollary 7.16)**.** *Let R be a Noetherian ring, $\mathfrak{m}$ a maximal ideal of R, $\mathfrak{q}$ any ideal of R. Then the following are equivalent:*

(1) $\mathfrak{q}$ is $\mathfrak{m}$-primary.
(2) The radical of $\mathfrak{q}$, $r(\mathfrak{q})$, is $\mathfrak{m}$.
(3) $\mathfrak{m}^n \subseteq \mathfrak{q} \subseteq \mathfrak{m}$ for some $n \geq 0$.

5.3. Integral Extensions.

Definition 5.6 (Integral Extensions)**.** Suppose we have a ring extension $R \subseteq S$. We say $s \in S$ is *integral over* R, or an *integral extension of* R, if there is $f \in R[x]$ such that $f(s) = 0$. We say S is an *integral extension of* R if every element in S is integral over R. We say S is *finite over* R, or a *finite extension of* R, if S is a finitely generated R module. It can be checked that finite extensions are always integral.

Theorem 5.7 (Lying over, [1] Theorem 5.10)**.** *Let $R \subseteq S$ be an integral extension, $\mathfrak{p}$ be a prime ideal in R. Then there is a prime ideal $\mathfrak{q}$ in S such that $\mathfrak{p} = \mathfrak{q} \cap R$.*

Theorem 5.8 (Going up, [1] Theorem 5.11)**.** *Let $R \subseteq S$ be an integral extension. Suppose there is a chain of prime ideals in R:*

$$\mathfrak{p}_0 \subseteq \mathfrak{p}_1 \subseteq \cdots \subseteq \mathfrak{p}_n$$

and a chain of prime ideals in S:

$$\mathfrak{q}_0 \subseteq \mathfrak{q}_1 \subseteq \cdots \subseteq \mathfrak{q}_m$$

with $m \leq n$ such that $\mathfrak{p}_i = \mathfrak{q}_i \cap R$ for $0 \leq i \leq m$. Then the chain can be extended to a chain of prime ideals

$$\mathfrak{q}_0 \subseteq \mathfrak{q}_1 \subseteq \cdots \subseteq \mathfrak{q}_n$$

with $\mathfrak{p}_i = \mathfrak{q}_i \cap R$ for $0 \leq i \leq n$.

Theorem 5.9 (Incomparability, [3] Corollary 4.18)**.** *Let $R \subseteq S$ be an integral extension. Suppose there are prime ideals $\mathfrak{q}, \mathfrak{q}'$ of S such that $\mathfrak{q} \cap R = \mathfrak{q}' \cap R$. Then they are incomparable, i.e., $\mathfrak{q} \nsubseteq \mathfrak{q}', \mathfrak{q}' \nsubseteq \mathfrak{q}$.*

5.4. **Artinian Rings.**

Definition 5.10 (Artinian Rings)**.** A ring R that satisfies the descending chain condition is called *Artinian*, i.e., every decreasing chain of ideals in R stabilizes.

Theorem 5.11 ([3] Theorem 2.19)**.** *Let R be Noetherian, and let $I \subseteq R$ be an ideal. Let $\mathfrak{p}$ be a prime ideal in R containing I. Then the following are equivalent:*

(1) $\mathfrak{p}$ is minimal amongst all prime ideals containing I.
(2) $R_\mathfrak{p}/I_\mathfrak{p}$ is Artinian.
(3) In the localization $R_\mathfrak{p}$, we have ${\mathfrak{p}_\mathfrak{p}}^n \subseteq I_\mathfrak{p}$ for sufficiently large n.

Theorem 5.12 ([1] Theorem 8.5)**.** *A ring R is Artinian if and only if it is Noetherian and $\dim(R) = 0$.*

Proposition 5.13 ([1] Proposition 8.6)**.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring. Then exactly one of the following two statements is true:*

(1) $\mathfrak{m}^n \neq \mathfrak{m}^{n+1}$ for all n;
(2) $\mathfrak{m}^n = 0$ for some n, in which case R is an Artinian local ring.

5.5. **Completion of a Ring.**

Definition 5.14 (Completion of a ring)**.** Let R be any ring, $\mathfrak{q}$ a proper ideal. The $\mathfrak{q}$-*adic completion* of R, or *completion of R with respect to* $\mathfrak{q}$, $\hat{R}_\mathfrak{q}$, is the invese limit of the quotient rings $R/\mathfrak{q}^i$, defined as

$$\hat{R} := \varprojlim(R/\mathfrak{q}^i) = \left\{ r = (r_1, r_2, \ldots) \in \prod_i R/\mathfrak{q}^i \mid r_j \equiv r_i \mod \mathfrak{q}^i \text{ for all } j > i \right\}.$$

Then we have $\hat{\mathfrak{q}}^i = \left\{ r = (r_1, r_1, \ldots) \in \hat{R} \mid r_j = 0 \text{ for all } j \leq i \right\}$, and it follows that $\hat{R}/\hat{\mathfrak{q}}^i = R/\mathfrak{q}^i$.

Observe that if $(R, \mathfrak{m})$ is a local ring, then the completion of R with respect to $\mathfrak{m}$ is again a local ring with unique maximal ideal $\hat{\mathfrak{m}}$.

Example 5.15. Let k be a field, $R = k[x_1, \ldots, x_n]$ a polynomial ring over k in n indeterminants. Put $\mathfrak{m} = (x_1, \ldots, x_n)$, then the completion of R with respect to $\mathfrak{m}$ is the formal power series $k[[x_1, \ldots, x_n]]$.

Theorem 5.16 (Krull's Intersection Theorem, [8] Lemma 10.5.4)**.** *Let R be a Noetherian local ring, $\mathfrak{q}$ a proper ideal of R, M a finitely generated R-module. Then $\bigcap_{n\geq 0}\mathfrak{q}^n M = 0$.*

Proposition 5.17 ([1] Proposition 10.15 iii))**.** *Let R be Noetherian, and $\hat{R}$ its $\mathfrak{q}$-adic completion. Then $\mathfrak{q}^n/\mathfrak{q}^{n+1} \cong \hat{\mathfrak{q}}^n/\hat{\mathfrak{q}}^{n+1}$.*

5.6. **Filtration and Artin-Rees Lemma.**

An (infinite) chain $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n \supseteq \cdots$, where M_n are all submodules of M, is called a *filtration* of M, denoted (M_n). Let $\mathfrak{q}$ be an ideal of R. The filtration is called a $\mathfrak{q}$-*filtration* if $\mathfrak{q}M_n \subseteq M_{n+1}$ for all n, and a *stable* $\mathfrak{q}$-*filtration* if $\mathfrak{q}M_n = M_{n+1}$ for all sufficiently large n.

Lemma 5.18 (Bounded Difference, [1] Lemma 10.6)**.** *If $(M_n), (M'_n)$ are stable $\mathfrak{q}$-filtrations of M, then they have* bounded difference*: there exists an integer n_0 such that $M_{n+n_0} \subseteq M'_n$ and $M'_{n+n_0} \subseteq M_n$ for $n \geq 0$.*

Proposition 5.19 ([1] Proposition 10.7)**.** *Let R be a graded ring. Then R is Noetherian if and only if R_0 is Noetherian and R is finitely generated as an R_0-algebra.*

Proposition 5.20 (Artin-Rees, [1] Proposition 10.9, 10.10)**.** *Let R be a Noetherian ring, $\mathfrak{q}$ an ideal in R, M a finitely generated R-module, (M_n) a stable $\mathfrak{q}$-filtration of M. If $M' \subseteq M$ is a submodule, then $(M' \cap M_n)$ is a stable $\mathfrak{q}$-filtration of M'. As a corollary, there exists an integer $c > 0$ such that for all $n \geq c$,*

$$\mathfrak{q}^n M \cap M' = \mathfrak{q}^{n-c}(\mathfrak{q}^c M \cap M').$$

5.7. **Associated Graded Rings.**

Definition 5.21 (Associated Graded Ring)**.** Define the *associated graded ring* of R with respect to $\mathfrak{q}$ as

$$G(R) = G_{\mathfrak{q}}(R) = \bigoplus_{n=0}^{\infty} \mathfrak{q}^n/\mathfrak{q}^{n+1}.$$

One may see that this is a graded ring, with multiplication defined as: for $x_n \in \mathfrak{q}^n$, let $\bar{x}_n$ denote its image in $\mathfrak{q}^n/\mathfrak{q}^{n+1}$, and define $\bar{x}_n\bar{x}_m = \overline{x_n x_m}$, i.e., the image of $x_n x_m$ in $\mathfrak{q}^{n+m}/\mathfrak{q}^{n+m+1}$. Similarly, if M is a R-module, and (M_n) is a $\mathfrak{q}$-filtration of M, then the associated graded ring of M with respect to $\mathfrak{q}$ is

$$G(M) = G_{\mathfrak{q}}(M) = \bigoplus_{n=0}^{\infty} M_n/M_{n+1},$$

which is a graded $G(A)$-module. Let $G_n(M)$ denote the degree n part, M_n/M_{n+1}.

Proposition 5.22 ([1] Proposition 10.22)**.** *Let R be a Noetherian ring, and $\mathfrak{q}$ an ideal of R. Then:*

(1) $G_{\mathfrak{q}}(R)$ is Noetherian;
(2) $G_{\mathfrak{q}}(R)$ and $G_{\hat{\mathfrak{q}}}(\hat{R})$ are isomorphic as graded rings;
(3) If M is a finitely generated A module, and (M_n) is a stable $\mathfrak{q}$-filtration of M, then $G(M)$ is a finitely generated graded $G_{\mathfrak{q}}(R)$-module.

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