ON MANIFOLDS OF NEGATIVE CURVATURE, GEODESIC FLOW, AND ERGODICITY

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Abstract. We discuss geodesic flow on manifolds of negative sectional curvature. We find that geodesic flow is ergodic on the tangent bundle of a manifold, and present the proof for both $n = 2$ on surfaces and general n.

Contents

1. Introduction

We want to understand the behavior of geodesic flow on a manifold M of constant negative curvature. If we consider a vector in the unit tangent bundle of M, where does that vector go (or not go) when translated along its unique geodesic path. In a sense, we will show that the vector goes "everywhere," or that the vector visits a full measure subset of T^1M.

2. Riemannian Manifolds

We first introduce some of the initial definitions and concepts that allow us to understand Riemannian manifolds.

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Definition 2.1. If M is a differentiable manifold and $\alpha : (-\epsilon, \epsilon) \to M$ is a differentiable curve, where $\alpha(0) = p \in M$, then the tangent vector to the curve α at $t = 0$ is a function $\alpha'(0) : D \to \mathbb{R}$, where

$$\alpha'(0)f = \frac{d(f \circ \alpha)}{dt}|_{t=0}$$

for $f \in D$, where D is the set of functions on M that are differentiable at p.

The collection of all such tangent vectors with $\alpha(0) = p$ is called T_pM. We call the collection of all tangent vectors for all $p \in M$

$$TM = \{(p, v)|p \in M, v \in T_pM\},$$

which has a vector bundle structure.

Remark. Suppose $P \in TM$ for some manifold M. We call the projection of P onto M the footprint of P.

Notice that if we only restrict to the unit-length vectors in each fiber, we obtain another vector bundle. This is called the unit tangent bundle and is denoted T^1M, and the restriction to p is denoted T^1_pM. We can interpret T^1_pM as the set of directions from a given p. If M is a surface, we can write the set of directions as S^1M.

Definition 2.2. A Riemannian metric on M assigns each $p \in M$ to an inner product $\langle , \rangle_p$ in the tangent space T_pM. The metric varies differentiably along every curve through p. If $\mathbf{x} : U \subset \mathbb{R}^n \to M$ is a system of coordinates around p, with $\mathbf{x}(x_1, x_2, \ldots, x_n) = q$ and $\frac{\partial}{\partial x_i}(q) = d\mathbf{x}_q(0, \ldots, 1, \ldots, 0)$, then $\langle \frac{\partial}{\partial x_i}(q), \frac{\partial}{\partial x_j}(q)\rangle_q = h_{ij}(x_1, \ldots, x_n)$ is a differentiable function on U.

We call h_{ij} the local representation of the Riemannian metric in the coordinate system $\mathbf{x} : U \subset \mathbb{R}^n \to M$. Since inner products are symmetric, we notice that the Riemannian metric is symmetric also ($h_{ij} = h_{ji}$).

There also exist affine connections on our manifold M. A connection ∇ allows operation that is analogous to the directional derivative in Euclidean space, which we call the covariant derivative. There is a unique affine connection ∇ that is both symmetric and compatible with the Riemannian metric which is called the Levi-Civita connection. For the remainder of this paper, we will consider only the Levi-Civita connection.

Definition 2.3. A vector field X on M assigns each $p \in M$ to a vector $X(p) \in T_pM$. If $X : M \to TM$ is differentiable, then the field is differentiable.

2.1. **Geodesic Flow.**

Definition 2.4. A parameterized curve $\gamma : I \to M$ is geodesic at $t_0 \in I$ if $\frac{D}{dt}(\frac{d\gamma}{dt}) = 0$ at t_0. If γ is geodesic at t for all $t \in I$, we say that γ is a geodesic.

Consider the complex plane where $z = x + iy$, $z, y \in \mathbb{R}$. We let the hyperbolic plane be the upper half plane: $\mathbb{H} = \{z \in \mathbb{C}|\text{im}(z) > 0\}$ equipped with the metric $ds = \frac{\sqrt{dx^2+dy^2}}{y}$. In the hyperbolic plane, the geodesics are either straight lines or semicircles orthogonal to the real axis. These are shown in 1.

On the upper half plane, each $p \in T^1M$ uniquely defines a geodesic. The geodesic flow on $p \in T^1M$ for a time t is the direction $q \in T^1M$ that results from the parallel transport in the direction p for a time t. We call p forward asymptotic to θ_+, if

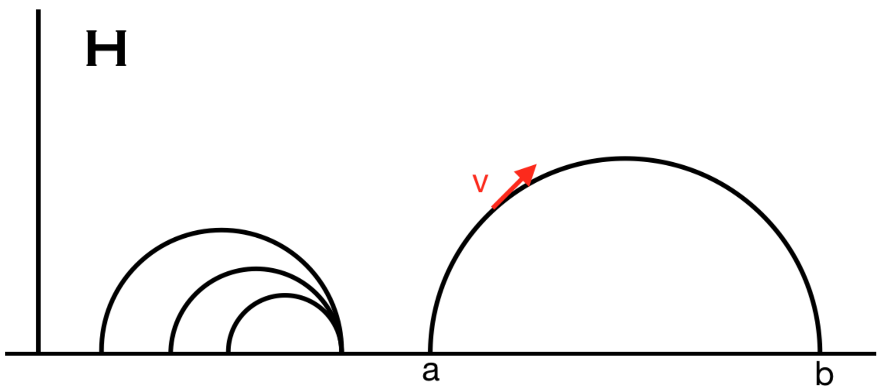

FIGURE 1. Left: Geodesics on the upper half plane $\mathbb{H}$, which are straight lines at a right angle to the real axis or half circles. If you consider the unit vector v, under geodesic flow the positive asymptote of v under the flow is b and the negative asymptote is a.

θ_+ is the intersection of $\partial\mathbb{H}$ with the geodesic through p. We will only work with manifolds negative sectional curvature and therefore we can always lift the flow to the universal cover of the manifold $\mathbb{H}^n$.

Proposition 2.5. *If X is a C^∞ vector field on the open set V in the manifold M and $p \in V$ then there exist an open set $V_0 \subset V$, $p \in V_0$, a number $\delta > 0$, and a C^∞ mapping $\varphi : (-\delta, \delta) \times V_0 \to V$ such that the curve $t \to \varphi(t, q)$, $t \in (-\delta, \delta)$, is the unique trajectory of X which at $t = 0$ passes through the point q for every $q \in V_0$.*

Remark. The mapping $\varphi_t : V_0 \to V$ given by $\varphi_t(q) = \varphi(t, q)$ is called the flow of X on V.

Definition 2.6. There exists a unique vector field G on TM whose trajectories are of the form $t \to (\gamma(t), \gamma'(t))$, where γ is a geodesic on M. The vector field G as defined above is called the geodesic field on TM and its flow is called the geodesic flow on TM.

If $|\gamma'(t)| = 1$, we call the geodesic a unit-speed geodesic. We also notate the geodesic flow of a vector $v \in TM$ for a time t as $g^t v$.

2.2. **Horospheres and Horocycle Flows.** A hyperbolic circle in $\mathbb{H}$ with center c and radius r, denoted $C(c, r) = \{z \in \mathbb{H} | d(z, c) = r\}$, is a Euclidean circle in $\mathbb{H}$. (Although the center of $C(c, r)$ is not c in Euclidean space.)

Definition 2.7. Let $v \in T_x\mathbb{H}$ for some x and let γ be geodesic at v. Then the circle $C = C(\gamma(r), r)$ will contain $\gamma(0)$ for all r. As $r \to \infty$, C converges to the Euclidean circle touching $\partial\mathbb{H}$ at the positive asymptote of γ and has v as the inward normal at c. We call $\lim_{r\to\infty} C$ the positive horosphere $S^+(v)$.

We define the negative horosphere $S^-(v)$ analogously, where $C = C(\gamma(-r), r)$. The vector v will be the outward normal at the negative asymptote of the geodesic.

We can also define the notion of flow on the boundary of a horocycle.

Definition 2.8. The horocycle flow $h_t^* : T^1\mathbb{H} \to T^1\mathbb{H}$ is the flow that slides the inward normal vectors to some $S^+(v)$ counterclockwise along $S^+(v)$ at unit speed.

The negative horocycle flow h_{t*} slides the outward normal vectors to $S^-(v)$ clockwise around $S^-(v)$.

2.3. **Curvature.**

Definition 2.9. Let $\chi(M)$ denote the vector fields on M. The curvature R of a Riemannian manifold is a mapping $R(X,Y) : \chi(M) \to \chi(M)$ given by $R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla[X,Y]Z$ for $X \in \chi(M)$.

Remark. We often write $\langle R(X,Y)Z, T\rangle$ as (X,Y,Z,T).

Definition 2.10. Let $\sigma \in T_pM$ be a two-dimensional subspace of T_pM and let $x, y \in \sigma$ where x, y are linearly independent vectors. Then, we define the sectional curvatures of M by $K(x,y)$, where we define $K(x,y) = \frac{\langle R(X,Y)X,Y\rangle}{|x \wedge y|^2}$. Note that the sectional curvature does not depend on the choice of x, y.

For a surface, the sign of the sectional curvature will always be the same as the sign of $\langle R(X,Y)Z, T\rangle$ at a given point on the surface. This is not necessarily true for manifolds of higher dimension. For example, if $M = S^1 \times \mathbb{H}$, we can choose mutually orthogonal vectors w, x, y, z, where $w, x \in TS^1$ and $y, z \in T\mathbb{H}$. The sectional curvature $K(w,x) = 1$, but $K(y,z) = -1$.

2.4. **Jacobi Fields.**

Definition 2.11. Let $\gamma : [0,a] \to M$ be a geodesic in a manifold M. A vector field J along γ is said to be a Jacobi field if it satisfies the following equation for all $t \in [0,a]$:

$$\frac{D^2 J}{dt^2} + R(\gamma'(t), J(t))\gamma'(t) = 0$$

The Jacobi fields measure how fast geodesics are diverging on a manifold M and are defined completely by the pair $(J(0), J'(0))$.

Definition 2.12. Let $\gamma : \mathbb{R} \to M$ be a unit speed geodesic. A Jacobi field J along γ is stable if $\|J(t)\| \leq C$ for all $t \geq 0$ and some constant $C \geq 0$.

3. Anosov Flows

Definition 3.1. A distribution B on a smooth manifold is a collection of subspaces $L_p \subset T_pM$ which depend smoothly on p. In particular, the dimension of all $L_a \in B$ is constant.

Definition 3.2. A partition W of M into connected k- dimensional C^1-submanifolds $W(x) \ni x$ is called a foliation if for every $x \in M$, there is a neighborhood $U = U_x \ni x$ and a homeomorphism $w = w_x : B^k \times B^{n-k} \to U_x$ such that:

(1) $w_x(0,0) = x$
(2) $w(B^k, z)$ is the connected component $W_U(w(0,z))$ of $W_U(w(0,z)) \cap U$ containing $w(0,z)$
(3) $w(\cdot, z)$ is a C^1 diffeomorphism of B^k onto $W_U(w(0,z))$ which depends continuously on $z \in B^{n-k}$ in the C^1-topology.

We call W a C^1-foliation if the homeomorphisms w_x are diffeomorphisms. We call $W(x)$ the leaves of the foliation. For a given open set U, we will denote the local leaves of U by $W_U(x)$.

Definition 3.3. A differentiable flow φ^t on a compact Riemannian manifold M is called Anosov if it has no fixed points and there are distributions $E^s, E^u \subset TM$ and constants C, $0 < \lambda < 1$ such that for each $x \in M$ and all $t \geq 0$ we have that:

- $E^s(x) \oplus E^u(x) \oplus E^c(x) = T_x M$
- $\|d\varphi_x^t v_s\| \leq C\lambda^t \|v_s\|$
- $\|d\varphi_x^{-t} v_u\| \leq C\lambda^t \|v_u\|$

We call E^u and E^s the stable and unstable distributions respectively.

The stable and unstable distributions of an Anosov flow integrable because there are stable and unstable foliations W^s and W^u whose tangent spaces are precisely E^s and E^u.

In the case of geodesic flow g^t, we will define our stable and unstable sets $W^s(x)$ and $W^u(x)$ by:

$$W^s(x) = \{y \in M | d(g^t x, g^t y) \to 0 \text{ as } t \to +\infty\} \tag{3.4}$$

$$W^u(x) = \{y \in M | d(g^t x, g^t y) \to 0 \text{ as } t \to -\infty\} \tag{3.5}$$

Consider some vector $v \in T^1M$. Then $T_v T^1 M$ decomposes orthogonally into the vertical and horizontal components $v^\perp \oplus T_p M$ where $v^\perp \in T_p M$ and is orthogonal to v. Consider the geodesic flow $g^t : T^1M \to T^1M$ and $dg^t : T_v T'M \to T_{g^t(v)} T'M$ where $X \in T_v T'M$ decomposes as (X_H, X_V). If we consider a geodesic γ_v through v where $\gamma(0) = \pi(0) = p$, then the Jacobi field J is completely defined as $J(0) = X_H$ and $J'(0) = X_v$. We then find that dg^t takes X to the vector whose decomposition is $(J(t), J'(t))$. The proposition below then follows.

Proposition 3.6. *Let $v \in T^1M$ and $w = g^t(v)$ where $p = \pi(v)$ and $q = \pi(w)$. Let $v^\perp$ and $w^\perp$ denote the subset of T_pM and T_qM that is orthogonal to v and w respectively. Then $dg^t : T_p^1 M \oplus v^\perp \to T_q^1 M \oplus w^\perp$ is uniquely defined by the condition that for any Jacobi field J along γ_v with $J'(0) \in v^\perp$ we have:*

$$d(g^t(J(0), J'(0)) = (J(t), J'(t))$$

Remark. This proposition implies that the unstable subspace E^u is spanned by $(J_u(0), J_u'(0))$ and the stable subspace E^s is spanned by $(J_s(0), J_s'(0))$. Therefore, the stable and unstable tangent spaces are precisely the stable and unstable Jacobi fields and the geodesic flow on a manifold M of negative sectional curvature is Anosov.

4. Ergodicity

In this section, we consider a topological space X, with a $\sigma-$algebra $\mathcal{B}$, and a measure μ. Unless otherwise stated, $T : X \to X$ is an invertible measure preserving map.

Definition 4.1. We call a measure preserving map T ergodic if whenever there exists an $f : X \to \mathbb{R}$ such that $f \circ T = f$, f is constant almost everywhere.

A simple example of a measure preserving transformation is the circle rotation $f_\theta : S^1 \to S^1$, where $f(x) = x + \theta$. If θ is rational, we can show that f_θ is a periodic map. However, if θ is irrational, f_θ is not periodic and is, in fact, ergodic.

A related nonexample can be given on the probability space $([0,1), \mathcal{B} \times \beta, \lambda \times \lambda)$ where β is the Lebesque $\sigma-$algebra on $[0,1)$ and λ is the Lesbegue measure normalized to 1. Let $\theta \in (0,1)$ be irrational and $T_\theta \times T_\theta : [0,1)\times[0,1) \to [0,1)\times[0,1)$ be the map that sends (x,y) to $(x+\theta \mod 1, y+\theta \mod 1)$. While, $T_\theta \times T_\theta$ is a measure preserving map, $T_\theta \times T_\theta$ is not ergodic.

Another simple ergodic transformation is $T_\beta : [0,1) \to [0,1)$ that is given by $T_\beta x = \beta x \mod (1) = \beta x - [\beta x]$, where β is a noninteger greater than 1.

Definition 4.2. We define the Liouville measure ν on the unit tangent bundle T^1M. The measure ν is given locally by the product of the Riemannian volume on M and the Lebesgue measure on the unit sphere.

Proposition 4.3. *Geodesic flow on a manifold M of negative sectional curvature preserves the Liouville measure on T^1M.*

Proof. We know that each geodesic at unit speed $b = (p, v)$ on $\mathbb{H}^n$ is uniquely defined by the footprint $\pi(b) = p$ and the direction of transport v. As we know that the speed $\|\dot{\gamma}_v(t)\|$ is constant and equal to $\|v\|$. Therefore, we see that there is a one to one correspondence between the unit tangent bundle T^1M given by $v \leftrightarrow \gamma_v$, so under geodesic flow, the Liouville measure on T^1M will be preserved. □

Theorem 4.4 (Birkhoff's Ergodic Theorem)**.** Let $f : X \to \mathbb{R}$ be from and let $p \in X$. Let

$$q^* = \lim_{\tau\to+\infty} \frac{1}{\tau}\int_0^\tau f(T^t p)dt \text{ and } q_* = \lim_{\tau\to-\infty} \frac{1}{\tau}\int_0^\tau f(T^t p)dt. \tag{4.5}$$

Then we know that $q^* = q_*$ almost everywhere, and then we know $q^* = \int f d\mu$. If q^* is also constant almost everywhere, then we know that T is ergodic.

Remark. We can summarize the Birkhoff Ergodic Theorem as "time average is equal to space average".

5. Surfaces of Negative Curvature

5.1. **Fuchsian Groups and Hyperbolic Surfaces.**

Consider the projective special linear group $\mathrm{PSL}(2,\mathbb{R})$, which can also be written as:

$$G = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid \det(g) = 1\}/\pm I.$$

We note that $\mathrm{PSL}(2,\mathbb{R})$ contains all transformations of the form $z \to az+b$ as well as $z \to -\frac{1}{z}$. Therefore, $\mathrm{PSL}(2,\mathbb{R})$ is a subset of the isometries of $\mathbb{H}$ and maps geodesics to geodesics. (In fact, all isometries of $\mathbb{H}$ are generated by $\mathrm{PSL}(2,\mathbb{R})$ and the map that sends z to $\overline{z}$.)

Isometries map geodesics to geodesics because isometries will commute with the covariant derivative and therefore preserve parallel transport.

Definition 5.1. A discrete subgroup of $Isom(\mathbb{H})$ is called a Fuchsian group if it consists of orientation-preserving transformations i.e. a Fuschian group is a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$.

We call a surface hyperbolic if the sectional curvature of the surface is negative everywhere. We can create any hyperbolic surface Σ by taking the quotient of $\mathbb{H}$ with a Fuchsian group G.

6. Geodesic Flow on Hyperbolic Surfaces

The proof of the ergodicity of geodesic flow for surfaces is less technical than the general case and gives one good intuition for the proof in higher dimensions. Therefore, we present it separately. For the case of surfaces, we follow an argument originally from Hopf [4], which considers the orbit spaces of unit vectors v under geodesic flow. We will notate the orbit space of v as Ω, which contains all $w \in T^1M$ where $g^t v = m$ for some t.

Fact 6.1. *If $p, p' \in T^1\mathbb{H}$ determine positively asymptotic geodesics, there exists $a \in \mathbb{R}$ (depending on p, p') such that $d(g^{t+a}p, g^t p') \to 0$ as $t \to \infty$. (There is an analogous fact in the case of a common negative asymptote.)*

Proof. On the upper half plane $\mathbb{H}$ we can write a metric as $ds^2 = (dx^2 + dy^2)/y^2$ where $p = (x, y) \in \mathbb{H}$. Let p, p' be positively asymptotic to some point θ. Then choose a so that $g^a p$ and p' have the same y coordinate. The distance between the footprint of $g^{t+a}p$ and $g^t p'$ then goes to zero as t approaches infinity. Consider the geodesic between the footprints of $g^{t+a}p$ and $g^t p'$, which we denote $\pi(g^{t+a}p) = z$ and $\pi(g^t p') = w$. The length of geodesic arc between z and w with parallel displacement of p along this path will tend to zero as t goes to infinity. We then know that as the distance between $g^{t+a}p$ and $g^t p'$ cannot be greater than the length of a path between these two elements, the distance between $g^{t+a}p$ and $g^t p'$ will also go to zero. □

Lemma 6.2. *Let Σ be a surface of negative sectional curvature. Then if P and P' determine geodesics which are positively asymptotic to each other, then there exists a number a depending on P, P' such that $d(T^{t+a}P, T^t P') \to 0$ as $t \to \infty$.*

Proof. Recall that $\Sigma = \mathbb{H}/G$ where G is some Fuchsian group. Let $P, P' \in \Sigma$ be positively asymptotic to the same θ. Then we know that $P \equiv Hx$ for some $x \in \mathbb{H}$ and any $H \in G$ as well as $H' \equiv G'x'$ for some $x' \in \mathbb{H}$ and any $H' \in G$. We can write the distance between a given point P, P' as $d(P, P') = \liminf_{H,H' \in G} d(Hx, H'x')$. We know that geodesics are preserved under the quotient of Fuschian groups, so if we choose a as in 6.1, we find that

$$\lim_{t\to\infty} d(g^{t+a}P, g^t P') = \liminf_{H,H' \in G, t\to\infty} d(g^{t+a}Hx, g^t H'x') = \lim_{t\to\infty} d(g^{t+a}x, g^t x') = 0.$$

□

Lemma 6.3. *Let Σ be a compact surface with negative sectional curvature and let Ω be the orbit space of $T^1\Sigma$ under geodesic flow. Let B_+ and B_- be $\nu-$measurable sets in Ω. Suppose that B_+ and B_- satisfy the following:*

(1) Each set is invariant under the geodesic flow.
(2) With each orbit in B_+, every orbit positively asymptotic to it is in B_+, and the same holds for B_- but with negative asymptotic.
(3) The measure of $B_+ \cap B_-^C$ is equal to the measure of $B_- \cap B_+^C$ which is 0.

Then B_+ and B_- are either both of measure 0 or of the full measure of $T^1\Sigma$.

Proof. We claim that we can write each set B_+ as the direct product between a set L_+ and a copy of S^1, where L_+ are the points on the real axis that are asymptotic to the same point for each $p \in L_+$. Since we are only considering the measure of B_+, we can write $B_+ = L_+ \times S^1$, because if $a \in L_+$ and b is any point in S^1, we can always find a geodesic from a to b, where a tangent vector will approach b by translation under g^t as $t \to +\infty$ and a by translation under g^t as $t \to -\infty$. By an identical argument, we can write $B_- = L_- \times S^1$.

Let μ be the product of the Lebesgue measure on $S^1 \times S^1$, normalized so that the full measure is 1. Suppose that the measure of B_+ is positive. We know that $\nu(B_+ \cap B_-^C) = \mu(L_+ \cap L_-^C) = 0$ and $\nu(B_- \cap B+^C) = \mu(L_- \cap L_+^C) = 0$ by the assumptions of the theorem. But we also know that the measure L_+ is greater than zero. Therefore, we have that L_-^C has measure 0, implying that B_- has full measure. By the same argument, B_+ must also have full measure. □

Theorem 6.4. The geodesic flow on a compact surface Σ of constant negative sectional curvature is ergodic.

Proof. Let f be any function from $T^1\Sigma \to \mathbb{R}$. We know that geodesic flow, which we denote g^t, is measure preserving on $T^1\Sigma$. The forward and backward Birkhoff averages along orbits are well-defined and thus equal almost everywhere by the Birkhoff Ergodic Theorem. Let $q^*(p)$ be the forward time average $\lim_{T\to\infty} \frac{1}{T}\int_0^T f(q^t p)dt$ and $q_*(p)$ be the backward time average. Let $B_+ = \{p \in T^1\Sigma | q^*(p) \geq c\}$ and $B_- = \{p \in T^1\Sigma | q_*(p) \geq c\}$ for some constant c. If we know that the measure of B_+ is always the measure of B_-, that is, either 1 or 0, we then show that q^* is constant almost everywhere. By 4.4, if q^* is constant almost everywhere then g^t is ergodic.

As a consequence of theorem 6.2, we find that $q^*(p) = q^*(p')$ for p and p' are the same if p and p' both have the same asymptotic point. Choose a as in theorem 6.2, then we know that $\frac{1}{T}\int_0^T f(q^{t+a}p)dt$ goes to $\frac{1}{T}\int_0^T f(q^t p)dt$ as T tends to ∞. So if we also notice that as $g^t p$ and p asymptotically approach the same point and that $q^* = q_*$ almost everywhere, that B_+ and B_- satisfy the hypothesis of 6.3. Therefore, we know that B_- and B_+ either have full or zero measure. □

7. The Ergodicity of Geodesic Flow on Compact Manifolds of Negative Sectional Curvature

The difference between cases of geodesic flow on a surface of negative curvature and a manifold of higher dimension is a result of pathologically bad foliations that are possible in higher dimensions. For an example, on the unit square, we can define a set of full measure S and a family of curves γ which intersect E at in at most one point. The family of curves can be seen in Figure 7.

We will use horospheres to foliate the manifold M. We use the horosphere as a function of the tangent vector v and the absolutely continuous nature of a map p (later defined) between two horospheres. The map p is in general only continuous. When $n = 2$ the horospheres are $1-$dimensional and bounded volume is equivalent to the Lipschitz continuity. This is not true in general.

We introduce the notion of transveral absolute continuity to avoid this issue. Recalling Birkhoff's Ergodic Theorem and the proof for surfaces of negative sectional curvature, we wish to show that an almost everywhere $g-$invariant function f is constant almost everywhere on M. As geodesic flow is Anosov, we can foliate M is with a stable foliation W^s and an unstable unstable foliation W^u. We prove first that if a function f is constant almost everywhere on the leaves two transversally absolutely continuous foliations, then f must also be constant almost everywhere on M. Subsequently, we we show that the foliations W^s and W^u are transversally absolutely continuous. We then combine these two facts to show that geodesic flow is ergodic on manifolds of negative sectional cruvature.

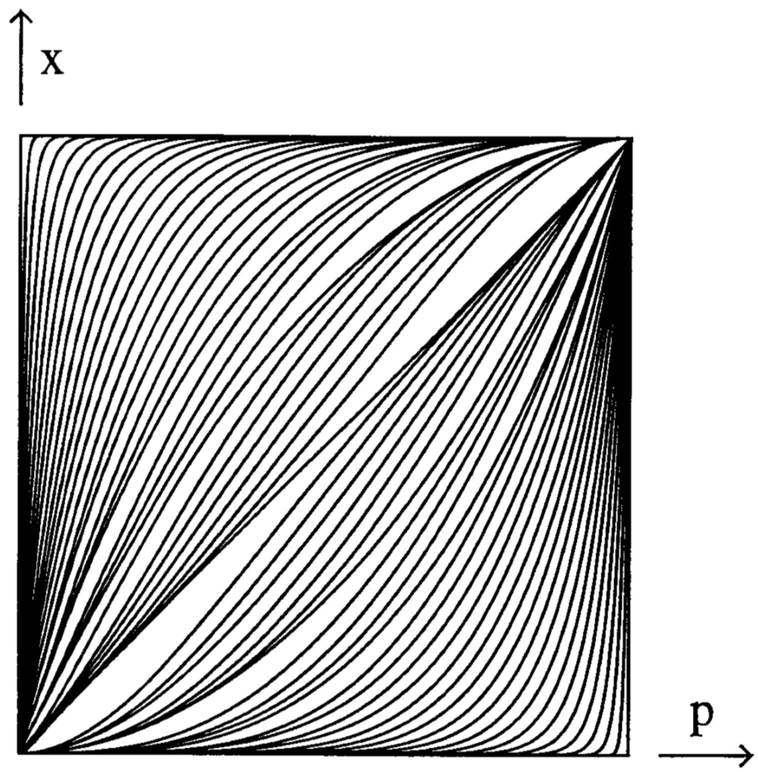

FIGURE 2. A family of curves γ that form a particularly poorly behaved foliation, taken from [7].

7.1. **Foliations and Absolute Continuity.**

Definition 7.1. We call L an $(n-k)-$dimensional open (local) transversal for a $k-$dimensional foliation W at $x \in M$, if $T_xM = T_xW(x) \oplus T_xL$ for all $x \in L$.

Definition 7.2. Let $U \subset M$ be an open set which is a union of local leaves for $k-$dimensional foliation W, i.e. $U = \cup_{x \in L} W_U(x)$, where $W_U(k) \approx B^k$ is the connected component of $W(x) \cap U$ containing x.

We call the foliation W absolutely continuous if for any L and U as defined above there is a measurable family of positive measurable functions $\delta_x : W_U(x) \to \mathbb{R}$ such that for any measurable set $A \subset U$, we have that:

$$m(A) = \int_L \int_{W_U(x)} \mathbb{1}_Z(x,y) dm_{W(x)}(y) dm_L(x)$$

where m is the measure on M and m_L and $m_{W(x)}$ are induced by the restriction of that measure to L and $W(x)$ respectively.

Definition 7.3. Let W be a foliation of M, $x_1 \in M$, $x_2 \in W(x_1)$ and let L_1, L_2 be two transversals to W. There are neighborhoods $U_1 \ni x_1, U_1 \subset L_1$ and the equivalent U_2 and a homeomorphism, called the Poincaré map, $p : U_1 \to U_2$ such that $p(x_1) = x_2$ and $p(y) \in W(y)$, $y \in U_1$.

The foliation W is transversally absolutely continuous if the Poincaré map p is absolutely continuous for any transversals L_i, that is, there exists a measurable, positive function $q : U_1 \to \mathbb{R}$ (called the *Jacobian* of p) such that for any measurable subset $A \subset U_1$ we have that:

$$m_{L_2}(p(A)) = \int_{U_1} \chi_A q(y) dm_{L_1}(y)$$

Remark. Transversal absolute continuity is a stronger condition than absolute continuity for a foliation W, i.e. if W is transversally absolutely continuous, then W is absolutely continuous.

We will borrow an example from [1] to illustrate this point. Consider two vertical intervals I_1, I_2 in the plane that make up opposite sides of the unit square. Let $C_1 \subset I_1$ be a thick Cantor set (one with positive measure) and $C_2 \subset I_2$ be the classical Cantor set. Then let $\alpha : I_1, \to I_2$ be an increasing homomorphism which is differentiable where $\alpha : (C_1) = C_2$. We can then obtain a foliation W by connecting x to $\alpha(x)$ by a straight line. This foliation is absolutely continuous, but it is not transversally absolutely continuous.

Lemma 7.4. *Let W be an absolutely continuous foliation of a manifold M and let $f : M \to \mathbb{R}$ be a measurable function which is almost everywhere constant on the leaves of W. Then for any transversal L to W there is a measurable subset $L' \subset L$ of full induced Riemannian volume m_L such that for each $x \in L'$, there is a subset $W'(x) \subset W(x)$ of full induced volume in $W(x)$ on which f is constant.*

Proof. Given a transversal L, we can consider portions of L which we call L_i. Consider neighborhoods U_i as defined in 7.3. As we know that f is constant almost everywhere on the leaves of W, there exists a set N so that the measure of N is zero and f is constant on W in $M \setminus N$. Let $N_i = N \cap U_i$. Since W is absolutely continuous, we know that there is a subset L_i' which has the same measure as L_i such that if $x \in L_i'$, we know that $W'_{U_i} = W_{U_i}(x) - N_i$ will also have full measure in the local leaf. □

Definition 7.5. We call two foliations W_1 and W_2 of a manifold M transversal if $T_x W_1(x) \cap T_x W_2(x) = \{0\}$ for each $x \in M$.

Definition 7.6. Two transversal foliations W_1, W_2 of dimensions d_1, d_2 are integrable of dimension $d_1 + d_2$ if $W(x) = \cup_{y \in W_1(x)} W_2(y) = \cup_{z \in W_2(x)} W_1(z)$. We call W the integrable hull.

Lemma 7.7. *Let M be a connected manifold and let W_1, W_2 be two transversal absolutely continuous foliations on M where $T_x M = T_x W_1(x) \oplus T_x W_2(x)$ for all $x \in M$. If f is a function that is almost everywhere constant on the leaves of W_1 and W_2, then f is constant almost everywhere in M.*

Proof. Let $N_1, N_2 \subset M$ be the measure 0 sets where f is constant on the leaves of W_i in $M_i = M \setminus N_i$. Let $x \in M$ and let U be a neighborhood containing x small enough that we can apply the previous lemma to find a y sufficiently close to x

such that $W_{U_1}(y)$ intersects M_1 by a set $M_1'(y)$ that have full measure, which we can apply as W_1 is absolutely continuous.

Since W_2 is also absolutely continuous, for almost all $z \in M_1'(y)$ then the intersection $M_1'(y) \cap W_2$ also has full measure. Therefore, we see that f is constant almost everywhere in a neighborhood of x. As M is connected, this then implies that f is constant almost everywhere on M. $\square$

Lemma 7.8. *Let W_1, W_2 be transversal integrable foliations of a manifold M with integral hull W such that W_1 is a C^1-foliation and W_2 is absolutely continuous. Then W is absolutely continuous.*

Proof. Let L be a transversal foliation for W. We can choose a neighborhood $U \subset M$ and $L' = \cup_{x\in L} W_{1U}(x)$ is a transversal for W_2. Since W_2 is absolutely continuous, we have than if $A \subset U$ is measurable then:

$$m(A) = \int_{L'} \int_{W_{2U}(y)} \chi_A(y,z)\delta_y(z) dm_{W_{2U}(y)} dm_{L'}(y)$$

Since L' is foliated by W_1 leaves, we then know that:

$$\int_{L'} dm_{L'}(x) = \int_{L'} \int_{W_{1U}(y)} j(x,y)\delta_y(z) dm_{W_{1U}(y)} dm_{L'}(y)$$

where j is positive and measurable. We then see that W is absolutely continuous as well. $\square$

Proposition 7.9. *Let φ^t be an Anosov flow. Then for every $\theta > 0$ there is a $C_1 > 0$ such that for any subspace $H \subset T_xM$ with the same dimension as $E^s(x)$ and $\theta-$transversal to $E^{cu}(x)$ and any $t \geq 0$:*

$$d(d\varphi^{-t}(x)H, E^s(\varphi^{-t}x)) \leq C_1\lambda^t d(H, E^s(x))$$

Remark. We can define an adjusted metric over M. Let g^t is an Anosov flow, $\beta \in (\lambda, 1)$ — as defined in the previous proposition — and $v_c \in E^c, v_s \in E^s$, and $v_u \in E^u$. We can then set the following:

$$|v_c| = \|v_0\|$$
$$|v_s| = \int_0^T \frac{\|dg^t v_s\|}{\beta^t} dt$$
$$|v_u| = \int_0^T \frac{\|dg^{-t} v_s\|}{\beta^t} dt$$
$$\text{and } |v_c + v_s + v_u|^2 = |v_c|^2 + |v_s|^2 + |v_u|^2$$

When T is large, we then have that $|dg^t v_s| \leq \beta^t |v_s|$. This inequality will be used in the following proof of the Hölder continuity of Anosov flows.

Proposition 7.10. *The distributions E^s, E^u, E^{cs}, D^{cu} of a C^2 Anosov flow are g^t Hölder continuous.*

Proof. As the direct product of two Hölder continuous distributions will be Hölder continuous, we need only show to that E^s is Hölder continuous. The proof for E^u is nearly identical. We can also use the adjusted metric as described above without loss of generality, as if a distribution is Hölder continuous in one metric it will be other metrics as well.

Let $x, y \in M$. If $g^m x$ and $g^m y$ are sufficiently close, $E^s(g^m x)$ and $E^s(g^m x)$ will also be close as E^s is continuous.

Let $Y \in (0,1)$ and fix q so that $Y^{q+1} < d(x,y) \leq Y^q$ and $D \geq \|dg\|$ and $q > 0$. Let m be the integer part of $\frac{\log \epsilon - q \log Y}{\log D}$ so $d(g^i x, g^i y) \leq d(x,y) D^i \leq Y^q D^i$. Then $(dg^i x, dg^i y) \leq \epsilon$ for $i \leq m$.

Assume that Y is small enough so that m is large. Let $U_i \subset V_i \ni g^i x$ be a system of coordinate neighborhoods that we identify with small balls in $Tg^i xM$ such that $g^{-1} V_i \subset U_{i-1}$. Let ϵ be small enough so that $g^i y \in V_i$ for $i \leq m$. We identify the tangent spaces $T_z M$, $z \in V_i$ with $T_{g^i x}$ and the derivatives $(d_z g^{-1})$ with matrices that can act on tangent vectors with footprints (the projection of the tangent vector to the manifold) anywhere in V_i.

We will estimate the distance between $E^s(x)$ and $E^s(y)$ under the parallel translation from $y \to x$. As distance is Lipschitz continuous, this will imply the Hölder continuity of E^s.

Let $v_y \in E^s(g^m y)$, $v_y \neq 0$, $v_k = dg^{ik} v_y = v_k^s + v_k^{cu}$, $v_s \in E^s(g^{m-k} x)$, and $v_{cu} \in E^{cu}(g^{m-k})$.

Let $\kappa \in (\sqrt{\beta}, 1)$, where β is chosen for the adjusted metric described above. We will use induction on k to show that $|v_k^{cu}|/|v_k|^s \leq \delta \kappa^k$ and small δ with small enough ϵ.

When $k = 0$, we have $|v_0^{cu}|/|v_0|^s \leq \delta\kappa$ because E^s is continuous.

Assume that the above inequality is true for some k. Let $A_k = (d_{g^{m-k}x} g)^{-1}$ and $B_k = (d_{g^{m-k}y} g)^{-1}$. By choice of Y, q, and D, we know that $\|A_k - B_k\| \leq C_1 Y^q D^{m-k} \leq C_2 \cdot \epsilon$ where C_1, C_2 are constants.

Then we have that $|A_k v_k^s| \leq \beta^{-1} |v_k^s|$ and $|A_k v_k^{cu}| \leq |v_k^{cu}|$. If we remember that $v_{k+1} = B_k v_k = A_k (v_k^s + v_k^c u) + (B_k - A_k) v_k$, we get that:

$$\begin{aligned}
\frac{|v_{k+1}^{cu}|}{|v_{k+1}^s|} &\leq \frac{|A_k V_k^{cu}| + |A_k - B_k| \cdot |v_k|}{|A_k V_k^s| - |A_k - B_k| \cdot |v_k|} \\
&= (\delta\kappa^k + (C_1 Y^q D^{m-k})\sqrt{\beta}) \frac{|v_k|}{|v_k|(1 - \beta C_1 D^{m-k} - \delta\kappa^k)\sqrt{\beta}}
\end{aligned}$$

So as $C_1 Y^q D^{m-k} \leq C_1 \epsilon D^{-k-1}$, we have that $|v_{k+1}^{cu}|/|v_{k+1}|^s \leq \delta\kappa^{k+1}$.

Finally, suppose that $k = m$. Then $|v_m^{cu}|/|v_m^s| \leq \delta\kappa^m$ and

$$\begin{aligned}
|v_m^{cu}|/|v_m^s| &\leq C_3 Y^{-q+1 \frac{\log \kappa}{\log D}} \\
&= C_4 d(x,y)^{-\frac{\log \kappa}{\log D}}
\end{aligned}$$

□

7.2. **Proof of Ergodicity.** We will use the following lemma to bound the Jacobians of a convergent series.

Proposition 7.11. *Let $(X, \mathcal{U}, \mu)$, $(Y, \mathcal{B}, \nu)$ be two compact metric spaces with Borel $\sigma-$algebras and $\sigma-$additive Borel measures and let $p_n : X \to Y$, $n \in \mathbb{N}$, with $p : X \to Y$ be continuous maps where:*

(1) each p_n and p are homeomorphisms onto their images,
(2) p_n converges to p uniformly as $n \to \infty$, and
(3) there is a constant C such that $\nu(p_n(A)) \leq C\mu(A)$ for $A \in \mathcal{U}$.

Then $\nu(p(A)) \leq C\mu(A)$ for any $A \in \mathcal{U}$.

Lemma 7.12. *Let M be a compact Riemannian manifold with a C^3-metric of negative sectional curvature. Then the foliations W^s and W^u of T^1M are transversally absolutely continuous with bounded Jacobians.*

Proof. We will consider only W^s, as the proof for W^u will be identical if we reverse the time of the geodesic flow g^t. Let L_1, L_2 be two C^1-transversals to W^s. Let U_i and p be as in definition of absolute continuity, so $U_i \subset L_i$, $x_i \in U_i$, $x_i \in L_i$, and $p(x_1) = (x_2)$, if $p(y) \in W(y)$, then $y \in U_1$.

Let Σ_n be the foliation of T^1M into 'inward' spheres, ie. each leaf of Σ_n is the set of unit vectors normal to a sphere of radius n that is pointing to the inside of the sphere. Let $V_i \subset U_i$ be a closed set where $p(V_1) \subset V_2$ and let p_n be the Poincare map for Σ_n with transversals restricted to v_i.

We can write $p_n = g^{-n} \circ P_0 \circ g^n$, where g^n is the flow along the transversal L_1 and $P_0 : g^n(L_1) \to g^n(L_2)$ is the Poincare map along the fibers $\pi : T^1M \to T^1M$. For n big enough, Σ_n becomes arbitrarily close to the stable horospheres. Σ_n is uniformly transverse and p_i is well defined on v_i by the construction of v_i.

Let $v_i \in V_i$ and let $p_n(v_1) = v_2$ then $\pi(g^n v_1) = \pi(g^n v_2)$, so we have that $P_0(g^n v_1) = g^n v_2$.

Let $J_k^i = |det(dg^1(g^k v_i))|_{T_k^i}|$, so that J_k^i is the Jacobian of g_i^1, $T_k^i = T_{g^k v_i} L_i$ at $g_{v_i}^k \in L_i$. Let J_0 be the Jacobian of P_0.

We then have that the Jacobian of p_n, q_n is equal to the following by linear algebra:

$$q_n(v_i) = \Pi_{k=0}^{n-1}(J_k^2)^{-1} \cdot J_0 \Pi_{k=0}^{n-1}(J_k^2) \tag{7.13}$$

$$= J_0(g^n v_1) \cdot \Pi_{k=0}^{n-1}(J_k^1/J_k^2) \tag{7.14}$$

For large enough n we have that $T_{g^n v_i} g^n(L_1)$ is close to $E^{so}(g^n v_1)$ by 7.9 and so is transverse to T_xM at $x = \pi(g^n v_1) = \pi(g^n v_2)$. As the unit spheres form a fixed smooth foliation of TM and as J_0 is uniformly bounded from above in v_1 we have that $d(g^k v, g^k v_2) \leq K \cdot \exp(-ak) \cdot d(v_1, v_2)$ where K and a are constants.

By 7.10 E^{cu} is Holder continuous, so we have that:

$$d(E^{cu}(g^k v_1), E^{cu}) \leq C \cdot d(g^k v_1, g^k v_2)^\alpha \leq C \cdot \exp(-ak\alpha)$$

If we again apply 7.9, we have that $d(T_k^1, T_k^2) \leq C \cdot \exp(-\beta k)$ for some $\beta > 0$. So as the determinants in exponentially close distributions will also be exponentially close $\left\|J_k^1 - J_k^2\right\| < c \cdot \exp(-\gamma k)$. We also know that M is compact and as all $\left\|J_k^i\right\| > 0$, there must exist a positive constant S so all $\left\|J_k^i\right\| > s > 0$.

We then find that $\Pi_{k=0}^{n-1}(J_k^1/J_k^2)$ is uniformly bounded in n and v_i. By 7.11 as q_n are bounded Jacobians, we know that the Jacobian q of p is bounded. □

Remark. We require the metric to be C^3 as we take three derivations: the first to T^1M to consider the unit speed geodesics (needs to be C^1), then working in TT^1M (needs to be C^2), and then evaluating the sequence of Jacobians (needs to be C^3).

Lemma 7.15. *Let W be an absolutely continuous foliation of a manifold Q and let $N \subset Q$ be a set of measure zero. Then there exists another set N_1 of measure 0 such that for any $x \in Q \setminus N_1$ the intersection $W(x) \cap N$ has conditional measure 0 in $W(x)$.*

Proof. Consider a local transversal foliation L to W. As W is absolutely continuous, there exists a subset L' of full measure in L such that $m_{M(x)}(N) = 0$ for $x \in L'$. We then have that $\cup_{x \in L'} W(x)$ will have full measure, implying that $W(x) \cap N$ must have measure zero. $\square$

Theorem 7.16. Let M be a compact Riemannian manifold with a C^3 metric of negative sectional curvature. Then the geodesic flow $g^t : T^1M \to T^1M$ is ergodic.

Proof. Let μ be the Liouville measure in TM. Let m be the Riemannian volume in M. Since M is compact, $m(M) < \infty$.

The local differential equations determining the geodesic flow g^t are $\dot{x} = v$ and $\dot{v} = 0$. We know that the Liouville measure is invariant under g^t. We can then apply Birkhoff's Ergodic Theorem and 7.11 to g^t.

We know that the unstable foliation W^u is absolutely continuous and that W^c is C^1 so integral hull W^{cu} is absolutely continuous. Let $f : TM \to \mathbb{R}$ be a measurable $g-$invariant function. By 7.4, we can correct f to a strictly $g-$invariant function f'. Additionally, there is a set of measure zero N_u such that f' is constant on the leaves of W^u in $TM \setminus N_u$ by 7.7.

By 7.15 we can obtain a set of measure zero N_1 where N_u is measure 0 in $W^{cu}(v)$ for any $v \in TM \setminus N_1$ and measure 0 in $W^u(v)$ for any $v \in TM \setminus N_1$. We then have that f is almost everywhere constant on the leaf $W^u(v)$ and $W^{cu}(v)$ because f is $g-$invariant. Therefore, f is constant almost everywhere on W^{cu}.

So by 7.7 we know that f is constant almost everywhere on W. Therefore by Birkhoff's Ergodic Theorem, we know that g^t is ergodic. $\square$

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