MARSTRAND'S THEOREM AND TANGENT MEASURES

ALEKSANDER SKENDERI

Abstract. This paper aims to prove and motivate Marstrand's theorem, which is a fundamental result in geometric measure theory. Along the way, we introduce and motivate the notion of tangent measures, while also proving several related results. The paper assumes familiarity with the rudiments of measure theory such as Lebesgue measure, Lebesgue integration, and the basic convergence theorems.

Contents

1. Introduction

Some of the most important objects in all of mathematics are smooth manifolds. Indeed, they occupy a central position in differential geometry, algebraic topology, and are often very important when relating mathematics to physics. However, there are also many objects of interest which may not be differentiable over certain sets of points in their domains. Therefore, we want to study more general objects than smooth manifolds; these objects are known as rectifiable sets. The following definitions and theorems require the notion of Hausdorff measure. If the reader is unfamiliar with Hausdorff measure, he should consult definition 2.10 of section 2.3 before continuing to read this section.

Definition 1.1. A k-dimensional Borel set $E \subset \mathbb{R}^n$ is called *rectifiable* if there exists a countable family $\{\Gamma_i\}_{i=1}^{\infty}$ of Lipschitz graphs such that $\mathcal{H}^k(E \setminus \cup \Gamma_i) = 0$.

From Rademacher's theorem, we know that Lipschitz graphs are differentiable almost everywhere. Therefore, we can associate a tangent plane to almost every point of our set E. One of the first results to characterize the geometry of rectifiable sets is the following theorem due to Besicovitch.

Theorem 1.2 (Besicovitch, 1928)**.** *Let $E \subset \mathbb{R}^2$ be such that $\mathcal{H}^1(E) < \infty$. If for $\mathcal{H}^1-$a.e. $x \in E$*

$$\lim_{r \to 0} \frac{\mathcal{H}^1(B_r(x) \cap E)}{2r} = 1,$$

then E is rectifiable.

This is the foundational result in geometric measure theory; it is the archetypal example of the idea that we use information about the measure supported on a set to study the geometry of that set. Marstrand, Mattila, and Preiss have generalized this theorem, but a great deal of work had to be done to get there. We include the statements of these theorems below. A further discussion of these theorems can be found in chapters five and six of [2].

Theorem 1.3 (Marstrand-Mattila Rectifiability Criterion)**.** *Let E be a Borel set such that $0 < \mathcal{H}^k(E) < \infty$ and assume that E is weakly linearly approximable at $\mathcal{H}^k-$a.e. $x \in E$. Then E is rectifiable.*

Theorem 1.4 (Preiss)**.** *Let m be an integer and μ a locally finite measure on $\mathbb{R}^n$ such that $0 < \theta^m_*(\mu, x) < \theta^{*m}(\mu, x) < \infty$ for $\mu-$a.e. $x \in E$. Then, μ is an $m-$dimensional rectifiable measure.*

This paper aims to give an introduction to geometric measure theory. In particular, we discuss and prove Marstrand's theorem, which is a fundamental result in this field, especially since its proof techniques are needed to establish the two theorems above. In section 2, we introduce the necessary measure theoretic notions that are needed to understand the main results of the paper. Finally, section 3 is devoted to developing intuition for the concept of tangent measures and proving Marstrand's theorem in such a way as to illuminate its central ideas.

2. Preliminaries on Measure Theory

2.1. **Weak Convergence of Measures.** As is common, we equip the space $C_c(\mathbb{R}^n)$ of continuous compactly supported functions on n-dimensional Euclidean space with the topology of uniform convergence on compact sets. That is, a function ϕ_j converges to ϕ if there exists a compact set K such that $\text{supp}(\phi_j) \subset K$ for all n, and if the convergence is uniform.

For any locally finite measure μ on $\mathbb{R}^n$, the map

$$\phi \mapsto \int \phi \, d\mu$$

induces a continuous linear functional on $C_c(\mathbb{R}^n)$. The following important theorem, whose proof can be found in a graduate text in analysis, tells us that the converse is true for any nonnegative linear functional.

Theorem 2.1 (Riesz Representation Theorem)**.** *Let $L : C_c(\mathbb{R}^n) \longrightarrow \mathbb{R}$ be a linear functional such that $L(\phi) \geq 0$ for every $\phi \geq 0$. Then there exists a locally finite nonnegative measure μ such that*

$$L(\phi) = \int \phi \, d\mu.$$

This enables us to equip the space of locally finite nonnegative Euclidean measures with the topology of the dual space of $C_c(\mathbb{R}^n)$. We now introduce a very important definition, which will play a crucial role in many results of this paper.

Definition 2.2. Let $\{\mu_j\}$ be a sequence of locally finite nonnegative measures on $\mathbb{R}^n$. We say that μ_j *converges weakly* to μ (and we write $\mu_j \overset{*}{\rightharpoonup} \mu$) if

$$\lim_{j\to\infty} \int \phi \, d\mu_j = \int \phi \, d\mu,$$

for every $\phi \in C_c(\mathbb{R}^n)$.

When comparing two different measures μ and ν on an open set A, it is frequently helpful to use the *total variation* of $\mu - \nu$ on A, which we denote by $|\mu - \nu|(A)$ and define as

$$|\mu - \nu|(A) := \sup_{\phi \in C_c(\mathbb{R}^n), |\phi| \leq 1} \int \phi \, d(\mu - \nu).$$

It can often be quite useful to use the total variation if we have an intuition for what the weak limit of some sequence of measures might be. Indeed, if for every bounded open set A we have $|\mu_j - \mu|(A) \to 0$, then $\mu_j \overset{*}{\rightharpoonup} \mu$.

Notice that if $\mu_j \overset{*}{\rightharpoonup} \mu$, then $\{\mu_j\}$ is locally uniformly bounded. That is, for every compact set K, there exists a constant C_K such that $\mu_j(K) \leq C_K$ for every $j \in \mathbb{N}$ (to see this, take ϕ in the above definition to be the indicator function on K and the conclusion quickly follows). Additionally, since the space of locally finite nonnegative Euclidean measures is the dual of the topological vector space $C_c(\mathbb{R}^n)$, the following compactness property holds:

Proposition 2.3. *Let $\{\mu_j\}$ be a sequence of uniformly locally bounded measures. Then there exists a subsequence $\{\mu_{j_i}\}$ and a locally finite measure μ such that $\mu_{j_i} \overset{*}{\rightharpoonup} \mu$.*

We now state a proposition that will be used in many instances throughout the paper. Its proof can be found in section 2.2 of [2].

Proposition 2.4. *Let $\{\nu_i\}$ be a sequence of measures such that $\nu_i \overset{*}{\rightharpoonup} \nu$. Then,*

$$\liminf_i \nu_i(A) \geq \nu(A) \text{ for every open set } A,$$
$$\limsup_i \nu_i(K) \leq \nu(K) \text{ for every compact set } K.$$

Therefore,

$$\nu_i(A) \to \nu(A) \text{ for every bounded open set } A \text{ such that } \nu(\partial A) = 0, \text{ and}$$

$$\text{for any point } x \text{ there exists an at most countable set } S_x \subset \mathbb{R}^+ \text{ such that}$$

$$\nu_i(B_\rho(x)) \to \nu(B_\rho(x)) \text{ for every } \rho \in \mathbb{R}^+ \backslash S_x.$$

2.2. **Differentiation of Measures and Covering Theorems.** So far, the only way we know of comparing two distinct measures μ and ν is by looking at their total variation over some set A. However, it often happens that we are interested in analyzing how the measures act locally around a point, as opposed to studying their behavior over a set. This leads to the following definition:

Definition 2.5. Let μ and ν be locally finite Borel measures on $\mathbb{R}^n$. The *upper and lower derivatives* of μ with respect to ν at a point $x \in \mathbb{R}^n$ are defined respectively by

$$\overline{D}(\mu,\nu,x) = \limsup_{r\to 0} \frac{\mu(B_r(x))}{\nu(B_r(x))},$$
$$\underline{D}(\mu,\nu,x) = \liminf_{r\to 0} \frac{\mu(B_r(x))}{\nu(B_r(x))}.$$

At the points x where the limit exists, we define the *derivative* of μ with respect to ν at x to be the common value, and denote it by $D(\mu,\nu,x)$.

In this paper, we will need to use a theorem, due to Besicovitch, on the differentiation of measures. We first state the main tool to prove this theorem (which will also be used later on), and then the theorem itself. Proofs of both of these statements may be found in chapter two of [1].

Theorem 2.6 (Besicovitch-Vitali Covering Theorem)**.** *Let A be a bounded Borel Euclidean set and $\mathcal{B}$ a collection of closed balls such that for every $x \in A$ and every $r > 0$ there exists a ball $\overline{B}_\rho(x) \in \mathcal{B}$ with radius $\rho < r$. If μ is a locally finite measure, then there exists a countable subset $\mathcal{C} \subset \mathcal{B}$ of pairwise disjoint balls such that $\mu(A \backslash \bigcup_{\overline{B}\in\mathcal{C}} \overline{B}) = 0$.*

This theorem tells us that if, given a set A, there exists a collection of closed balls of arbitrarily small radii centered at points in A, then we can cover A almost everywhere by a countable, pairwise disjoint subcollection of these closed balls. We now state the aforementioned theorem on the differentiation of measures.

Theorem 2.7 (Besicovitch Differentiation of Measures)**.** *Let ν and μ be locally finite Euclidean measures. Then, $D(\nu,\mu,x)$ exists and is finite at $\mu-$a.e. point $x \in supp(\mu)$. Moreover, the Radon-Nikodym decomposition of ν with respect to μ is given by $\nu = D(\nu,\mu,x)\cdot\mu + \nu \llcorner E$, where*

$$E := (\mathbb{R}^n \backslash \mathrm{supp}(\mu)) \cup \{x \in \mathrm{supp}(\mu) : D(\nu,\mu,x) = \infty\}.$$

Recall that $f\mu(A) = \int_A f\, d\mu$. Thus, the theorem provides us with intuition as to why we call the quantity $D(\nu,\mu,x)$, the derivative: it is the analog of the derivative that we are familiar with from calculus. Note also how μ and ν are symmetric in the formulation of the theorem and so the roles of the two measures can be reversed. Before we state an application of this theorem that will be useful later, we need the following definition:

Definition 2.8. Let f be a μ-measurable function. We say that f is *Lebesgue continuous at a point x with respect to the measure μ* if

$$\lim_{r\to 0} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f(y) - f(x)| dy = 0.$$

Proposition 2.9. *If μ is a locally finite measure and $f \in L^1(\mu)$, then for $\mu-$a.e. x, f is Lebesgue continuous at x with respect to μ.*

2.3. **Hausdorff Measure and Densities.** Recall that the Lebesgue measure is the standard way of assigning measure to subsets of n-dimensional Euclidean space. For $n = 1, 2, 3$, it coincides with our notions of length, area, and volume, respectively. Although the Lebesgue measure allows us to gain valuable information

about many sets in Euclidean space that we would otherwise not be able to say much about, it still has its limitations. For example, $\mathcal{L}^1$ assigns measure zero to both a set consisting of a single point and the Cantor set, although they are quite different sets. Therefore, we require a generalization of Lebesgue measure that will allow us to properly study the geometry of sets that Lebesgue measure tells us little about. This is known as the Hausdorff measure.

Definition 2.10. Let $E \subset \mathbb{R}^n$. The $\alpha -$ *dimensional Hausdorff measure* of E is denoted by $\mathcal{H}^\alpha(E)$ and defined by

$$\mathcal{H}^\alpha(E) := \lim_{\delta \to 0} \mathcal{H}^\alpha_\delta(E),$$

where $\mathcal{H}^\alpha_\delta(E)$ is given by

$$\mathcal{H}^\alpha_\delta(E) := \omega_\alpha \inf \left\{ \sum_{i \in I} \left(\frac{\text{diam } (E_i)}{2} \right)^\alpha \ : \ \text{diam } (E_i) < \delta, E \subset \bigcup_{i \in I} E_i \right\}.$$

When α is a natural number, the constant ω_α is equal to the $\mathcal{L}^\alpha$ measure of the Euclidean unit ball in $\mathbb{R}^\alpha$. We can then generalize ω_α for all nonnegative real numbers α using the gamma function, but this is not very important. Now that we have defined Hausdorff measure, we conclude our preliminary discussion of measure theory with the definition of α-densities.

Definition 2.11. Let μ be a locally finite Euclidean measure and α a nonnegative number. Then we define the *upper (respectively lower)* $\alpha -$ *density* of μ at x as

$$\theta^{*\alpha}(\mu, x) := \limsup_{r \to 0} \frac{\mu(B_r(x))}{\omega_\alpha r^\alpha},$$
$$\theta^\alpha_*(\mu, x) := \liminf_{r \to 0} \frac{\mu(B_r(x))}{\omega_\alpha r^\alpha}.$$

When the two quantities coincide, we speak of the $\alpha -$ *density* of μ at x and denote it by $\theta^\alpha(\mu, x)$.

If we restrict the $\alpha-$dimensional Hausdorff measure to some $\alpha-$plane, then the measure of a ball $B_r(x)$ is $\omega_\alpha r^\alpha$. Therefore, the α-density compares the locally finite measure μ to the α-dimensional Hausdorff measure in the neighborhood of a point. As we will see later, it is very useful when the α-density is constant for μ almost all points in a given set. Indeed, this tells us, intuitively speaking, that the measure μ behaves, up to a constant, like $\alpha-$dimensional Hausdorff measure.

3. Marstrand's Theorem

3.1. **Tangent measures and uniform measures.** Having introduced α-densities, we are now able to state the main result of the paper.

Theorem 3.1 (Marstrand's Theorem)**.** *Let μ be a measure on $\mathbb{R}^n$, α a nonnegative real number, and E a Borel set with $\mu(E) > 0$. Assume further that*

$$0 < \theta^\alpha_*(\mu, x) = \theta^{*\alpha}(\mu, x) < \infty \quad \text{for } \mu - a.e.\ x \in E.$$

Then, α is an integer.

This is a very beautiful result. It tells us that if the $\alpha-$density of μ at x exists for almost all $x \in E$, then μ behaves locally like $\mathcal{L}^\alpha$ (up to a constant, which is the

$\alpha-$density at the point x). Indeed, since $\mathcal{H}^\alpha$ coincides with $\mathcal{L}^\alpha$ on $\mathbb{R}^\alpha$ for natural numbers α, we see that

$$c_x := \theta^\alpha(\mu, x) = \lim_{r\to 0} \frac{\mu(B_r(x))}{\mathcal{H}^\alpha(B_r(x))} = \lim_{r\to 0} \frac{\mu(B_r(x))}{\mathcal{L}^\alpha(B_r(x))},$$

implies that for r sufficiently small,

$$\mu(B_r(x)) \approx c_x \mathcal{L}^\alpha(B_r(x)).$$

Naturally, we ask ourselves the following question: what is a good way of tackling this problem? Since our density is finite, when analyzing the ratio

$$\frac{\mu(B_r(x))}{\omega_\alpha r^\alpha},$$

we might as well omit the constant ω_α and only look at the behavior of

$$\frac{\mu(B_r(x))}{r^\alpha}$$

as $r \to 0$. One way of doing this is by seeing if the sequence of measures $\frac{\mu}{r_i^\alpha}$ (or something similar) converges weakly to some measure as the sequence $\{r_i\}$ decreases to 0. This leads to the following definition.

Definition 3.2 (Tangent Measures)**.** Let μ be a measure, $x \in \mathbb{R}^n$, and r a positive real number. We define the measure $\mu_{x,r}$ by

$$\mu_{x,r}(A) = \mu(x + rA) \quad \text{for all Borel sets } A \subset \mathbb{R}^n.$$

For any nonnegative real number α, we denote the set of $\alpha-$*tangent measures to* μ *at* x by $\mathrm{Tan}_\alpha(\mu, x)$, and define it as the set of all measures ν for which there exists a sequence r_i decreasing to 0 such that

$$\frac{\mu_{x,r_i}}{r_i^\alpha} \overset{*}{\rightharpoonup} \nu.$$

$\mathrm{Tan}_\alpha(\mu, x)$ is actually a subset of $\mathrm{Tan}(\mu, x)$, the set of *tangent measures to* μ *at* x. However, for our purposes, $\mathrm{Tan}_\alpha(\mu, x)$ contains all the information we need, and so there is no need to go into the (slightly) more general case.

Tangent measures have a very nice geometric interpretation. They can be viewed as a generalization of the concept of tangent planes to a C^1 submanifold of $\mathbb{R}^n$. Indeed, let Γ be a $k-$dimensional C^1 submanifold of $\mathbb{R}^n$, set $\mu := \mathcal{H}^k \llcorner \Gamma$ and fix some $x \in \Gamma$. Consider some set A containing x. Then,

$$\begin{aligned} r^{-k}\mu_{x,r}(A) &= r^{-k}\mu(x + rA) = r^{-k}(\mathcal{H}^k \llcorner \Gamma)(x + rA) \\ &= r^{-k}\mathcal{H}^k(\Gamma \cap (x + rA)) = r^{-k}\mathcal{H}^k(\Gamma_r), \end{aligned}$$

where we define $\Gamma_r := \{y \in A \mid x + ry \in \Gamma\}$. Since the set Γ is C^1, as $r \to 0$, the sets Γ_r begin to look like the tangent plane T_x to Γ at x. In fact,

$$\mathcal{H}^k \llcorner \Gamma_r \overset{*}{\rightharpoonup} \mathcal{H}^k \llcorner T_x.$$

For a proof of this statement, refer to section four of [2]. We now prove a useful result about tangent measures.

Proposition 3.3. *Let* μ *be a measure on* $\mathbb{R}^n$ *and* $f \in L^1(\mu)$ *a Borel nonnegative function. Then,*

$$\mathrm{Tan}_\alpha(f\mu, x) = f(x)\mathrm{Tan}_\alpha(\mu, x) \quad \text{for } \mu - a.e.\ x. \tag{3.4}$$

Proof. Consider the set of points

$$S := \left\{ x \in \mathbb{R}^n \ : \ \lim_{r \to 0} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f(y) - f(x)| d\mu(y) = 0 \right\}. \tag{3.5}$$

By Proposition 2.9, $\mu(\mathbb{R}^n \setminus S) = 0$. Therefore, it suffices to prove that equality (3.4) holds for all x in S. Fix $x \in S$, $\nu \in \mathrm{Tan}_\alpha(\mu, x)$ and a sequence $\{r_i\}$ decreasing to 0 such that

$$\nu_i := \frac{\mu_{x,r_i}}{r_i^\alpha} \overset{*}{\rightharpoonup} \nu. \tag{3.6}$$

If we define

$$\nu_i^{'} := \frac{(f\mu)_{x,r_i}}{r_i^\alpha},$$

then for every ball $B_\rho(0)$, we obtain

$$\begin{aligned}
|f(x)\nu_i - \nu_i^{'}|(B_\rho(0)) &= \left| \frac{f(x)\mu_{x,r_i}}{r_i^\alpha} - \frac{(f(y)\mu)_{x,r_i}}{r_i^\alpha} \right| (B_\rho(0)) \\
&= \frac{1}{r_i^\alpha} \left[\sup_{\phi \in C_c(B_\rho(0)), |\phi| \le 1} \int_{B_\rho(0)} \phi \ d(f(x)\mu_{x,r_i} - (f(y)\mu)_{x,r_i}) \right] \\
&\le \frac{1}{r_i^\alpha} \int_{B_\rho(0)} \chi_{B_\rho(0)} d(f(x)\mu_{x,r_i} - (f(y)\mu)_{x,r_i}) \\
&= \frac{1}{r_i^\alpha} (f(x)\mu_{x,r_i} - (f(y)\mu)_{x,r_i})(B_\rho(0)) \\
&= \frac{1}{r_i^\alpha} \int_{B_{\rho r_i}(x)} (f(x) - f(y)) \ d\mu(y) \\
&\le \frac{1}{r_i^\alpha} \int_{B_{\rho r_i}(x)} |f(y) - f(x)| d\mu(y) \\
&= \frac{\mu(B_{\rho r_i}(x))}{r_i^\alpha} \cdot \frac{1}{\mu(B_{\rho r_i}(x))} \int_{B_{\rho r_i}(x)} |f(y) - f(x)| d\mu(y).
\end{aligned}$$

But, the quantity

$$\frac{1}{\mu(B_{\rho r_i}(x))} \int_{B_{\rho r_i}(x)} |f(y) - f(x)| d\mu(y)$$

vanishes in the limit as $r_i \to 0$ because $x \in S$, whereas the term

$$\frac{\mu(B_{\rho r_i}(x))}{r_i^\alpha}$$

is bounded because of (3.6). It follows that $|f(x)\nu_i - \nu_i^{'}|(B_\rho(x)) \to 0$ for every $\rho > 0$ and therefore $\nu_i^{'} \overset{*}{\rightharpoonup} f(x)\nu$. This shows that $\mathrm{Tan}_\alpha(f\mu, x) \subset f(x)\mathrm{Tan}_\alpha(\mu, x)$. The reverse inclusion follows by similar reasoning. □

Remark 3.7. As a corollary of Proposition 3.3, we obtain the very intuitive fact that, for every Borel set B,

$$\mathrm{Tan}_\alpha(\mu \llcorner B, x) = \mathrm{Tan}_\alpha(\mu, x) \ \text{ for } \mu - a.e.\ x \in B.$$

Before returning to Marstrand's theorem, we need one more definition.

Definition 3.8 ($\alpha-$uniform measures)**.** We say that a measure μ is $\alpha - uniform$ if

$$\mu(B_r(x)) = \omega_\alpha r^\alpha \ \text{ for every } x \in \mathrm{supp}(\mu) \text{ and every } r > 0.$$

We let $\mathcal{U}^\alpha(\mathbb{R}^n)$ denote the set of $\alpha-$uniform measures μ such that $0 \in \mathrm{supp}(\mu)$.

3.2. **Proving Marstrand's Theorem.** Proving Marstrand's theorem can be reduced to proving the following two propositions:

Proposition 3.9. *Let μ be as in Theorem 3.1. Then, for $\mu-$a.e. $x \in E$, we have*

$$\emptyset \neq \mathrm{Tan}_\alpha(\mu, x) \subset \{\theta^\alpha(\mu, x)\nu \ : \ \nu \in \mathcal{U}^\alpha(\mathbb{R}^n)\}.$$

Proposition 3.10. *If $\mathcal{U}^\alpha(\mathbb{R}^n) \neq \emptyset$, then α is a nonnegative integer less than or equal to n.*

Proving Proposition 3.10 will comprise the vast majority of our work. Before analyzing the statement of this proposition more carefully, we prove Proposition 3.9.

Proof of Proposition 3.9. To clarify the technical details of the proof, we split the proof into three parts. The main purpose of the proposition is to show that $\mathrm{Tan}_\alpha(\mu, x) \subset \{\theta^\alpha(\mu, x)\nu \ : \ \nu \in \mathcal{U}^\alpha(\mathbb{R}^n)\}$. Thus, if $\nu \in \mathrm{Tan}_\alpha(\mu, x)$, our goal is to find a bound on

$$\left|\nu(B_r(y)) - \theta^\alpha(\mu, x)\omega_\alpha r^\alpha\right|$$

that can become as small as desired. Since we wish to show the result for $\mu - a.e.$ $x \in E$, we employ the common technique of writing E in a crafty way as a countable union of sets. Bearing this in mind, we begin the proof.

Part 1 For every $i, j, k \in \mathbb{N}$ consider the sets

$$E^{i,j,k} := \left\{x : \frac{(j-1)\omega_\alpha}{i} \leq \frac{\mu(B_r(x))}{r^\alpha} \leq \frac{(j+1)\omega_\alpha}{i} \ \text{ for all } r \leq \frac{1}{k}\right\}.$$

Since the $\alpha-$density is, by hypothesis, finite almost everywhere in E, we obtain for every fixed i,

$$A \subset \bigcup_{j,k} E^{i,j,k}, \tag{3.11}$$

where $A \subset E$ is the set of points where the $\alpha-$density is finite. Indeed, for arbitrary $x \in A$, $\epsilon > 0$, and $r > 0$ sufficiently small, we obtain

$$(\theta^\alpha(\mu, x) - \epsilon)\omega_\alpha \leq \frac{\mu(B_r(x))}{r^\alpha} < (\theta^\alpha(\mu, x) + \epsilon)\omega_\alpha.$$

For suitable k and j, the above inequalities show that x is contained in some $E^{i,j,k}$. We claim that, for $\mu - a.e.$ $x \in E^{i,j,k}$ the following holds: For every $\nu \in \mathrm{Tan}_\alpha(\mu \llcorner E^{i,j,k}, x)$, we have the inequality

$$\left|\nu(B_r(y)) - \theta^\alpha(\mu, x)\omega_\alpha r^\alpha\right| \leq \frac{2\omega_\alpha}{i} \ \text{ for every } y \in \mathrm{supp}(\nu) \text{ and } r > 0. \tag{3.12}$$

We will prove this claim in the next part of the proof. For now, we show how it implies that $\mathrm{Tan}_\alpha(\mu, x) \subset \{\theta^\alpha(\mu, x)\nu \ : \ \nu \in \mathcal{U}^\alpha(\mathbb{R}^n)\}$. Notice that combining this

claim with Remark 3.7 shows that if we fix i, then for every j and k, the inequality (3.12) holds for $\mu - a.e.\ x \in E^{i,j,k}$, for all $\nu \in \mathrm{Tan}_\alpha(\mu, x)$.

From (3.11) we see that for $\mu - a.e.\ x \in E$, the bound (3.12) holds for every $\nu \in \mathrm{Tan}_\alpha(\mu, x)$. Since i runs through the set of natural numbers, we conclude that for all such x and ν,

$$\nu(B_r(y)) = \theta^\alpha(\mu, x)\omega_\alpha r^\alpha \quad \text{for every } y \in \mathrm{supp}(\nu) \text{ and } r > 0.$$

That is, $\frac{1}{\theta^\alpha(\mu,x)}\nu$ is an $\alpha-$ uniform measure. To conclude that $\frac{1}{\theta^\alpha(\mu,x)}\nu \in \mathcal{U}^\alpha(\mathbb{R}^n)$, it suffices to show that $0 \in \mathrm{supp}(\nu)$. By Proposition 2.4,

$$\rho^{-\alpha}\nu(\overline{B_\rho(0)}) \geq \rho^{-\alpha} \limsup_{i\to\infty} \mu_{x_i,r_i}(B_\rho(0)) = \limsup_{i\to\infty} \frac{\mu(B_{\rho r_i}(x_i))}{(\rho r_i)^\alpha} \geq \omega_\alpha \theta^\alpha_*(\mu, x).$$

Letting ρ increase to r, we conclude that $r^{-\alpha}\nu(B_r(0)) \geq \omega_\alpha\theta^\alpha_*(\mu, x)$. Thus, $\theta^\alpha_*(\nu, 0) \geq \theta^\alpha_*(\mu, x) > 0$, as desired.

Part 2 We now have to prove (3.12) for $\mu - a.e.\ x \in E^{i,j,k}$. To simplify the notation, set

$$F := E^{i,j,k} \quad ; \quad F_1 := \left\{ x \in F \ : \ \lim_{r\to 0} \frac{\mu(B_r(x) \setminus F)}{r^\alpha} = 0 \right\}.$$

By Proposition 2.9, we have $\mu(F \setminus F_1) = 0$ and therefore it suffices to prove (3.12) in the case when $x \in F_1$. Fix $x \in F_1, \nu \in \mathrm{Tan}_\alpha(\mu \llcorner F, x)$, and $r_i \to 0$ such that

$$\nu^i := \frac{(\mu \llcorner F)_{x,r_i}}{r_i^\alpha} \overset{*}{\rightharpoonup} \nu.$$

Notice that for every $y \in \mathrm{supp}(\nu)$ there exists a sequence of points $\{x_i\} \subset F$ such that

$$y_i := \frac{x_i - x}{r_{n_i}} \to y,$$

for some subsequence $\{r_{n_i}\}$ of $\{r_i\}$. Since $y \in \mathrm{supp}(\nu)$, for all $s > 0$, $\nu(B_s(y)) > 0$. Choose $s_i \downarrow 0$. Since

$$\lim_{j\to\infty} (\mu \llcorner F)_{x,r_j}(B_{s_i}(y)) > 0,$$

there exists an N such that $j \geq N$ implies that

$$\mu(B_{s_i r_j}(x + r_j y) \cap F) > 0.$$

Therefore, for these j, we have

$$B_{s_i r_j}(x + r_j y) \cap F \neq \emptyset.$$

This shows that there exists a subsequence $\{r_{n_i}\}$ and a sequence of points $\{x_i\}$ such that

$$x_i \in B_{s_i r_{n_i}}(x + r_{n_i} y) \cap F.$$

Thus, there exists a sequence of points $\{y_i^{'}\}$ with $|y_i^{'}| < s_i r_{n_i}$ such that

$$x_i = x + r_{n_i} y + y_i^{'} \Longrightarrow \frac{x_i - x}{r_{n_i}} = y + \frac{y_i^{'}}{r_{n_i}}.$$

As $i \to \infty$, $\frac{y_i^{'}}{r_{n_i}} \to 0$ and so the above statement is valid.

We now claim that there exists a set $S \subset \mathbb{R}$, which is at most countable and such that

$$\lim_{i\to\infty} \nu^i(B_\rho(y_i)) = \nu(B_\rho(y)) \text{ for every } \rho \in \mathbb{R}^+ \setminus S. \tag{3.13}$$

If we define $\xi^i := \nu^i_{y_i-y,1}$, then we see that $\xi^i \overset{*}{\rightharpoonup} \nu$. Indeed, notice that for every bounded open set A,

$$\xi^i(A) = \nu^i_{y_i-y,1}(A) = \nu^i((y_i - y) + A) \to \nu^i(A), \text{ as } i \to \infty.$$

Hence,

$$\begin{aligned} |\xi^i - \nu|(A) &= \sup_{\phi \in C_c(\mathbb{R}^n), |\phi| \le 1} \int_A \phi \; d(\xi^i - \nu) \\ &\le \int_A \chi_A \; d(\xi^i - \nu) = (\xi^i - \nu)(A). \end{aligned}$$

But as $i \to \infty$, we know that $\xi^i(A) \to \nu^i(A)$. Moreover, since $\nu^i \overset{*}{\rightharpoonup} \nu$, we see that $|\xi^i - \nu|(A) \to 0$. Hence, $\xi^i \overset{*}{\rightharpoonup} \nu$, as desired. Therefore, by Proposition 2.4, there exists an at most countable set $S \subset \mathbb{R}$ such that

$$\lim_{i\to\infty} \nu^i(B_\rho(y_i)) = \lim_{i\to\infty} \nu^i_{y_i-y,1}(B_\rho(y)) = \lim_{i\to\infty} \xi^i(B_\rho(y)) = \nu(B_\rho(y)) \text{ for every } \rho \in \mathbb{R}^+ \setminus S.$$

This is precisely (3.13). We now compute

$$\lim_{i\to\infty} \nu^i(B_\rho(y_i)) = \lim_{i\to\infty} \frac{\mu(B_{\rho r_i}(x_i) \cap F)}{r_i^\alpha}.$$

Since $\frac{x_i - x}{r_i} \to y$, we have that $|x_i - x| \le C r_i$ for some sufficiently large constant C. Thus,

$$\lim_{i\to\infty} \frac{\mu(B_{\rho r_i}(x_i) \setminus F)}{r_i^\alpha} \le \lim_{i\to\infty} \frac{\mu(B_{(C+\rho) r_i}(x) \setminus F)}{r_i^\alpha} = 0,$$

with the last equality following from the fact that $x \in F_1$. Hence,

$$\lim_{i\to\infty} \nu^i(B_\rho(y_i)) = \lim_{i\to\infty} \frac{\mu(B_{\rho r_i}(x_i))}{r_i^\alpha}.$$

From this equality and the definition of F, we see that inequality (3.12) holds for every $\rho \in \mathbb{R}^+ \setminus S$. Since S is countable, for every $\rho \in S$, there exists a sequence $\{\rho_j\} \subset \mathbb{R}^+ \setminus S$ increasing to S. Therefore, $\nu(B_\rho(y)) = \lim_{j\to\infty} \nu(B_{\rho_j}(y))$ and therefore inequality (3.14) holds for every $r \in \mathbb{R}^+$.

Part 3 Thus far, we have shown that

$$\mathrm{Tan}_\alpha(\mu, x) \subset \{\theta^\alpha(\mu, x)\nu \; : \; \nu \in \mathcal{U}^\alpha(\mathbb{R}^n)\}.$$

What remains to show is that for $\mu - a.e.$ $x \in \mathbb{R}^n$ the set $\mathrm{Tan}_\alpha(\mu, x)$ is nonempty. Fix any x such that $\theta^{*\alpha}(\mu, x) < \infty$. Then, for every $\rho > 0$, the set of numbers

$$r^{-\alpha}\mu(B_{\rho r}(x)) = r^{-\alpha}\mu_{x,r}(B_\rho(0)) \;\; \text{where } r \le 1$$

is uniformly bounded. Hence, the family of measures $\{r^{-\alpha}\mu_{x,r}\}_{r\le 1}$ is locally uniformly bounded. By Proposition 2.3, there exists a sequence $\{r_j\}$ decreasing to 0 and a measure $\tilde{\mu}$ such that $\mu_{x,r_j} \overset{*}{\rightharpoonup} \tilde{\mu}$. Hence, $\tilde{\mu} \in \mathrm{Tan}_\alpha(\mu, x)$, completing the proof. □

We now return to Proposition 3.10. We first provide an outline of the proof, before introducing the technical lemmas necessary to prove the proposition.

Outline of the proof of Proposition 3.10. There are three main parts to the proof.

Part 1 The Besicovitch Differentiation Theorem (Theorem 2.7) implies that $\mathcal{U}^\alpha(\mathbb{R}^k) = \emptyset$ for every $\alpha > k$. Indeed, fix some $\mu \in \mathcal{U}^\alpha(\mathbb{R}^k)$. Since both μ and

$\mathcal{L}^k$ are locally finite measures on $\mathbb{R}^k$, we can apply the Besicovitch Differentiation Theorem to obtain that

$$D(\mu, \mathcal{L}^k, x) = \lim_{r\to 0} \frac{\mu(B_r(x))}{\mathcal{L}^k(B_r(x))} = \lim_{r\to 0} \frac{\omega_\alpha r^\alpha}{\omega_k r^k} \tag{3.14}$$

exists for $\mathcal{L}^k - a.e.\ x \in \operatorname{supp}(\mu)$. Notice that the support of μ is all of $\mathbb{R}^k$. Moreover, if $\alpha \geq k$, then the set of points where the derivative $D(\mu, \mathcal{L}^k, x)$ is infinite is empty. It follows that

$$\mu = D(\mu, \mathcal{L}^k, x)\mathcal{L}^k.$$

That is, for every measurable set $E \subset \mathbb{R}^k$, we have

$$\mu(E) = \int_E D(\mu, \mathcal{L}^k, x) d\mathcal{L}^k.$$

But if $\alpha > k$, then (3.14) shows that the derivative is 0, and so μ reduces to the trivial measure. Thus, $\mathcal{U}^\alpha(\mathbb{R}^k) = \emptyset$ whenever $\alpha > k$.

Part 2 This step, which is at the heart of the proof, aims to show that if $\alpha < k$, then

$$\mathcal{U}^\alpha(\mathbb{R}^k) \neq \emptyset \Longrightarrow \mathcal{U}^\alpha(\mathbb{R}^{k-1}) \neq \emptyset. \tag{3.15}$$

Part 3 Finally, arguing by contradiction, we assume that $\mathcal{U}^\alpha(\mathbb{R}^n) \neq \emptyset$ for some $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$. Letting $k := \lfloor \alpha \rfloor < \alpha < n$ and applying (3.15) $n - \lfloor \alpha \rfloor$ times, we conclude that $\mathcal{U}^\alpha(\mathbb{R}^k) \neq \emptyset$, which contradicts the conclusion of part 1. □

Therefore, we have reduced proving Proposition 3.10, which implies Marstrand's Theorem, to proving the claim in part 2 above. The proof of this claim is split up into the following three lemmas. The first is highly similar to Proposition 3.9.

Lemma 3.16. *Let* $\alpha \geq 0, \mu \in \mathcal{U}^\alpha(\mathbb{R}^k)$, and $x \in \operatorname{supp}(\mu)$. *Then*

$$\emptyset \neq \operatorname{Tan}_\alpha(\mu, x) \subset \mathcal{U}^\alpha(\mathbb{R}^k).$$

The second is a geometric observation characterizing the support of a tangent measure.

Lemma 3.17. *Let* $0 \leq \alpha < k$ *and* $\mu \in \mathcal{U}^\alpha(\mathbb{R}^k)$. *Then there exists* $y \in \operatorname{supp}(\mu)$ *and a system of coordinates* $x_1, ..., x_k$ *on* $\mathbb{R}^k$ *such that*

$$\operatorname{supp}(\nu) \subset \{x_1 \geq 0\} \text{ for every } \nu \in \operatorname{Tan}_\alpha(\mu, y).$$

The final lemma builds upon the previous one and is the core of Marstrand's proof.

Lemma 3.18. *Let* $0 \leq \alpha < k$ *and* $\nu \in \mathcal{U}^\alpha(\mathbb{R}^k)$. *If* $\operatorname{supp}(\nu) \subset \{x_1 \geq 0\}$, *then*

$$\operatorname{supp}(\tilde{\nu}) \subset \{x_1 = 0\} \text{ for every } \tilde{\nu} \in \operatorname{Tan}_\alpha(\nu, 0).$$

The result of the last lemma deserves special attention. The fact that the support of every tangent measure $\tilde{\nu}$ of ν is contained in the hyperplane $\{x_1 = 0\}$ shows that we can construct tangent measures of uniform measures (which by Lemma 3.16 are themselves uniform) whose support lives in a set of dimension exactly one smaller than that of our initial measure ν. This means that $\tilde{\nu}$ can be viewed in a natural way as an element of $\mathcal{U}^\alpha(\mathbb{R}^{n-1})$, which is exactly what we wanted! We now proceed to prove these lemmas.

Proof of Lemma 3.16. The argument given in part 3 of the proof of Proposition 3.9 shows that $\mathrm{Tan}_\alpha(\mu, x) \neq \emptyset$ for every $x \in \mathrm{supp}(\mu)$. Fix some arbitrary $x \in \mathrm{supp}(\mu)$ and any $\nu \in \mathrm{Tan}_\alpha(\mu, x)$, and let $\{r_i\}$ be a sequence decreasing to 0 such that

$$r_i^{-\alpha}\mu_{x,r_i} \overset{*}{\rightharpoonup} \nu.$$

Given any $y \in \mathrm{supp}(\nu)$, the same argument as in part 2 of the proof of Proposition 3.9 shows both that there exists a sequence $\{x_i\} \subset \mathrm{supp}(\mu)$ such that

$$y_i := \frac{x_i - x}{r_i} \to y,$$

and that there exists a countable set $S \subset \mathbb{R}^+$ such that

$$\lim_{i\to\infty} r_i^{-\alpha}\mu_{x,r_i}(B_\rho(y_i)) = \nu(B_\rho(y)) \ \text{ for every } \rho \in \mathbb{R}^+ \setminus S.$$

Another way of saying the second statement is that for every $\rho \in \mathbb{R}^+ \setminus S$ we have

$$\nu(B_\rho(y)) = \lim_{i\to\infty} \frac{\mu(B_{\rho r_i}(x_i))}{r_i^\alpha} = \omega_\alpha \rho^\alpha.$$

For every $\rho \in \mathbb{R}^+$, there exists a sequence $\{\rho_j\} \in \mathbb{R}^+ \setminus S$ such that $\rho_j \uparrow \rho$. This implies that $\nu(B_\rho(y)) = \omega_\alpha\rho^\alpha$ for every $\rho > 0$. Since y was arbitrary, we conclude that $\nu \in \mathcal{U}^\alpha(\mathbb{R}^n)$. □

We now prove Lemma 3.17.

Proof of Lemma 3.17. Set $E := \mathrm{supp}(\mu)$. We first want to show that $E \neq \mathbb{R}^k$. In fact, since $\alpha < k$, it makes sense that even $B_1(0)$ is not contained in E. Indeed, by the Besicovitch-Vitali Covering Theorem (Theorem 2.6), we can cover $\mathcal{L}^n-$almost all $B_1(0)$ by a countable collection of pairwise disjoint closed balls $\{\overline{B}_{r_j}(x_j)\}$ contained in $B_1(0)$ and with $r_j < 1$ for all j. If we had $B_1(0) \subset E$, then we could obtain

$$\begin{aligned}\mu(B_1(0)) &\geq \sum_{j=1}^\infty \mu(\overline{B}_{r_j}(x_j)) = \omega_\alpha \sum_{j=1}^\infty r_j^k > \omega_\alpha \sum_{j=1}^\infty r_j^k = \frac{\omega_\alpha}{\omega_k}\sum_{j=1}^\infty \mathcal{L}^k(\overline{B}_{r_j}(x_j)) \\ &= \frac{\omega_\alpha}{\omega_k}\mathcal{L}^k(B_1(0)) = \omega_\alpha.\end{aligned}$$

This contradicts the fact that $\mu(B_1(0)) = \omega_\alpha$. Notice that we used the identity $\mu(\overline{B}_{r_j}(x_j)) = \omega_\alpha r_j^\alpha$; even though in the definition of $\alpha-$uniform measures this identity holds for open balls, it is a quick exercise to see that it holds for closed balls as well.

Since $E \neq \mathbb{R}^k$, fix some $y \notin E$. Because E is a nonempty closed set, there exists a $z \in E$ such that $\mathrm{dist}(y, E) = |y - z| =: a$. Without loss of generality, we assume that z is the origin and we fix a system of coordinates $x_1, x_2, ..., x_k$ such that $y = (-a, 0, ..., 0)$. From the definition of $\mathrm{dist}(y, E)$, we see that E is contained in the set

$$\tilde{E} := \mathbb{R}^n \setminus B_a(y) = \big\{x : (a + x_1)^2 + x_2^2 + ... + x_n^2 \geq a^2\big\}.$$

Fix some $\nu \in \mathrm{Tan}_\alpha(\mu, 0)$ and a sequence $r_i \downarrow 0$ such that

$$\nu_i := \frac{\mu_{0,r_i}}{r_i^\alpha} \overset{*}{\rightharpoonup} \nu.$$

Then, the support of ν_i is given by

$$E_i := \frac{E}{r_i} \subset \tilde{E}_i := \left\{x : (a + r_i x_1)^2 + r_i^2(x_2^2 + ... + x_n^2) \geq a^2\right\}.$$

Notice that for any $x \in \{x_1 < 0\}$ there exists $N > 0$ and $\rho > 0$ such that $B_\rho(x) \cap \tilde{E}_i = \emptyset$ for $i \geq N$. Therefore, $\nu_i(B_\rho(x)) = 0$, which yields $x \notin \text{supp}(\nu)$. Therefore, $\text{supp}(\nu) \subset \{x_1 \geq 0\}$, as desired. □

Before presenting the proof of lemma 3.18, we make a small remark that will be valuable in the proof.

Remark 3.19. Let $\mu \in \mathcal{U}^\alpha(\mathbb{R}^n)$ and let $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$ be a simple function. Then, we can write f as

$$f = \sum_{i=1}^{N} a_i \chi_{[0,r_i)}$$

for some choice of $N \in \mathbb{N}, r_i > 0$ and $a_i \in \mathbb{R}$. Then, for any $y \in \text{supp}(\mu)$, we have

$$\int f(|z|)d\mu(z) = \sum_{i=1}^{N} a_i \mu(B_{r_i}(0)) = \sum_{i=1}^{N} a_i \mu(B_{r_i}(y)) = \int f(|z-y|)d\mu(z).$$

Therefore, if ϕ is a radial function (meaning its value at a point depends only on the distance between that point and the origin), then an approximation argument using simple functions shows that, for all $y \in \text{supp}(\mu)$, we have

$$\int \phi(z)d\mu(z) = \int \phi(z-y)d\mu(z). \tag{3.20}$$

Proof of Lemma 3.18. We first define the quantity

$$b(r) := \frac{\omega_\alpha}{\nu(B_r(0))} \int_{B_r(0)} z d\nu(z) = r^{-\alpha} \int_{B_r(0)} z d\nu(z). \tag{3.21}$$

Let $(b_1(r), b_2(r), ..., b_k(r))$ denote the components of the vector $b(r)$. Notice that the quantity $b(r)$ is ω_α multiplied by the center of mass of the measure $\nu \llcorner B_r(0)$. By hypothesis, we know that $\text{supp}(\nu) \subset \{x_1 \geq 0\}$, and hence $b_1(r) \geq 0$. Moreover, $b_1(r) = 0$ would imply the claim of the lemma. For if this were not the case, then there would exist some $\tilde{\nu} \in \text{Tan}_\alpha(\nu, 0)$ such that the support of $\tilde{\nu}$ is not contained in $\{x_1 = 0\}$. Therefore, there exists some $x = (x_1, ..., x_k)$ such that (without loss of generality) $x_1 > 0$ and $\tilde{\nu}(B_\rho(x)) > 0$ for some ρ such that $B_\rho(x) \subset \{x_1 > 0\}$. Let $\{r_i\}$ be a sequence decreasing to zero such that

$$\frac{\nu_{0,r_i}}{r_i^\alpha} \overset{*}{\rightharpoonup} \tilde{\nu}.$$

Then,

$$\int_{B_\rho(x)} z \, d\tilde{\nu}(z) = \lim_{i\to\infty} \int_{B_\rho(x)} z \, d\left(\frac{\nu_{0,r_i}}{r_i^\alpha}\right) = \lim_{i\to\infty} r_i^{-\alpha} \int_{B_{\rho r_i(x)}} z \, d\nu. \tag{3.22}$$

For r sufficiently large, $B_{\rho r_i(x)} \subset B_r(0)$. Therefore, the first component of $b(r)$, for this r, is at least as large as that of the first integral in (3.22). It follows that the first component of this integral must be zero. This is impossible, since $B_\rho(x) \subset \{x_1 > 0\}$ and the first component of $f(z) = z$ is positive on this ball. Therefore, if $b_1(r) = 0$, then the claim of the lemma is true.

When $b_1(r) > 0$, the main idea of the proof is to study the limiting behavior of the quantity $b(r)$ as r decreases to 0. More explicitly, given some $\tilde{\nu} \in \mathrm{Tan}_\alpha(\nu, 0)$ we define

$$c(r) := r^{-\alpha} \int_{B_r(0)} z d\tilde{\nu}(z), \tag{3.23}$$

which plays exactly the same role for $\tilde{\nu}$ that $b(r)$ does for ν. We want to show that $c(r) = 0$ for all r. Since $\mathrm{supp}(\tilde{\nu}) \subset \{x_1 \geq 0\}$ this would immediately imply that $\mathrm{supp}(\tilde{\nu}) \subset \{x_1 = 0\}$, concluding the proof. We split this into two parts.

Part 1 We first prove the following claim:

$$|\langle b(r), y\rangle| \leq C(\alpha)|y|^2 \quad \text{for every } y \in \mathrm{supp}(\nu) \cap B_{2r}(0). \tag{3.24}$$

We wish to show this because, if we can find a bound on an inner product with the quantity $b(r)$, then we can also find one on an inner product with $c(r)$, seeing as $c(r)$ and $b(r)$ are related through a limiting process. This will, in turn, help us show that $c(r) = 0$. From the identity

$$2\langle x, y\rangle = |y|^2 + (r^2 - |x-y|^2) - (r^2 - |x|^2),$$

we compute

$$\begin{aligned} 2|\langle b(r), y\rangle| &= r^{-\alpha}\left| \int_{B_r(0)} 2\langle x, y\rangle \, d\nu(x)\right| \\ &= r^{-\alpha}\left| |y|^2 \nu(B_r(0)) + \int_{B_r(0)} (r^2 - |x-y|^2) d\nu(x) - \int_{B_r(0)} (r^2 - |x|^2) d\nu(x)\right|. \end{aligned} \tag{3.25}$$

For $y \in \mathrm{supp}(\nu)$, Remark 3.19 shows that

$$\begin{aligned} &\int_{B_r(0)} (r^2 - |x-y|^2) d\nu(x) - \int_{B_r(0)} (r^2 - |x|^2) d\nu(x) \\ &= \int_{B_r(0)} (r^2 - |x-y|^2) d\nu(x) - \int_{B_r(y)} (r^2 - |x-y|^2) d\nu(x) \\ &= \int_{B_r(0)\setminus B_r(y)} (r^2 - |x-y|^2) d\nu(x) - \int_{B_r(y)\setminus B_r(0)} (r^2 - |x-y|^2) d\nu(x). \end{aligned} \tag{3.26}$$

Combining (3.25) and (3.26) we obtain

$$\begin{aligned} 2|\langle b(r), y\rangle| \leq \omega_\alpha |y|^2 &+ r^{-\alpha} \int_{B_r(0)\setminus B_r(y)} \left|r^2 - |x-y|^2\right| d\nu(x) \\ &+ r^{-\alpha} \int_{B_r(y)\setminus B_r(0)} \left|r^2 - |x-y|^2\right| d\nu(x). \end{aligned} \tag{3.27}$$

For $x \in B_r(0) \setminus B_r(y)$ we have

$$0 \leq |x-y|^2 - r^2 \leq |x-y|^2 - x^2 = (|x-y| + |x|)(|x-y| - |x|) \leq 4r|y|,$$

whereas for $x \in B_r(y) \setminus B_r(0)$ we similarly obtain

$$0 \leq r^2 - |x-y|^2 \leq x^2 - |x-y|^2 = (|x-y| + |x|)(|x| - |x-y|) \leq 4r|y|.$$

Therefore, (3.27) yields

$$\begin{aligned} 2|\langle b(r), y\rangle| &\leq \omega_\alpha |y|^2 + \frac{4r|y|}{r^\alpha}\Big[\nu(B_r(y)\backslash B_r(0)) + \nu(B_r(0)\backslash B_r(y))\Big] \\ &= \omega_\alpha |y|^2 + \frac{4r|y|}{r^\alpha}\nu\Big[(B_r(y)\backslash B_r(0)) + (B_r(0)\backslash B_r(y))\Big]. \end{aligned} \tag{3.28}$$

If $|y| < r$, then one verifies that

$$\big(B_r(y) \setminus B_r(0)\big) \cup \big(B_r(0) \setminus B_r(y)\big) \subset \big(B_{r+|y|}(0) \setminus B_{r-|y|}(y)\big).$$

Therefore, we obtain

$$\begin{aligned} 2|\langle b(r), y\rangle| &\leq \omega_\alpha |y|^2 + \frac{4|y|}{r^{\alpha-1}}\Big[\nu(B_{r+|y|}(0) - \nu(B_{r-|y|}(y))\Big] \\ &= \omega_\alpha |y|^2 + \frac{4|y|\omega_\alpha}{r^{\alpha-1}}\Big[(r+|y|)^\alpha - (r-|y|)^\alpha\Big] \\ &= \omega_\alpha |y|^2 + \frac{4|y|\omega_\alpha}{r^{\alpha-1}} \cdot r^\alpha \left[\left(1+\frac{|y|}{r}\right)^\alpha - \left(1-\frac{|y|}{r}\right)^\alpha\right]. \end{aligned} \tag{3.29}$$

By Newton's Binomial Theorem (notice that $\frac{|y|}{r} < 1$), this becomes

$$\begin{aligned} &\omega_\alpha |y|^2 + 4r|y|\omega_\alpha \left[\sum_{k=0}^{\infty}\binom{\alpha}{k}\left(\frac{|y|}{r}\right)^k - \sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}\left(\frac{|y|}{r}\right)^k\right] \\ &= \omega_\alpha |y|^2 + 4r|y|\omega_\alpha \left[\sum_{k \text{ odd}} 2\binom{\alpha}{k}\left(\frac{|y|}{r}\right)^k\right] \\ &= \omega_\alpha |y|^2 + 8\binom{\alpha}{k}|y|^2\omega_\alpha \left[\sum_{k \text{ odd}, k\geq 3} 2\binom{\alpha}{k}\left(\frac{|y|}{r}\right)^k\right]. \end{aligned}$$

Notice that the sum in the brackets can be bounded by 2^α, since both the terms in the brackets in equation (3.29) are less than 2^α. Therefore,

$$2|\langle b(r), y\rangle| \leq C(\alpha)|y|^2, \text{ where } C(\alpha) = \omega_\alpha\left(1 + 2^{(\alpha+3)}\binom{\alpha}{k}\right).$$

This yields (3.24) for $|y| \leq r$. For $r \leq |y| \leq 2r$ we use

$$B_r(y) \setminus B_r(0) \cup \ B_r(0) \setminus B_r(y) \subset B_{r+|y|}(0)$$

and a similar computation.

Part 2 To reach the desired conclusion, fix $\tilde{\nu} \in \mathrm{Tan}_\alpha(\nu, 0)$ and a sequence $r_i \downarrow 0$ such that

$$\nu_i := \frac{\nu_{0,r_i}}{r_i^\alpha} \overset{*}{\rightharpoonup} \tilde{\nu}.$$

Moreover, let $b(r)$ and $c(r)$ be the quantities defined in (3.21) and (3.23). By Proposition 2.4, there is a set $S \subset \mathbb{R}$ which is at most countable and such that

$$c(\rho) = \lim_{r_i \to 0} b(\rho r_i) \ \text{ for } \rho \in \mathbb{R}^+ \setminus S.$$

Let $\rho \in \mathbb{R}^+ \setminus S$ and $z \in \operatorname{supp}(\tilde{\nu}) \cap B_\rho(0)$. Then there exists a sequence $\{z_i\}$ converging to z such that $y_i := r_i z_i \in \operatorname{supp}(\nu)$. Using (3.24) we obtain

$$|\langle c(\rho), z\rangle| = \lim_{r_i \to 0} \frac{|\langle b(\rho r_i), y_i\rangle|}{r_i} \leq C(\alpha) \lim_{r_i \to 0} \frac{|y_i|^2}{r_i} = 0.$$

Since $\langle c(\rho), z\rangle = 0$ for every $z \in \operatorname{supp}(\tilde{\nu}) \cap B_\rho(0)$, we find that

$$0 = \rho^{-\alpha} \int_{B_\rho(0)} \langle c(\rho), z\rangle \, d\tilde{\nu}(z) = |c(\rho)|^2.$$

This holds for every $\rho \in \mathbb{R}^+ \setminus S$. Since for $\rho \in S$ there exists a sequence $\{\rho_i\}$ increasing to ρ, we conclude that

$$c(\rho) = \lim_{i \to \infty} c(\rho_i) = 0.$$

Therefore, $c(\rho) = 0$ for every $\rho > 0$, completing the proof. □

Acknowledgments

I would like to thank my mentor Dr. Dali Nimer for introducing me to geometric measure theory, providing me with great materials to read and study, and for helping me throughout the process of writing this paper. I would also like to thank Isabella Scott and Aygul Galimova for providing me with feedback on the paper. Additionally, I am very grateful to all the faculty members who gave lectures throughout the REU. Finally, I would like to express my sincerest gratitude to Professor May for organizing such a wonderful REU program, which provided me with a very formative learning experience.

References

[1] Mattila, P. *Geometry of Sets and Measures in Euclidean Space: Fractals and Rectifiability* Volume 44 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 1995.

[2] De Lellis, C. *Lecture Notes on Rectifiable Sets, Densities, and Tangent Measures*