HOMOTOPY TYPES OF SPACES WITH 7 OR FEWER POINTS

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Abstract. This paper classifies the homotopy types of spaces with 7 or fewer points. Particularly, we find that up to homeomorphism, there exist 25 minimal connected spaces with 7 points.

Contents

1. Introduction and Basic Definitions

Let X and Y be finite topological spaces. We say that X and Y are homotopy equivalent if there exist two continuous functions, $f : X \to Y$ and $g : Y \to X$, such that there is a continuous function $h : X \times [0,1] \to Y$, where $h(x,0) = f(x)$ and $h(x,1) = g(x)$.

In this paper, we classify spaces with 7 points or fewer up to homotopy equivalence. First, we establish basic definitions. Then, we discuss the notion of minimal spaces, and finally, we use reductions to classify homotopy types of spaces with 7 points or fewer.

First, we give some basic definitions about finite spaces.

Definition 1.1. A space consists of a set X and a set, called a topology on X, of subsets of X, called open, where:

- The empty set and X are open.
- The intersection of finitely many open sets is open.
- The union of open sets is open.

Definition 1.2. Let f be a function mapping the points (elements) of a space X to a space Y. If, for every open set $S \in Y$, the preimage $f^{-1}(S) = \{x \in X | f(x) \in S\}$ is open in X, we say that f is continuous.

Definition 1.3. In the metric topology of $[0,1]$, a set is open if it can be realized as the union of open balls $\{x \in [0,1] : a < x < b\}$ for real numbers a, b.

For the following definitions, Let X and Y be spaces.

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Definition 1.4. In the product topology of $X \times Y$, a set is open if it can be realized as the union of cartesian products of open sets in X and Y.

Definition 1.5. Let $f : X \to Y$ and $g : X \to Y$ be continuous functions. f and g are **homotopic** if there exists a continuous function $h : X \times [0, 1] \to Y$, such that $h(x, 0) = f$, and $h(x, 1) = g$.

Definition 1.6. If there exist continuous functions $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ is homotopic with the identity on X and $g \circ f$ is homotopic with the identity on Y, then X and Y are called **homotopy equivalent**.

It should be clear that homotopy equivalence is an equivalence relation.

2. Properties of Finite Spaces

In this section, we provide definitions and establish useful properties of finite spaces.

Definition 2.1. Let X be a finite space. Let a and b be points in X. Then, say $a \geqslant b$ if for all open sets S, $a \in S$ implies $b \in S$. Further, if $a \geqslant b$, but $b \ngeqslant a$, we say $a > b$.

Definition 2.2. If X and Y are finite spaces, and $f : X \to Y$ and $g : X \to Y$ are functions such that for all $x \in X$, $f(x) \geqslant g(x)$, then we say $f \geqslant g$.

Definition 2.3. Let X be a finite space, and let x be a point in X. We define the minimum neighborhood of x, denoted U_x, as the open set containing x with minimum cardinality. Note that U_x is also the intersection of all open sets containing x.

Lemma 2.4. ([1]) *Let X and Y be finite spaces and let $f : X$ and $g : X$ be continuous functions. If f and g are homotopic, then there exists a chain of functions $f_1, f_2, f_3, f_4 ..., f_n$ homotopic to both f and g such that $f = f_1$, $g = f_n$, and for all $1 \leqslant i \leqslant n$, either $f_i \leqslant f_{i+1}$ or $f_i \geqslant f_{i+1}$.*

In this paper, we will draw finite spaces as directed graphs. Let the set of points in a space be the vertex set of its graph. Then, draw a directed edge from x to y if $x \geqslant y$ and there does not exist $z \neq x$ such that $x \geqslant z > y$.

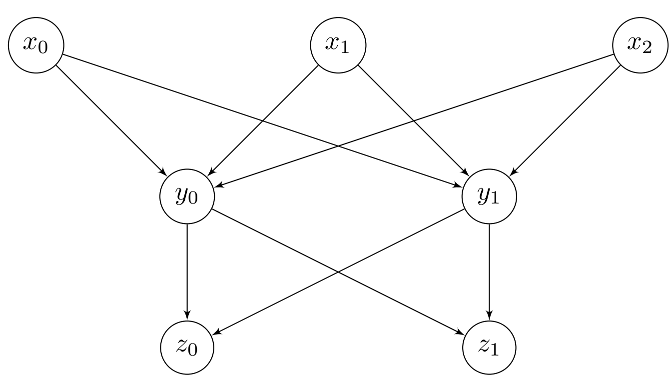

Figure 1. A graph representation of a minimal space.

Definition 2.5. Let X be a finite space. Let x be a point in X.

- If there exists $y \geqslant x$, $y \neq x$, such that for all points z, if $z > x$, $z \geqslant y$, x is upbeat with y.
- If there exists $y \leqslant x$, $y \neq x$ such that for all points z, if $z < x$, $z \leqslant y$, x is downbeat with y.
- If x is either upbeat or downbeat, it is beat.

Definition 2.6. If a space contains no beat points, we call it minimal.

Observation 2.7. *A space X is minimal if and only if in the graph representation of X, no vertex has an in-degree or out-degree of 1, and there are no 2-cycles. This is because any point with in-degree or out-degree 1 is necessarily beat with the singular point adjacent with it. For example, while the space in figure 1 is minimal, in figure 2, z_1 is upbeat with y_1.*

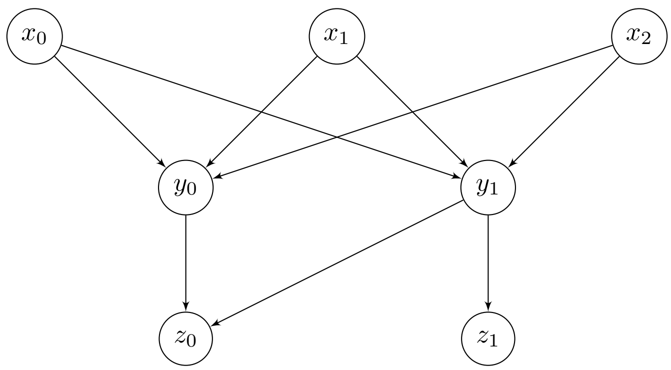

FIGURE 2. A space where z_1 is upbeat with y_1

Theorem 2.8. ([1]) *Every finite space is homotopy equivalent to a unique minimal space.*

Proof. Suppose a space X isn't minimal. Then it contains a beat point. Say it contains an upbeat point x. Then, there exists a point $y \geqslant x$, $y \neq x$, such that $z > x \implies z \geqslant y$. Define $f : X \to X \setminus \{x\}$ by $f(e) = e$ if $e \neq x$, and $f(x) = y$. Then, define $g : X \setminus \{x\} \to X$ by $g(e) = e$. We have that $f \circ g$ is equal to, and therefore homotopic to the identity. Then, define the function $h : X \times [0, 1] \to X$ by $h(x, i) = x$ if $i < 1$, and $h(x, i) = g \circ f(x)$ if i=1. This is a homotopy, since the preimage of every open set S other than U_x under h is $S \times [0, 1]$, and the preimage of U_x is $U_x \times [0, 1] \setminus (x, 1)$. Thus, X is homotopy equivalent to $X \setminus \{x\}$. A similar argument proves that if x is a downbeat point, X is homotopy equivalent to $X \setminus x$.

Now, we have that every space with beat points is homotopy equivalent to a space with one fewer beat point. Thus, after a finite number of iterations, any finite space is homotopy equivalent to a minimal space.

Now, we prove uniqueness. Let X be a minimal finite space. Suppose $f : X \to X$ is homotopic to the identity. We want to show that f is equal to the identity as well. Note that by *Lemma* 2.4, it is sufficient to show this for $f \geqslant id$ and $f \leqslant id$. First, suppose $f \geqslant id$. If x is maximal, clearly $f(x) = x$. Then, inductively suppose that for some x, for all $y > x$, $f(y) = y$. Then, if $f(x) \neq x$, then for all $y > x$,

$y \geqslant f(x)$. Thus x is beat with $f(x)$, a contradiction. By a similar argument, if $f \leqslant id$, $f = id$.

□

Theorem 2.8 motivates us to count and analyze minimal finite spaces, since classifying homotopy classes is equivalent to classifying minimal spaces. The following definitions will be useful for this.

Definition 2.9. Let X be a finite space. Let x be a point in X.

- If there exist no points y such that $x \geqslant y$, x is minimal.
- If there exist no points y such that $x \leqslant y$, x is maximal.

Definition 2.10. Let X be a connected finite space. Let Y be the set of maximal points in X. Define a subspace, called the reduction of X, and notated $r(X)$, of vertices of X not in Y. A set S is open on $r(X)$ if $S \cup Y$ is open in X.

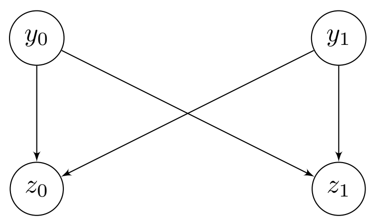

Figure 3. The reduction of the space in Figure 1.

3. Minimal Spaces with fewer than 7 points

Definition 3.1. A space X is connected if there exist no open sets S where the complement of S is open.

In this section, we describe all minimal spaces with less than 7 points. Note that any minimal finite space can be expressed as the disjoint union of finite minimal connected spaces, so in this paper, only the connected spaces are discussed.

There is clearly only one minimal connected space with one point. Note that the disjoint union of arbitrarily many copies of this space is also minimal. We call such a space discrete. There are no connected minimal spaces with two or three points, but there exists one connected minimal space with 4 points. We call this space the 4-point loop or 4-point circle. (because it is weak homotopy equivalent to a circle) There exist exactly two minimal connected spaces with 5 points. One has 3 minimal points and two maximal points, and the other has 2 minimal points and 3 maximal points.

Lemma 3.2. *If a minimal space X has less than 7 points, its reduction is minimal.*

Proof. Let X be a minimal space with less than 7 points.

(1) If X has less than 6 points, then $r(X)$ is discrete, so it clearly contains no beat points.
(2) If X has 6 points, then there is only one space such that $r(X)$ is not discrete, shown below. The reduction of this space is the 4-point circle.

□

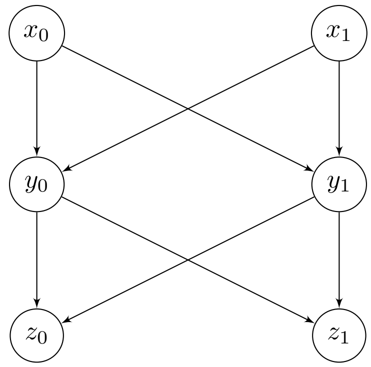

FIGURE 4. The 6-point space whose reduction is not discrete

Theorem 3.3. ([1]) *There exist 7 minimal connected 6-point spaces.*

Proof. We classify these spaces by the cardinalities of their reductions.

- If $r(X)$ has 2 points: There only exists one space X such that $r(X)$ has two points. By Lemma 3.2, $r(X)$ is discrete. All 4 other points must be maximal, and, since X has no beat points, all 4 other points are greater than both points of $r(X)$.
- If $r(X)$ has 3 points: There exist 4 minimal spaces such that $r(X)$ has 3 minimal points. $r(X)$ must be the discrete space, by Lemma 3.2. Each point in $r(X)$ must be less than at least two of the maximal points, and 0, 1, 2, or 3 can be less than all of the maximal points.
- If $r(X)$ has 4 points: There exist two minimal spaces such that $r(X)$ has two points: the space where $r(X)$ is the discrete 4 point space, and the space where $r(X)$ is the 4-point circle.

□

4. Connected Minimal Spaces with 7 points

Lemma 4.1. *There exist only 2 minimal, connected 7-point spaces such that their reductions are not minimal.*

Proof. Suppose a space X is a minimal, connected 7-point space, such that $r(X)$ has a beat point.

Case 1: Suppose X has two maximal points. Then, clearly, every point must be less than both maximal points, so, if a point is beat in $r(X)$, it must be beat in the original 7 point space. Thus, since X is minimal, $r(X)$ is minimal, a contradiction.

Case 2: Suppose X has 5 maximal points. Then, since every maximal point must be greater than both minimal points, $r(X)$ is discrete and thus minimal, a contradiction.

Case 3: Suppose X has 4 maximal points. Clearly, X cannot have 3 minimal points, since this implies that the $r(X)$ is discrete, and thus has no beat points. Then, X has two minimal points. Let x be the non-minimal, non-maximal point in X. Note that every maximal point must be greater than both minimal points. Otherwise, there must exist a beat point in X. Consider 3 cases.

(1) x is not greater than either of the minimal points. Then, x is minimal, a contradiction.
(2) x is greater than one of the two minimal points. Then, x is downbeat with that point, since the minimal point x is greater than is the only point less than x.
(3) x is greater than both of the minimal points. Then, either x is maximal, or, x is less than at least one of the maximal points, y. Then, y is downbeat with x.

Case 4: Suppose X has 3 maximal points. Call these y_1, y_2, and y_3. Call the 4 non-maximal points x_1, x_2, x_3, and x_4. Because $r(X)$ must have a beat point, at least one point is greater than another, so without loss of generality, say $x_1 > x_2$. Then, because x_1 is not downbeat with x_2 in X, another point must be less than x_1. Without loss of generality, say $x_1 > x_3$. Because x_3 is not upbeat with x_1 in X, there exists some maximal point, say y_1, such that $y_1 > x_3$, but $y_1 \not> x_1$. Further, since x_1 is neither maximal nor upbeat in X, $x_1 < y_2$, and $x_1 < y_3$. Then, since y_2 and y_3 are not downbeat with x_1, $y_2 > x_4$, and $y_3 > x_4$. Note now that if $x_4 > x_2$, and $x_4 > x_3$, then $r(X)$ is the 4-point loop, and thus minimal. Further, if x_4 is greater than just one of either x_2 and x_3, then it is also beat with that point in X. This leaves only two possibilities, which are shown in the figures below. □

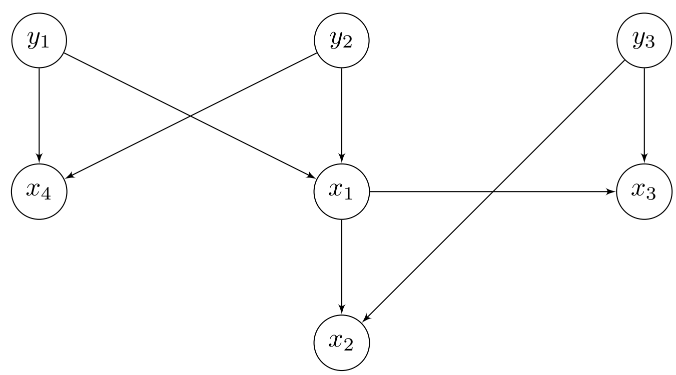

FIGURE 5. One of the two 7-point spaces whose reduction isn't minimal

Theorem 4.2. *There exist 25 minimal connected 7 point spaces.*

Proof. Again, we classify these spaces using the cardinalities of their reductions.

Claim 1: There exist 4 minimal connected 7 point spaces with two maximal points. Note that by 4.1, for every 7 point space with two maximal points, $r(X)$ is minimal. There exist four 5-point minimal spaces: The discrete space, the disjoint union of 4-point circle and a discrete point, and the two connected 5-point minimal spaces. Further, for every 5-point minimal space, there exists only one 7-point connected minimal space with the 5-point space as its reduction. This is because every maximal point of the 5-point reduction must be less than both maximal points in the 7-point space.

Claim 2: There exist 12 minimal connected 7 point spaces with 3 maximal points.

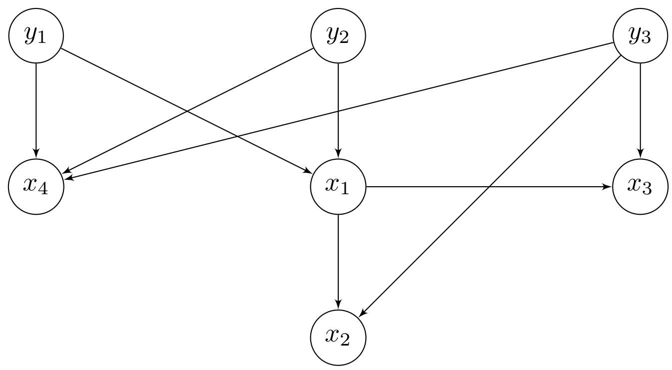

FIGURE 6. The other of the two 7-point spaces whose reduction isn't minimal

Claim 2a: There exist 4 minimal connected 7 point spaces X with 3 maximal points for which $r(X)$ is not discrete. These 4 spaces are the two spaces with non-minimal reductions described in Lemma 4.1, and the 2 spaces whose reductions are the 4-point loop. In the first of these spaces, all of 3 of the maximal points are greater than both of the maximal points of the 4-point loop. In the second space, one of the maximal points is greater than neither of the maximal points of the 4-point loop, but instead only greater than both of the minimal points.

Claim 2b: There exist 8 connected 7 point spaces X with 3 maximal points such that $r(X)$ is the discrete 4 point space. Note that each maximal point must be greater than at least two minimal points, and each minimal point must be less than at least two maximal points. There are 8 possible different combinations of numbers of points the maximal points can be greater than:

$$(4,4,4),(4,4,3),(4,4,2),(4,3,3),(4,3,2),(4,2,2),(3,3,3),(3,3,2)$$

In our graph drawing, these values are the out-degrees of the maximal points. For each possibility, it isn't very difficult to check that up to homeomorphism, there only exists 1 corresponding minimal space. For instance, for $(3,3,2)$, since each of the 4 minimal vertices must be less than at least 2 maximal points, the following space is the only corresponding minimal space.

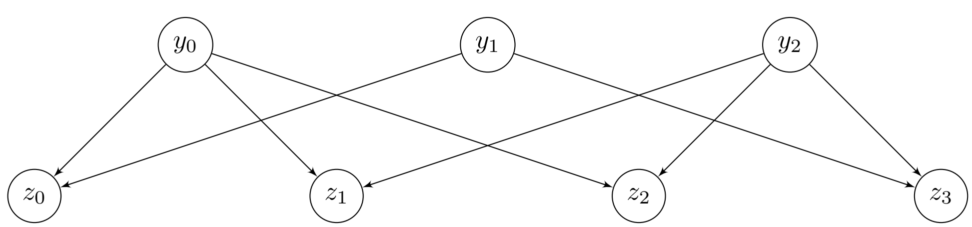

FIGURE 7. The 7-point minimal connected space whose reduction is discrete and whose maximal points are greater than 3, 3, and 2 points.

Claim 3: There exist 8 minimal finite spaces with 4 maximal points. By 4.1, the reduction of any minimal 7-point space with 4 maximal points must be minimal, so it must be discrete. Then, it must have 4 maximal points and 3 minimal points. By the same argument as Claim 2b, there are 8 such spaces.

Claim 4: There exists 1 minimal finite space with 5 maximal points. This space has two minimal points, which must be less than every maximal point.

□

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References

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