AN EXPOSITION OF MONSKY'S THEOREM

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Abstract. Since the 1970s, the problem of dividing a polygon into triangles of equal area has been a surprisingly difficult yet rich field of study. This paper gives an exposition of some of the combinatorial and number theoretic ideas used in this field. Specifically, this paper will examine how these methods are used to prove Monsky's theorem which states only an even number of triangles of equal area can divide a square.

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1. Introduction

If one tried to divide a square into triangles of equal area, one would see in Figure 1 that an even number of such triangles would work, but could an odd number work? Fred Richman [3], a professor at New Mexico State, considered using this question in an exam for graduate students in 1965. After being unable to find any reference or answer to this question, he decided to ask it in the American Mathematical Monthly.

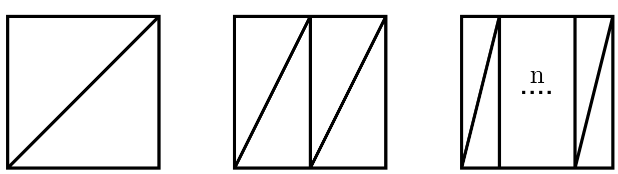

Figure 1. Dissection of squares into an even number of triangles

After 5 years, Paul Monsky [4] finally found an answer when he proved that only an even number of triangles of equal area could divide a square, which later came to be known as Monsky's theorem. Formally, the theorem states:

Theorem 1.1 (Monsky [4]). *Let S be a square in the plane. If a triangulation of S into m triangles of equal area is given, then m is even.*

The theorem appears to be simple, but its proof brilliantly applies ideas from number theory and combinatorics. Furthermore, there are theorems which generalize Monsky's theorem to other polygons and polytopes [5][6], and their proofs are similar to the proof for Monsky's theorem. These theorems fall under a field known as equidissections where one tries to see if a polygon or polytope could be divided into triangles or simplices of the same area or volume.

In this paper, we will discuss the two main components of the proof of Monsky's theorem. We will first look at the p-adic absolute value in Section 2. In Section 3, we will study Sperner's lemma in order to relate the boundary of the square to the triangles in its dissection. Finally, in Section 4, we will see how Sperner's lemma and the p-adic absolute value are used to prove Monsky's theorem.

2. The p-adic Valuation and Absolute Value

The p-adic numbers can be defined as a set of formal power series in base p. For example, 35 can be written out in base 2 as $35 = 1 \cdot 2^5 + 0 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0$. In 2-adic notation, we denote 35 as $35 = 100011_2$. Moreover, there is also a p-adic absolute value which measures "size" and has some similarities to the Euclidean absolute value. Although p-adic numbers are applied to many areas of mathematics, we shall exhibit a clever application of them to Monsky's theorem. However, before we apply the p-adic absolute value, we will first define the p-adic valuation and the p-adic absolute value.

Definition 2.1. Let p be a prime number. For each non-zero rational number x, it can be written uniquely as $x = p^n \frac{a}{b}$, where $a, b \in \mathbb{Z}$ are coprime to p and $n \in \mathbb{Z}$. Then the **p-adic valuation** is a surjective map on $\mathbb{Q}$ defined as

$$v_p : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$$
$$v_p(x) = v_p(p^n \frac{a}{b}) := n$$
$$\text{and } v(0) = \infty.$$

Definition 2.2. An **absolute value** on a field K is a map

$$|.| : K \to \mathbb{R}$$

that satisfies the following:

(1) $|x| > 0$ for all $x \in K \setminus \{0\}$
(2) $|x| = 0$ if and only if $x = 0$
(3) $|xy| = |x||y|$ for all $x, y \in K$
(4) $|x + y| \leq |x| + |y|$ for all $x, y \in K$.

If an absolute value also satisfies Axiom (4') $|x + y| \leq \max\{|x|, |y|\}$ for all $x, y \in K$, then we say that the absolute value is **non-archimedean**.

Proposition 2.3. *For all $x, y \in K$, if $|.|$ is non-archimedean and $|x| > |y|$, then $|x + y| = |x|$.*

Proof. Assume $|x| > |y|$. Then $|x + y| \leq \max\{|x|, |y|\} = |x|$. On the other hand, $|x| = |x + y - y| \leq \max\{|x + y|, |y|\}$. Assume $\max\{|x + y|, |y|\} = |y|$. Then $|x| \leq |y| < |x|$. Therefore, $|x + y| = |x|$ and the chain of inequalities shows this. $\square$

Definition 2.4. Proceeding from our definition of the p-adic valuation, we define the non-archimedean **p-adic absolute value** on $\mathbb{Q}$ to be

$$|.|_p : \mathbb{Q} \to \mathbb{R}$$
$$|x|_p := p^{-v_p(x)}.$$

Essentially, this means that if a large power of p divides a rational number x, then the p-adic absolute value of x would be very small.

Example 2.5. If we fix $p = 2$, then we will denote the 2-adic absolute value as $|x|_2$, where $x \in \mathbb{Q}$. When $x = 1$, we see $|1|_2 = 2^{-0} = 1$. However, in the case where $x = 2$, we get $|2|_2 = 2^{-1} = \frac{1}{2}$. This means 2 is considered to be "smaller" in comparison to 1, but this would not be the case for any other prime p.

Remark 2.6. We note $n \in \mathbb{Z}$ and $|n|_2 < 1$ if and only if n is even. This is a key fact that we will use in the proof of Monsky's Theorem.

With the p-adic absolute value, we are capable of proving a weaker version of Monsky's theorem. In particular, we can prove that only an even number of triangles can divide a square that has vertices in $\mathbb{Q}^2$ [1]. However, in order to prove Monsky's theorem holds for any square in the plane, we need a function which extends the domain of the p-adic absolute value to the real numbers. By a theorem from Claude Chevalley [2], there exists such a function which extends the p-adic absolute value to $\mathbb{R}$, and we will denote this function as $|x|_p'$ where $|x|_p' := p^{-v_p(x)}$.

3. Sperner's Lemma

Before we prove Monsky's theorem, we shall first define what it means to divide something into triangles.

Let R be a polygon.

Definition 3.1. A **dissection** of R is defined as a collection of triangles whose union is R and the intersection of the interiors of any two distinct triangles is empty. An **equidissection** of R is a dissection of R into triangles such that all of the triangles have the same area.

Definition 3.2. Given a polygon R in the plane, a **triangulation** of R is a set of vertices in the polygon connected to partition R into triangles.

For an example of a triangulation of a polygon, see the figure below.

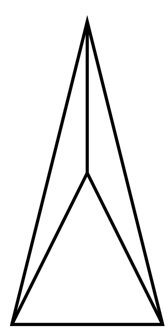

FIGURE 2. Triangulation of a polygon

For Monsky's theorem, we only consider triangulations of the square.

We will now prove Sperner's lemma which will prove the existence of a special type of triangle in the triangulation of a polygon.

Given a polygon R and a triangulation of R, label each vertex of the triangulation either P_0, P_1, or P_2. We define a "complete edge" to be a line segment whose vertices are labeled P_1 and P_0. Furthermore, we shall define a "complete triangle" to be a triangle whose vertices are labeled P_0, P_1, and P_2 up to permutation.

Lemma 3.3 (Sperner [3])**.** *Consider a triangulation of a polygon R such that each vertex is labeled either P_0, P_1, or P_2. Then the number of complete triangles in R and the number of complete edges on the boundary of R have the same parity.*

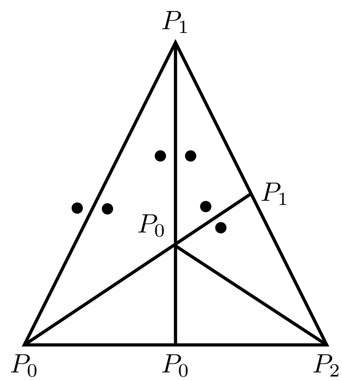

Figure 3. Triangulation of a triangle

Proof. We shall apply a double counting argument. Place a dot on each side of a complete edge. First count the number of dots in the interior of the polygon and note that each interior complete edge contributes 2 dots while other interior edges contribute 0. Furthermore, each complete edge on the boundary of R contributes 1 dot to the interior whilst other boundary segments contribute 0. Next, count the number of dots in the interior of each triangle in the dissection. By construction, complete triangles contain one dot while the rest contain an even number of dots. Thus, the parity of the number of dots in the interior of all the triangles is the same as the number of dots contributed by all of the complete edges on the boundary. Therefore, the parity of the number of complete edges on the boundary of R is the same as the number of complete triangles in R. $\square$

If we look back at Monsky's theorem, then we see that Sperner's lemma can be used to relate the boundary of the square to the triangles in its dissection. However, in order to apply this lemma, we will need a method of labelling the vertices of the triangulation. Conveniently, the p-adic absolute value can serve this purpose.

4. Proof of Monsky's Theorem

We will start by labelling the points of the plane. Since the p-adic absolute value can be extended to $\mathbb{R}$ [2], we shall fix $p = 2$ and label the points in the plane in the following way:

$$P_0 := \{(x,y) \ : \ |x|_2^{'} < 1, |y|_2^{'} < 1\}$$
$$P_1 := \{(x,y) \ : \ |x|_2^{'} \geq 1, |x|_2^{'} \geq |y|_2^{'}\}$$
$$P_2 := \{(x,y) \ : \ |y|_2^{'} \geq 1, |y|_2^{'} > |x|_2^{'}\}.$$

To see that this a partition of the plane, see Figure 4 below.

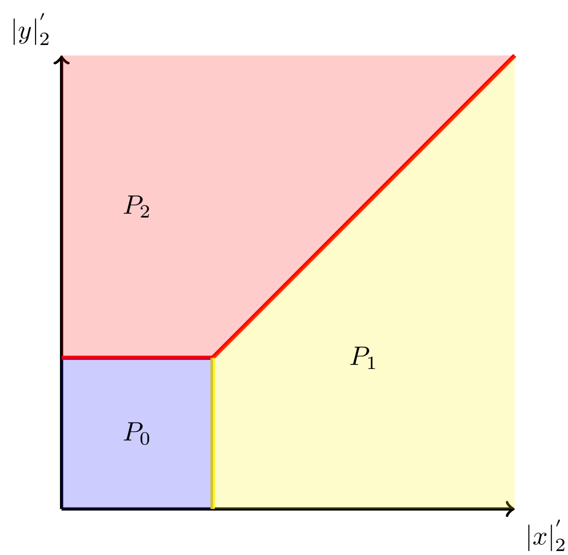

FIGURE 4. Image of $\mathbb{R}^2$ mapped by the sets above

Proposition 4.1. *Let* $(x_0, y_0) \in P_0$*,* $(x_1, y_1) \in P_1$*, and* $(x_2, y_2) \in P_2$*. Then* (x_1, y_1) *and* (x_2, y_2) *are translation-invariant with respect to* (x_0, y_0)*.*

Proof. Suppose (x_1, y_1) is translated by the point (x_0, y_0). Due to the labelling above and the non-archimedean property of the p-adic absolute value, $(x_1+x_0, y_1+y_0) \in P_1$. The case for (x_2, y_2) is proven similarly. Thus, the result follows. ☐

We will now prove a lemma which describes the area of a complete triangle.

Lemma 4.2. *Let* T *be a triangle in the plane complete with respect to the labeling above. Then its area satisfies the inequality below*

$$|area\ T|_2^{'} > 1.$$

Proof. Due to the labelling being translation-invariant with respect to points in P_0, we may translate T to the origin which is a point in P_0. Without loss of generality, assume the vertices of T are $(0,0) \in P_0$, $(x_1, y_1) \in P_1$, and $(x_2, y_2) \in P_2$. With the coordinates of the vertices, we may compute the area of the triangle with the following formula

$$\text{area } T = \frac{1}{2}\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} = \frac{x_1y_2 - x_2y_1}{2}.$$

Due to the partitioning of the plane, we see that $|x_1|_2' \geq |y_1|_2'$ and $|y_2|_2' > |x_1|_2'$, so $|x_1 y_2|_2' > |x_2 y_1|_2'$. Therefore, by Proposition 2.3, it can be simplified as $|x_1y_2 - x_2y_1|_2' = |x_1y_2|_2'$. Thus, the 2-adic absolute value of the area of T is

$$|\text{area } T|_2' = \left|\frac{1}{2}\right|_2' |x_1y_2 - x_2y_1|_2' = 2|x_1y_2|_2' \geq 2$$

since $|x_1|_2', |y_2|_2' \geq 1$. □

With Lemma 4.2, we are now able to prove Monsky's Theorem.

Theorem 4.3 (Monsky [4])**.** *Let S be a square in the plane. If a triangulation of S into m triangles of equal area is given, then m is even.*

Proof. Suppose a square S in the plane is triangulated into m triangles of equal area. We will now label the vertices of the triangulation according to the sets P_0, P_1, and P_2 which are defined above Figure 4. Without loss of generality, by Proposition 4.1, we may translate and dilate S such that it is the unit square translated to the origin. In Figure 5, we see one example of an equidissection of the square in the plane and how the four vertices of the square are labeled according to the sets P_0, P_1, and P_2.

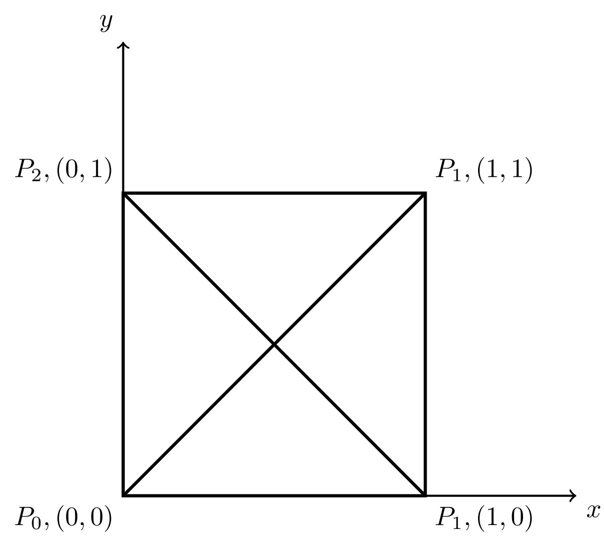

FIGURE 5. Unit square at the origin

We claim that complete edges occur only on the boundary segment with vertices (0,0) and (1,0). For the boundary segment with vertices (0,0) and (0,1), it does not contain points in P_1 since $|x|_2' = 0$ always. For the boundary segment with vertices (0,1) and (1,1), it does not contain points in P_0 since $|y|_2' = 1$ always. For the boundary segment with vertices (1,1) and (1,0), it does not contain vertices in P_0 since $|x|_2' = 1$ always. Finally, the boundary segment with vertices (0,0) and (1,0) does not contain points in P_2 since $|y|_2' = 0$ always. Thus, complete edges only occur on the boundary segment with vertices (0,0) and (1,0).

In addition, we also note that there are an odd number of complete edges on the boundary of S. If we look at the boundary segment where complete edges occur, then we see that it begins with a point labeled P_0 and ends with a point labeled P_1. Then there will always be an odd number of changes in labels from $(0,0)$ to $(1,0)$ no matter how many points of P_0 or P_1 are placed on this boundary segment. Therefore, by Lemma 3.3, there exists at least one complete triangle T in the triangulation of the square.

By Lemma 4.2, we know that $|\text{area } T|_2' > 1$. However, S is triangulated into m triangles of equal area, so $m \cdot \text{area } T = 1$. Since the p-adic absolute value is multiplicative, this implies that $|m|_2' < 1$. Thus, since m is an integer, m is even. □

Remark 4.4. One interesting fact about Monsky's theorem is that the proof above is currently the only known proof of this theorem [3].

Although Monsky's theorem is easy to understand, the components of its proof draws from different areas of mathematics in an elegant way. Furthermore, this theorem's proof has inspired similar theorems to be made which also mainly rely on the p-adic absolute value and Sperner's lemma. For example, the following two theorems generalize Monsky's theorem:

Theorem 4.5 (Mead [5])**.** *Let C be the unit hypercube in $\mathbb{R}^n$. C can be equidissected into m simplices if and only if m is a multiple of $n!$.*

Theorem 4.6 (Kasimatis [6])**.** *Let $n \geq 5$ be an integer. A regular n-gon can be decomposed into m triangles of equal area if and only if m is a multiple of n.*

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