THE HOPF INVARIANT ONE PROBLEM

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Abstract. In this paper we present the Hopf invariant one problem and its solution in terms of complex topological K-theory. Our paper introduces K-theory starting from the definition of a vector bundle and also discusses its properties as a cohomology theory.

Contents

1. Introduction

The Hopf invariant one problem is a classical question in algebraic topology. The *Hopf invariant* is an integer associated with the homotopy class of a function $S^{2n-1} \to S^n$. The Hopf invariant one problem asks for which n there exist maps of Hopf invariant ± 1. The problem was first solved in 1958 by Adams [1].

Theorem 1.1 (Adams)**.** *There exists a map $S^{2n-1} \to S^n$ of Hopf invariant ± 1 only for $n = 2, 4,$ and 8.*

The Hopf invariant one problem has several consequences of elementary interest.

Theorem 1.2. *The following statements hold only for $n = 1, 2, 4,$ and 8:*

(1) The vector space $\mathbb{R}^n$ can be endowed with the structure of a real division algebra.
(2) The sphere S^{n-1} is parallelizable.
(3) The sphere S^{n-1} can be endowed with an H-space structure.

Date: August 2019.

Adams's original proof of Theorem 1.1 employed secondary cohomology operations in ordinary cohomology. In 1964, Adams and Atiyah provided a simpler proof using complex topological K-theory [2], a cohomology theory defined in terms of vector bundles. In this paper we follow the second approach.

In Section 2, we construct K-theory and define the Hopf invariant. In Section 3 we prove a technical result called the splitting principle needed for the proof of Adams's theorem, which we present in Section 4. In Section 5, we prove Theorem 1.1 and examine the cases of $n = 1, 2, 4,$ and 8.

1.1. **Preliminaries.** All maps are assumed to be continuous. The symbol $\cong$ denotes homeomorphism or isomorphism and $\simeq$ denotes homotopy equivalence. Brackets $[X, Y]$ denote homotopy class of maps $X \to Y$ and $[X, Y]_*$ denotes homotopy classes of based maps. If X is a space, then X^+ is X with a disjoint basepoint. A compact pair (X, A) is a compact space X and a compact subspace A.

2. Construction of K-Theory

2.1. **Cohomology Theories.** The Eilenberg-Steenrod axioms define cohomology theories as certain contravariant functors. As we will later show, topological K-theory is a cohomology theory on compact spaces.

Definitions 2.1. A *cohomology theory* h^* is a sequence of contravariant functors h^n from the homotopy category of compact pairs to the category of abelian groups along with natural transformations $\delta^n : h^n(A, \emptyset) \to h^{n+1}(X, A)$ such that

(1) If $f : (X, A) \to (Y, B)$ is a weak homotopy equivalence, then
$$f^* : h^n(Y, B) \to h^n(X, A)$$
is an isomorphism.

(2) For a pair (X, A) and inclusions $i : (A, \emptyset) \to (X, \emptyset)$ and $j : (X, \emptyset) \to (X, A)$, the induced sequence is exact:
$$\cdots \xrightarrow{\delta^{n-1}} h^n(X, A) \xrightarrow{j^*} h^n(X, \emptyset) \xrightarrow{i^*} h^n(A, \emptyset) \xrightarrow{\delta^n} h^{n+1}(X, A) \xrightarrow{j^*} \cdots$$

(3) If (X, A) is a pair and the closure of U lie in the interior of A, then the inclusion $(X \setminus U, A \setminus U) \to (X, A)$ induces an isomorphism for every h^n.

(4) If $(X, A) = \coprod_\alpha (X_\alpha, A_\alpha)$ and $i_\beta : (X_\beta, A_\beta) \to (X, A)$ are inclusion maps, then the product of induced maps is an isomorphism for every n:
$$h^n(X, A) \xrightarrow{\prod_\alpha i_\alpha^*} \prod_\alpha h^n(X_\alpha, A_\alpha)$$

A *reduced cohomology theory* $\widetilde{h}^*$ is a sequence of contravariant functors $\widetilde{h}^n$ from the homotopy category of based compact spaces to the category of abelian groups such that

(1) If $f : X \to Y$ is a weak homotopy equivalence, then
$$f^* : \widetilde{h}^n(Y) \to \widetilde{h}^n(X)$$
is an isomorphism.

(2) For every n, there is a natural isomorphism
$$\widetilde{h}^n(X) \longrightarrow \widetilde{h}^{n+1}(\Sigma X).$$

(3) For an inclusion $i : A \to X$ that is a cofibration and a quotient $j : X \to X/A$, the induced sequence is exact:

$$\widetilde{h}^n(X/A) \xrightarrow{j^*} \widetilde{h}^n(X) \xrightarrow{i^*} \widetilde{h}^n(A)$$

(4) If $X = \vee_\alpha X_\alpha$ and $i_\beta : X_\beta \to X$ are inclusions, then the product of induced maps is an isomorphism for every n:

$$\widetilde{h}^n(X) \xrightarrow{\prod_\alpha i^*_\alpha} \prod_\alpha \widetilde{h}^n(X_\alpha).$$

It can be shown that reduced and unreduced cohomology theories are in unique correspondence via $h^*(X) = \widetilde{h}^*(X_+)$. An important property of reduced cohomology theories is that they are representable functors.

Definition 2.2. An *Ω-prespectrum* is a sequence of based CW-complexes $\{K_n\}$ together with homotopy equivalences $K_n \to \Omega K_{n+1}$.

Theorem 2.3. [6, §4.3] *If $\{K_n\}$ is an Ω-prespectrum, then the functors*

$$\widetilde{h}^n = [-, K_n]_*$$

define a reduced cohomology theory $\widetilde{h}^*$. Conversely, if $\tilde{h}^*$ is a reduced cohomology theory, then it is represented by some Ω-prespectrum.

The latter result is known as the *Brown representability theorem* [6, §4.E].

2.2. **Vector Bundles.** Topological K-theory is constructed from the set of vector bundles over a space.

Definitions 2.4. A *vector bundle* over a space X consists of a topological space E called the *total space* and a *projection map* $p : E \to X$ onto a space X called the *base space* which satisfies the following properties:

(1) Each fiber $E_x = p^{-1}(x)$ for $x \in X$ has the structure of a finite-dimensional vector space over $\Bbbk$.
(2) *(Local Triviality)* There is an open cover $\{U_\alpha\}$ of X such that for every U_α there exists a homeomorphism $\phi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times \Bbbk^n$ which restricts on each fiber $p^{-1}(x)$ to an isomorphism of vector spaces:

$$\phi_\alpha : E_x \xrightarrow{\cong} \{x\} \times \Bbbk^n.$$

If $\Bbbk = \mathbb{R}$, then E is a *real vector bundle* and if $\Bbbk = \mathbb{C}$, then E is a *complex vector bundle*. The number n is locally constant and is the *dimension* of E on each connected component. A vector bundle of dimension one is called a *line bundle*. A *sub-bundle* G of E is a vector bundle over X whose fiber G_x is always a subspace of E_x. A *section* is a map $s : X \to E$ such that $p \circ e = \mathrm{id}_X$.

A *homomorphism* of vector bundles is a function $\phi : E \to F$ between bundles over the same base space X such that

$$\begin{tikzcd} E \arrow[rr, "\phi"] \arrow[dr, "p_E"'] & & F \arrow[dl, "p_F"] \\ & X & \end{tikzcd}$$

commutes and whose restriction onto each fiber of E is a linear map $E_x \to F_x$. An *isomorphism* $\phi : E \to F$ is a homomorphism whose inverse is also a homomorphism.

Examples 2.5. The following are examples of vector bundles:

(1) The *trivial bundle* of dimension n over X is the product $X \times \Bbbk^n$ with the vector space structure of $\Bbbk^n$ on each fiber $\{x\} \times \Bbbk^n$. When the base space and field is understood, it is denoted merely by n.
(2) The *tangent bundle* TM of an n-manifold M is a real vector bundle of dimension n.
(3) The *tautological bundle* γ_n^k over the Grassmanian $G(n,k)$ of k-dimensional subspaces of $\mathbb{C}^n$ is the sub-bundle of $G(n,k) \times \mathbb{C}^n$ given by
$$\gamma_n^k = \{(V,v) \in G(n,k) \times \mathbb{C}^n \mid v \in V\}.$$

In the rest of this paper we will assume that all bundles are complex over compact Hausdorff base spaces. Let $\mathrm{Vect}^n(X)$ refer to the set of isomorphism classes of complex vector bundles of dimension n over X and $\mathrm{Vect}(X) = \cup_{n\geq 0} \mathrm{Vect}^n(X)$.

Constructions on vector spaces extend to vector bundles by being carried out separately on each fiber. For many such constructions, the result can be topologized so that the resulting space is a vector bundle. The details are given in [3, §1.2].

Definitions 2.6. Let E and F be n- and m-dimensional bundles over X.

(1) The *dual bundle* E^* of E is an n-dimensional bundle whose fibers are the dual spaces of the fibers of E: $(E^*)_x = E_x^*$.
(2) The *direct sum* of E and F is an $(n+m)$-dimensional bundle $E \oplus F$ over X whose fibers are $(E \oplus F)_x = E_x \oplus F_x$.
(3) The *tensor product* of E and F is an nm-dimensional bundle $E \otimes F$ whose fibers are $(E \otimes F)_x = E_x \otimes F_x$.
(4) The *k-th exterior power* of E is a bundle $\lambda^k(E)$ is an $\binom{n}{k}$-dimensional bundle whose fibers are $\lambda^k(E)_x = \lambda^k(E_x)$. The exterior powers satisfy the formula $\lambda^k(E \oplus F) \cong \oplus_{i=0}^k (\lambda^i(E) \otimes \lambda^{k-i}(F))$.
(5) A *Hermitian form* on E is the continuous assignment of a nondegenerate Hermitian form to each fiber of E. If G is a k-dimensional sub-bundle of E, then a Hermitian form defines an $(n-k)$-dimensional *orthogonal complement* bundle $G^\perp$ whose fibers are the orthogonal complements of the fibers of G: $(G^\perp)_x = G_x^\perp$. All complex vector bundles over compact spaces can be endowed with a nondegenerate Hermitian form [3, Lem 1.4.10].
(6) If $f : Y \to X$ is a map, then the *pullback bundle* f^*E is an n-dimensional bundle over Y given by
$$f^*E = \{(y,v) \in Y \times E \mid v \in E_{f(y)}\}.$$
This defines a map $f^* : \mathrm{Vect}(X) \to \mathrm{Vect}(Y)$. It is a fact that homotopic maps $Y \to X$ induce isomorphic pullback bundles [3, Lem. 1.4.3].
(7) If E is a bundle over X and F is a bundle over Y, then the *external product* is the bundle $p_X^* E \otimes p_Y^* F$ over $X \times Y$.
(8) Let $f : S^{m-1} \to \mathrm{GL}_n(\mathbb{C})$ be a map. We can write $S^m = D_-^m \cup D_+^m$ for two disks $D_\pm^m$ such that $D_-^m \cap D_+^m = S^{m-1}$. The *clutching construction* builds an n-dimensional vector bundle E over S^m given by
$$E = ((D_-^m \times \mathbb{C}^n) \sqcup (D_+^m \times \mathbb{C}^n))/\sim$$
where $(x_-, v) \sim (x_+, f(x)v)$ for $v \in \mathbb{C}^n$ and points $x_- \in D_-^m$ and $x_+ \in D_+^m$ which correspond to the same point x in $D_-^m \cap D_+^m = S^{m-1}$.

It can be shown that E depends only on the homotopy class of f. In fact, the clutching construction defines a bijection between $\mathrm{Vect}^n(S^m)$ and homotopy classes $[S^{m-1}, \mathrm{GL}_n(\mathbb{C})]$. For a proof, see [3, Lem 1.4.9].

Example 2.7. By the clutching construction, n-dimensional bundles over S^1 correspond to homotopy classes $[S^0, \mathrm{GL}_n(\mathbb{C})]$. However, $\mathrm{GL}_n(\mathbb{C})$ is connected, so there is only one such homotopy class for each n. Hence all bundles over S^1 are trivial.

The dual of the tautological line bundle $H = \gamma_2^1$ over $S^2 \cong \mathbb{CP}^1 = G(2,1)$ is called the *canonical line bundle* H^*. Its clutching map $S^1 \to \mathrm{GL}_1(\mathbb{C})$ is given by $z \mapsto z$. One can show explicitly that the clutching matrices

$$\begin{pmatrix} z^2 & 0 \\ 0 & 1 \end{pmatrix} \qquad\qquad \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$$

corresponding to the bundles $(H^* \otimes H^*) \oplus 1$ and $H^* \oplus H^*$ are homotopic. This implies that the bundles are isomorphic.

Vector bundles over arbitrary spaces can also be classified in terms of homotopy theory. This requires an important lemma.

Lemma 2.8. [3, Cor. 1.4.14] *Every vector bundle over a compact space is a sub-bundle of a trivial bundle. As a result, for every bundle E, there is a bundle G such that $E \oplus G$ is trivial.*

Definition 2.9. The *classifying space* $BU(n)$ is a topological space constructed as the colimit of the inclusions

$$\begin{aligned} G(n, n+k) &\hookrightarrow G(n, n+k+1) \\ V &\mapsto V \oplus \mathbb{C} \end{aligned}$$

for all $k = 0, 1, 2, \ldots$ The *tautological bundle* γ_n over $BU(n)$ is given by

$$\gamma_n = \{(U, v) \mid v \in U \in G(n, n+k) \text{ for some } k \geq 0\}.$$

Proposition 2.10. *The mapping*

$$\begin{aligned} [X, BU(n)] &\longrightarrow \mathrm{Vect}^n(X) \\ f &\longmapsto f^*\gamma_n \end{aligned}$$

is a bijection.

Proof. Since homotopic maps induce isomorphic pullback bundles, the function in question is well-defined. To show surjectivity, consider a bundle E of dimension n. It is a sub-bundle of a trivial bundle N, so each fiber of E is a subspace of $\mathbb{C}^N$. This defines a map $f : X \to G(n, N) \to BU(n)$. By the definition of the tautological bundle, $f^*\gamma_n \cong E$. Injectivity follows if we can show that the homotopy class of f is independent of N. Suppose that E can be realized as a sub-bundle of two trivial bundles of dimension N and N'. Then E is a sub-bundle of a trivial bundle of dimension $N + N'$. Via a permutation of coordinates that is homotopic to the identity, we can interlace the two copies of E in the coordinates of the trivial bundle. A linear homotopy then shows that the two maps $f, f' : X \to G(n, N + N')$ defined by these realization are homotopic. $\square$

2.3. ***K*-Theory.** The set of isomorphism classes of vector bundles over a space X forms a semiring $(\mathrm{Vect}(X), \oplus, \otimes)$ under direct sum and tensor product.

Definition 2.11. Let X be a compact Hausdorff space. The *K-theory ring* $K^0(X)$ is defined to be the group completion of the semiring $(\mathrm{Vect}(X), \oplus, \otimes)$.

Elements of $K^0(X)$ are formal differences $E - F$ of vector bundles over X. By construction, equality $E = F$ in $K^0(X)$ means that there is a bundle G such that $E \oplus G \cong F \oplus G$. The trivial line bundle is the multiplicative identity. By Lemma 2.8, a bundle F has a bundle G such that $F \oplus G \cong n$, so every element $E - F \in k(X)$ can be written as $E' - n = (E \oplus G) - n$.

The K-theory ring defines a contravariant functor from the category of compact Hausdorff spaces to the category of rings. A map $f : X \to Y$ of spaces gives a semiring homomorphism

$$f^* : (\mathrm{Vect}(Y), \oplus, \otimes) \longrightarrow (\mathrm{Vect}(X), \oplus, \otimes).$$

By the functoriality of group completion this induces a ring homomorphism

$$f^* : K^0(Y) \longrightarrow K^0(X).$$

Since homotopic maps give isomorphic vector bundles, K^0 is a homotopy invariant.

Examples 2.12. Since all bundles over a point are trivial, $K^0(*) \cong \mathbb{Z}$ and $K^0(S^0) \cong \mathbb{Z} \oplus \mathbb{Z}$. From Example 2.7, we know that all bundles over S^1 are trivial, so $K^0(S^1) \cong \mathbb{Z}$. We also know that $(1 - H^*)^2 = 0$ in $K^0(S^2)$.

Definition 2.13. Let X be a based space with basepoint x_0. The *reduced K-theory ring* $\widetilde{K}^0(X)$ is the kernel of the homomorphism $i^* : K^0(X) \to K^0(x_0)$ induced by the inclusion $i : \{x_0\} \to X$.

Note that the retraction $X \to \{x_0\}$ induces a splitting of the exact sequence

$$0 \longrightarrow \widetilde{K}^0(X) \longrightarrow K^0(X) \longrightarrow K^0(x_0) \longrightarrow 0$$

from which it follows that $K^0(X) \cong \widetilde{K}^0(X) \oplus \mathbb{Z}$.

Since $\widetilde{K}^0(X)$ is an ideal of $K^0(X)$, it does not possess an identity. Like K^0, reduced K-theory is a homotopy invariant. The elements of $\widetilde{K}^0(X)$ are formal differences of vector bundles of the same dimension; they can all be written as $E - n$ for a bundle E of dimension n.

To extend the functor K^0 to a cohomology theory, we will exhibit the Ω-prespectrum for K-theory.

Definitions 2.14. The classifying space BU is the topological colimit of the inclusions $BU(n) \hookrightarrow BU(n+1)$ induced by the maps

$$\begin{aligned} G(n, n+k) &\hookrightarrow G(n+1, n+k+1) \\ V &\mapsto V \oplus \mathbb{C}. \end{aligned}$$

The *K-theory spectrum* KU_n is

$$KU_n = \begin{cases} BU \times \mathbb{Z} & n \text{ is even,} \\ \Omega(BU \times \mathbb{Z}) & n \text{ is odd.} \end{cases}$$

For KU to be an Ω-prespectrum, there must exist homotopy equivalences $KU_n \simeq \Omega KU_{n+1}$ for all n. For n odd, these exist by construction. For n even, the homotopy equivalence

$$BU \times \mathbb{Z} \xrightarrow{\simeq} \Omega^2(BU \times \mathbb{Z})$$

is the famous result of *Bott periodicity*. We will assume it without proof.

The space $BU \times \mathbb{Z}$ possesses addition and multiplication maps induced by the effects of the direct sum and tensor product on the Grassmanians from which BU is constructed. In fact, $BU \times \mathbb{Z}$ will have the structure of an E_∞-*ring* space and KU will be an E_∞-*ring spectrum*. Further information can be found in [10].

Proposition 2.15. *For every space X, there is a natural ring isomorphism*

$$K^0(X) \cong [X_+, BU \times \mathbb{Z}]_*.$$

Proof. By Proposition 2.10, there is a bijection $\mathrm{Vect}^n(X) \to [X, BU(n)]$ sending a bundle to a classifying map f. We define the map $K^0(X) \to [X_+, BU \times \mathbb{Z}]_*$ by $E - n \mapsto (f, m-n)$, for E of dimension m. This will be a natural ring isomorphism for the ring structure induced on $[X_+, BU \times \mathbb{Z}]_*$ by the structure on $BU \times \mathbb{Z}$. □

We can now extend K-theory to other indices.

Definitions 2.16. For $n \in \mathbb{Z}$ and (X, A) a compact pair,

$$\widetilde{K}^n(X) = \begin{cases} \widetilde{K}^0(X) & n \text{ is even,} \\ \widetilde{K}^0(\Sigma X) & n \text{ is odd,} \end{cases}$$

$$K^n(X, A) = \widetilde{K}^n(X/A), \qquad K^n(X) = \widetilde{K}^n(X_+).$$

We also let $\widetilde{K}^*(X) = \bigoplus_{i\in\mathbb{Z}} \widetilde{K}^i(X)$, $K^*(X) = \bigoplus_{i\in\mathbb{Z}} K^i(X)$, and $K^*(X, A) = \bigoplus_{i\in\mathbb{Z}} K^i(X, A)$.

By Proposition 2.15 and Theorem 2.3, $\widetilde{K}^*$ is a 2-periodic reduced cohomology theory represented by the Ω-prespectrum KU corresponding to a cohomology theory K^*. All the Eilenberg-Steenrod axioms hold for K-theory. In particular, $\widetilde{K}^n(X) \cong \widetilde{K}^{n+1}(\Sigma X)$. Additionally, for a compact pair (X, A), there is an induced long exact sequence

$$\cdots \xrightarrow{\delta^{n-1}} K^n(X, A) \xrightarrow{j^*} \widetilde{K}^n(X) \xrightarrow{i^*} \widetilde{K}^n(A) \xrightarrow{\delta^n} K^{n+1}(X, A) \xrightarrow{j^*} \cdots$$

which also holds for K^*.

Lemma 2.17. *If X and Y are based spaces, then*

$$\widetilde{K}^n(X \vee Y) \cong \widetilde{K}^n(X) \oplus \widetilde{K}^n(Y),$$
$$\widetilde{K}^n(X \times Y) \cong \widetilde{K}^n(X \wedge Y) \oplus \widetilde{K}^n(X) \oplus \widetilde{K}^n(Y).$$

Proof. Since X is a retract of $X \vee Y$, it induces a splitting of the exact sequence for the pair $(X \vee Y, X)$. This gives the first result. The second result comes from seeing that the exact sequence of $(X \times Y, X \vee Y)$ splits. □

The external product of vector bundles induces a linear map $K^0(X) \otimes K^0(Y) \to K^0(X \times Y)$ which restricts to a map $\widetilde{K}^0(X) \otimes \widetilde{K}^0(Y) \to \widetilde{K}^0(X \times Y)$. By the previous lemma, we can then project to obtain a product $\widetilde{K}^0(X) \otimes \widetilde{K}^0(Y) \to \widetilde{K}^0(X \wedge Y)$. The suspension isomorphisms then give products

$$\widetilde{K}^i(X) \otimes \widetilde{K}^j(Y) \to \widetilde{K}^{i+j}(X \wedge Y), \qquad K^i(X) \otimes K^{i+j}(Y) \to K^j(X \times Y).$$

If (X, A) and (Y, B) are compact pairs, the homeomorphism $X/A \wedge Y/B \cong X \times Y/(X \times B \cup A \times Y)$ allows us to define a relative product

$$K^i(X, A) \otimes K^j(Y, B) \to K^{i+j}(X \times Y, X \times B \cup A \times Y).$$

If we compose with the induced maps in K-theory of the diagonal maps $X \to X \times X$, $X \to X \wedge X$, and $X/(A \cup B) \to X/A \wedge X/B$, we define products

$$\widetilde{K}^i(X) \otimes \widetilde{K}^j(X) \to \widetilde{K}^{i+j}(X), \qquad K^i(X) \otimes K^j(X) \to K^{i+j}(X)$$

$$K^i(X, A) \otimes K^j(X, B) \to K^{i+j}(X, A \cup B).$$

With these products, $\widetilde{K}^*(X)$ and $K^*(X)$ are $\mathbb{Z}$-graded rings.

The Bott periodicity isomorphism in K-theory $\widetilde{K}^0(X) \cong \widetilde{K}^{-2}(X) \cong \widetilde{K}^0(S^2 \wedge X)$ can be realized as multiplication on K-theory. By Bott periodicity, $\widetilde{K}^0(S^2) \cong \mathbb{Z}$ as a group. The element $b = 1 - H^* \in \widetilde{K}^0(S^2)$ is called the *Bott element*.

Proposition 2.18. *The group homomorphism*

$$\widetilde{K}^0(X) \longrightarrow \widetilde{K}^0(X) \otimes \widetilde{K}^0(S^2) \longrightarrow \widetilde{K}^0(S^2 \wedge X) \cong \widetilde{K}^{-2}(X)$$

given by multiplication by $b \in \widetilde{K}^0(S^2)$ is the isomorphism of Bott periodicity.

The proof of this result is carried out in [5]. We can now deduce the ring structure on the K-theory of spheres.

Proposition 2.19. *As groups, $\widetilde{K}^0(S^{2n-1})$ is trivial and $\widetilde{K}^0(S^{2n})$ is freely generated by the n-fold external product of the Bott element with itself. As nonunital rings, $\widetilde{K}^0(S^0) \cong \mathbb{Z}$ and for $n > 0$, the multiplication on $\widetilde{K}^0(S^n)$ is trivial.*

Proof. The group structure of $\widetilde{K}^0(S^n)$ follows from Example 2.12 and the iterated product in Proposition 2.18 beginning with $X = S^2$. If we write S^n as the union of two contractible hemispheres $D^n_+ \cup D^n_-$ with a common basepoint, then the product on $\widetilde{K}^*(S^n) \cong \widetilde{K}^*(S^n, D^n_\pm)$ factors as

$$\widetilde{K}^*(S^n) \otimes \widetilde{K}^*(S^n) \to \widetilde{K}^*(S^n, D^n_-) \otimes \widetilde{K}^*(S^n, D^n_+) \to \widetilde{K}^*(S^n, D^n_- \cup D^n_+) \to \widetilde{K}^*(S^n)$$

However, since the $D^n_\pm$ cover S^n, the penultimate ring is zero and hence the multiplication on $\widetilde{K}^*(S^n)$ is trivial for $n \neq 0$. □

2.4. **The Hopf invariant.** The Hopf invariant of a map $S^{2n-1} \to S^n$ can be defined in several equivalent ways as either a linking number of smooth submanifolds, an invariant coming from integral cohomology, or an invariant in K-thory. Since it can be shown that all these definitions are equivalent, it will suffice for our purposes to define it in terms of K-theory.

Definition 2.20. Consider the cofiber Cf of a map $f : S^{2n-1} \to S^n$. The exact sequence of the pair (Cf, S^n) is

$$\widetilde{K}^{-1}(S^n) \longrightarrow \widetilde{K}^0(S^{2n}) \xrightarrow{j^*} \widetilde{K}^0(Cf) \xrightarrow{i^*} \widetilde{K}^0(S^n) \longrightarrow 0$$

since $\widetilde{K}^{-1}(S^{2n}) = \widetilde{K}^{-1}(Cf, S^n) \cong 0$. Let $\alpha \in \widetilde{K}^0(Cf)$ be the image under j^* of the generator of $\widetilde{K}^0(S^{2n}) \cong \mathbb{Z}$. Let $\beta \in \widetilde{K}^0(Cf)$ map to the generator of $\widetilde{K}^0(S^n)$. Since the square of any element in $\widetilde{K}^0(S^n)$ is zero, $i^*(\beta^2) = 0$, so by exactness, $\beta^2 = h\alpha$ for some $h \in \mathbb{Z}$. The integer h is the *Hopf invariant* of f.

Let us see why the Hopf invariant is well-defined. The element β is defined up to a multiple of α. All squares vanish in $\widetilde{K}^0(S^{2n})$, so $\alpha^2 = 0$, and hence $(\beta + k\alpha)^2 = \beta^2 + 2k\alpha\beta$. If n is odd, then $\widetilde{K}^0(S^n) \cong 0$, so $i^*(\beta) = 0$ and hence $\beta = m\alpha$ for $m \in \mathbb{Z}$, which implies that $\alpha\beta = m\alpha^2 = 0$. Otherwise, let n be even. The image of $\alpha\beta$ under i^* vanishes since $i^*(\alpha) = 0$. Thus $\alpha\beta = \ell\alpha$ for some $\ell \in \mathbb{Z}$ and $\ell^2\alpha = \ell\alpha\beta = \alpha\beta^2 = h\alpha^2 = 0$. Yet because n is even, $\widetilde{K}^{-1}(S^n) \cong 0$. It follows that j^* is injective and therefore $\alpha\beta = 0$.

Remark 2.21. As noted in the preceding paragraph, for n odd, β is a multiple of α. Consequently, $\beta^2 = 0$ and the Hopf invariant vanishes.

3. Splitting Principle

Given a bundle E over X, the splitting principle provides a space $F(E)$ and a map $F(E) \to X$ under which E pulls back to the direct sum of line bundles. This result will be used in the following section to construct operations in K-theory.

Definitions 3.1. A *fiber bundle* with fiber F consists of a *total space* E, a *base space* X, and a map $p : E \to X$ such that there exists a cover $\{U_\alpha\}$ of X along with homeomorphisms $\phi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times F$ sending each $p^{-1}(x)$ to $\{x\} \times F$.

If E is an n-dimensional vector bundle over X, then the *projective bundle* $P(E)$ is the quotient space of E by the scalar action of $\mathbb{C}$ on each of its fibers. The *tautological bundle* over $P(E)$ is the n-dimensional vector bundle $L = \{(x, v) \mid \mathbb{C}v = x\}$. The *flag bundle* of E is the subspace $F(E) \subset P(E)^n$ of n-tuples of orthogonal lines in the fibers of E.

The projective bundle of E is a fiber bundle over X with fiber $\mathbb{CP}^{n-1}$. The flag bundle of E is also a fiber bundle. Its fibers are spaces called *flag manifolds*.

Let us note that if E is a fiber bundle over X with projection $p : E \to X$, then $K^*(E)$ carries the structure of a $K^*(X)$-module given by the product

$$\begin{aligned} K^*(X) \otimes K^*(E) &\longrightarrow K^*(E) \\ \delta \otimes \gamma &\longmapsto (p^*\delta)\gamma. \end{aligned}$$

Proposition 3.2 (Splitting Principle)**.** *If E is a bundle over X and $p : F(E) \to X$ is the projection map, then p^*E splits as the sum of line bundles and $p^* : K^*(X) \to K^*(P(E))$ is injective.*

The splitting of p^*E is straightforward. The nontrivial content of the splitting principle is the injectivity of p^*. To deduce this, we will need to calculate $K^*(P(E))$.

Lemma 3.3. *If X is a finite CW-complex with n cells, then $K^*(X)$ is a group generated by at most n generators. If all the cells of X have even dimension, then $K^{-1}(X) \cong 0$ and $K^0(X)$ is a free abelian group with one generator for each cell.*

Proof. These results follow by induction on the number of cells: if X is obtained from A by gluing a k-cell, then the exact sequences in K^*, K^{-1}, and $\widetilde{K}^0$ for the pair (X, A) will imply that the result holds for X if it holds for A. □

Proposition 3.4. *As rings, $K^0(\mathbb{CP}^n) \cong \mathbb{Z}[L]/(L-1)^{n+1}$ where $L = \gamma^n_{n+1}$ is the tautological bundle over $\mathbb{CP}^n$.*

Proof. By considering the inclusions $\mathbb{CP}^n \hookrightarrow \mathbb{CP}^{n+1}$, we can find that the CW-structure for $\mathbb{CP}^n$ has one cell of dimension $0, 2, \cdots, 2n$ each.

We can also describe $\mathbb{CP}^n$ as the quotient of $\partial D^{2n+1} \subset \mathbb{C}^{n+1}$ under the scalar action of the unit circle. If we write

$$\partial D^{2n+1} = \textstyle\bigcup_i D^2_0 \times \cdots \times \partial D^2_i \times \cdots \times D^2_n,$$

then action of the unit circle respects this decomposition. The orbits of each term in the decomposition are spaces

$$C_i = D^2_0 \times \cdots \times D^2_{i-1} \times D^2_{i+1} \times \cdots \times D^2_n,$$

each homeomorphic to D^{2n}, which together compose $\mathbb{CP}^n = \bigcup_i C_i$. The boundary of the first component C_0 decomposes as $\partial C_0 = \bigcup_i \partial_i C_0$ for

$$\partial_i C_0 = D^2_1 \times \cdots \times \partial D^2_i \times \cdots \times D^2_n.$$

For these disks, there are inclusions $(D^2_i, \partial D^2_i) \subset (C_0, \partial_i C_0) \subset (\mathbb{CP}^n, C_i)$. They fit within a commutative diagram

$$\begin{tikzcd}
D^2_1/\partial D^2_1 \wedge \cdots \wedge D^2_n/\partial D^2_n \arrow[d, "\simeq"'] & & \\
C_0/\partial_1 C_0 \wedge \cdots \wedge C_0/\partial_n C_0 \arrow[d] & C_0/\partial C_0 \arrow[l, "\Delta"'] \arrow[ul, "\cong"'] \arrow[d, "\cong"] & \\
\mathbb{CP}^n/C_1 \wedge \cdots \wedge \mathbb{CP}^n/C_n & \mathbb{CP}^n/(C_1 \cup \cdots \cup C_n) \arrow[l, "\Delta"'] & \mathbb{CP}^n/\mathbb{CP}^{n-1} \arrow[l, "\simeq"'] \\
\mathbb{CP}^n \wedge \cdots \wedge \mathbb{CP}^n \arrow[u] & \mathbb{CP}^n \arrow[l, "\Delta"'] \arrow[u] \arrow[ur, "j"'] &
\end{tikzcd}$$

On the far right, $\mathbb{CP}^{n-1}$ sits inside the last n coordinates of $\mathbb{CP}^n$ disjoint from C_0. This produces the homotopy equivalence $\mathbb{CP}^n/\mathbb{CP}^{n-1} \to \mathbb{CP}^n/(C_1 \cup \cdots \cup C_n)$.

Let us pass to K^0 and compose on the left of the leftmost column by the tensor product of K-theory groups. By the commutativity of the diagram, the elements $c_i \in K^0(\mathbb{CP}^n/C_i)$ mapping down to $L-1 \in K^0(\mathbb{CP}^n)$ will map up to the generators of $K^0(C_0/\partial_i C_0) \cong K^0(S^2)$. Then $c_1 \cdots c_n$ is the generator of $K^0(\mathbb{CP}^n, C_1 \cup \cdots \cup C_n)$. By the commutativity of the lower triangle, $(L-1)^n$ generates the image of the map $j^* : K^0(\mathbb{CP}^n, \mathbb{CP}^{n-1}) \to K^0(\mathbb{CP}^n)$.

The exact sequence for $(\mathbb{CP}^n, \mathbb{CP}^{n-1})$ reads

$$0 \longrightarrow K^0(\mathbb{CP}^n, \mathbb{CP}^{n-1}) \xrightarrow{j^*} K^0(\mathbb{CP}^n) \xrightarrow{i^*} K^0(\mathbb{CP}^{n-1}) \longrightarrow 0.$$

Since $(L-1)^n$ generates the kernel of i^*, it vanishes in $K^0(\mathbb{CP}^{n-1})$. Replacing n by $n+1$, we deduce that $(L-1)^{n+1}$ vanishes in $K^0(\mathbb{CP}^n)$. If we assume inductively that $K^0(\mathbb{CP}^{n-1}) \cong \mathbb{Z}[L]/(L-1)^n$, we conclude that $K^0(\mathbb{CP}^n) \cong \mathbb{Z}[L]/(L-1)^{n+1}$. □

The following important result allows us to relate the K-theory of a fiber bundle to the K-theory of its base and fiber in certain cases. Since the proof is a tedious induction on the cells of CW-complexes, we will omit it.

Theorem 3.5 (Leray-Hirsch)**.** [7, Thm. 2.25] *Let E be a fiber bundle over X with fiber F and projection $p : E \to X$. Suppose that $K^*(F)$ is a free abelian group and that there exist elements $c_1, \dots, c_k \in K^*(E)$ that restrict to a basis for $K^*(F)$ on each fiber of E. If X is a finite CW-complex or F is a finite CW-complex of only even-dimensional cells, then $K^*(E)$ is free over $K^*(X)$ with basis $\{c_1, \dots, c_k\}$.*

Corollary 3.6. *If E is an n-dimensional vector bundle over X, then $K^*(P(E))$ is free over $K^*(X)$ with basis $\{1, L, \dots, L^{n-1}\}$.*

We can now establish the splitting principle.

Proof of Proposition 3.2. Since $p^*\colon K^*(X) \to K^*(P(E))$ defines the $K^*(X)$-module structure on $K^*(P(E))$ and 1 lies in the basis for $K^*(P(E))$, p^* must be injective.

The pullback p^*E contains the tautological bundle L as a sub-bundle, so p^*E splits orthogonally as $L \oplus E'$. The space $P(E')$ consists of pairs of orthogonal lines in the fibers of E. We can repeat the process by pulling back E' under the composite projection $P(E') \to P(E) \to X$, split off a line bundle, and produce a space $P(E'')$ of triples of orthogonal lines. After n iterations, we obtain the flag bundle $F(E)$, and the pullback of E under the composite projection will split as the sum of n line bundles. The pullback map $K^*(X) \to K^*(F(E))$ will be the composite of the injective maps and hence injective itself. $\square$

4. Proof of Adams's Theorem

The proof of Adams's Theorem relies on operations in K-theory constructed from symmetric polynomials. The fundamental theorem of symmetric polynomials states that any symmetric polynomial in n variables of degree k has a unique representation as a polynomial of degree k in the elementary symmetric polynomials $\sigma_i = \prod_{j_1 < \dots < j_i} x_{j_1} \cdots x_{j_i}$. For the polynomial $x_1^k + \cdots x_n^k$, let s_k be such that

$$x_1^k + \cdots x_n^k = s_k(\sigma_1, \dots, \sigma_k).$$

The polynomial s_k is independent of n for $n \geq k$ since $x_1^k + \cdots x_n^k$ is of degree k.

Definition 4.1. The *Adams operations* are ring homomorphisms $\psi^k : K^0(X) \to K^0(X)$ induced by maps on vector bundles $\psi^k(E) = s_k(\lambda^1(E), \dots, \lambda^k(E))$.

Proposition 4.2. *The Adams operations satisfy the following properties:*

(1) ψ^k is a natural transformation from the functor K^0 to itself.
(2) $\psi^k(L) = L^k$ if L is a line bundle.
(3) $\psi^k \circ \psi^l = \psi^{k+l}$.
(4) If p is prime, then $\psi^p(\alpha) \equiv \alpha^p \pmod p$.

The essence of the proof is the use of the splitting principle to reduce to the case of line bundles.

Proof. In addition to the properties listed, it still remains to show that the Adams operations, as defined on vector bundles, induce ring homomorphisms.

First, let $f : X \to Y$ and E be a vector bundle over X. The fact that $f^*\lambda^i(E) = \lambda^i(f^*E)$ implies that $\psi^k(f^*E) = f^*(\psi^k(E))$. This proves the naturality in *(1)*.

Now let us show that ψ^k is additive and multiplicative on line bundles. Recall the formula $\lambda^k(E \oplus F) \cong \oplus_{i=0}^k(\lambda^i(E) \otimes \lambda^{k-i}(F))$. If $L_1, \dots, L_n$ are line bundles, then $\lambda^i(L_j) = 0$ for $i > 1$. It follows that

$$\lambda^k(L_1 \oplus \cdots \oplus L_n) \cong \sigma_k(L_1, \dots, L_n)$$

Therefore, by the definition of s_k,

(4.3) $$\psi^k(L_1 \oplus \cdots \oplus L_n) = s_k(\sigma_1(L_1, \ldots, L_n), \ldots, \sigma_k(L_1, \ldots, L_n)) = L_1^k \oplus \cdots \oplus L_n^k.$$

This proves *(2)* as a special case. It follows immediately from *(2)* that ψ^k is multiplicative on line bundles.

For the general case, the splitting principle offers maps $p_1 : F(E) \to X$ and $p_2 : F(p_1^*F) \to F(E)$ such that $(p_2 \circ p_1)^*E$ and $(p_2 \circ p_1)^*F$ split as sums of line bundles and $(p_2 \circ p_1)^* : K^0(X) \to K^0(F(p_1^*F))$ is injective. By the linearity for line bundles in (4.3),

$$(p_2 \circ p_1)^*\psi^k(E \oplus F) = \psi^k((p_2 \circ p_1)^*(E \oplus F)) = (p_2 \circ p_1)^*(\psi^k(E) \oplus \psi^k(F)).$$

Injectivity implies that $\psi^k(E \oplus F) = \psi^k(E) \oplus \psi^k(F)$. We conclude that ψ^k is multiplicative by similar arguments. As a result, the ψ^k define ring homomorphisms.

Property *(2)* implies *(3)* for line bundles. A splitting principle argument extends it to all of $K^0(X)$.

By properties of binomial coefficients,

$$\psi^p(L_1 \oplus \cdots \oplus L_n) = L_1^p \oplus \cdots \oplus L_n^p \equiv (L_1 \oplus \cdots \oplus L_n)^p \pmod{p}.$$

Then, by the splitting principle, property *(4)* extends to all bundles. □

The naturality of the Adams operations implies that they also restrict to operations on $\widetilde{K}^0$.

Lemma 4.4. *The action of $\psi^k : \widetilde{K}^0(S^{2m}) \to \widetilde{K}^0(S^{2m})$ is multiplication by k^m.*

Proof. Recall that $\widetilde{K}^0(S^2)$ is generated by $b = 1 - H^*$ and that $b^2 = 0$. Then

$$\psi^k(b) = 1 - (H^*)^k = 1 - (b-1)^k = kb.$$

Let us write $b^{(m)} \in \widetilde{K}^0(S^{2m})$ for the m-fold external product of b with itself. By Proposition 2.19, $b^{(m)}$ is a generator for $\widetilde{K}^0(S^{2m})$. The naturality of the Adams operations implies that they are homomorphisms with respect to the external product on $\widetilde{K}^0$. The result then follows by induction. □

We can now prove Adams's Theorem.

Proof of Theorem 1.1. By Remark 2.21, we may suppose that $n = 2m$. Consider the exact sequence defining the Hopf invariant. Since α is the image under j^* of the generator of $\widetilde{K}^0(S^{2n})$, and ψ^k acts on $\widetilde{K}^0(S^{2n})$ via multiplication by k^n, the naturality of ψ^k implies that $\psi^k(\alpha) = k^{2m}\alpha$. Similarly, because β is mapped by i^* to the generator of $\widetilde{K}^0(S^n)$, $\psi^k(\beta) = k^m\beta + \mu_k\alpha$ for some $\mu_k \in \mathbb{Z}$. Then

$$\psi^k\psi^\ell(\beta) = k^m\ell^m\beta + (k^{2m}\mu_\ell + \ell^m\mu_k)\alpha.$$

From the identity $\psi^k\psi^\ell(\beta) = \psi^\ell\psi^k(\beta)$, we can interchange k and ℓ to deduce that

$$(k^{2m} - k^m)\mu_\ell = (\ell^{2m} - \ell^m)\mu_k.$$

Let us set $k = 3$ and $\ell = 2$. By property *(4)* of Proposition 4.2, $\psi^2(\beta) \equiv \beta^2 = h\alpha \pmod{2}$. Thus $\mu_2\alpha \equiv h\alpha \pmod{2}$. If $h = \pm 1$, then μ_2 is odd. Since $3^m(3^m - 1)\mu_2 = 2^m(2^m - 1)\mu_3$, it follows that 2^m divides $3^m - 1$. It can be shown via elementary number theory that this is only the case for $m = 1, 2$, and 4. □

5. Consequences of Adams's Theorem

Let us show how Adams's Theorem implies the results of Theorem 1.2.

Definition 5.1. A *real division algebra* is a real vector space D with a product $D \times D \to D$ such that the maps $x \mapsto ax$ and $x \mapsto xa$ are linear for every $a \in D$ and invertible for nonzero a.

Definition 5.2. An *H-space* structure on a topological space X is a continuous product map $X \times X \to X$ which possesses an identity element $e \in X$.

Note that the products on a real division algebra or an H-space may not be associative.

Definition 5.3. A manifold M is *parallelizable* if its tangent bundle is trivial.

We can now now reduce the propositions of Theorem 1.2 to a single claim.

Lemma 5.4. *The following hold:*

(1) If $\mathbb{R}^n$ can be endowed with the structure of a real division algebra, then S^{n-1} has an H-space structure.
(2) If $\mathbb{R}^n$ can be endowed with the structure of a real division algebra, then S^{n-1} is parallelizable.
(3) If S^{n-1} is parallelizable, then S^{n-1} has an H-space structure.

Proof. Let $\mathbb{R}^n$ be a real division algebra and e be a vector of unit norm. We can compose the multiplication $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ with an invertible map $\mathbb{R}^n \to \mathbb{R}^n$ so that $e^2 = e$. With this modified multiplication, let $\alpha(x) = ex$ and $\beta(x) = xe$. We can then define a new division algebra product $x \odot y = \alpha^{-1}(x)\beta^{-1}(y)$. Then $e \odot x = x$ and $x \odot e = x$. This product has no zero divisors, so the map $(x, y) \mapsto (x \odot y)/|x \odot y|$ defines an H-space structure on $S^{n-1} \subset \mathbb{R}^n$ with identity e.

Let $\mathbb{R}^n$ be a real division algebra. By the previous argument, we can suppose that there is an identity e. If $\{e, b_1, \ldots, b_{n-1}\}$ is a basis for $\mathbb{R}^n$, then for any $v \in S^{n-1}$, $e \cdot v, b_1 \cdot v, \ldots, b_{n-1} \cdot v$ are all linearly independent. Since none of the $b_i \cdot v$ are parallel to v, we can project them onto the tangent plane of S^{n-1} at v. This defines $n-1$ tangent vector fields on S^{n-1} and a trivialization of TS^{n-1}.

If S^{n-1} is parallelizable via an isomorphism $\phi : S^{n-1} \times \mathbb{R}^n \to TS^{n-1}$, then there are $n-1$ linearly independent sections $v_1, \ldots, v_{n-1}$ of TS^{n-1} defined by $v_i(x) = \phi(x, e_i)$ for the standard basis $\{e_i\}$ of $\mathbb{R}^n$. By the Gram-Schmidt process, we can take these sections to be orthogonal. By applying a rotation to S^{n-1} if needed, we may also assume that $v_i(e_1) = e_{i+1}$. Let R_x be the orthogonal linear map sending the standard basis to $(x, v_1(x), \ldots, v_{n-1}(x))$. The map $(x, y) \mapsto R_x(y)$ defines an H-space structure on S^{n-1} with identity e_1. □

To relate the H-space structure on S^{n-1} to the Hopf invariant one problem, we construct an associated map $S^{2n-1} \to S^n$ which is to have Hopf invariant one.

Construction 5.5. Let $g : S^{n-1} \times S^{n-1} \to S^{n-1}$. Let us identify

$$S^{2n-1} = \partial(D^n \times D^n) \cong (\partial D^n \times D^n) \cup (D^n \times \partial D^n)$$

and $S^n = D^n_+ \cup D^n_-$. The *Hopf construction* is the map $\hat{g} : S^{2n-1} \to S^n$ given by

$$\hat{g}|_{\partial D^n \times D^n}(x, y) = |y| \cdot g(x, y/|y|) \in D^n_+,$$
$$\hat{g}|_{D^n \times \partial D^n}(x, y) = |x| \cdot g(x/|x|, y) \in D^n_-.$$

Lemma 5.6. *Let $n > 1$. If $\mu : S^{n-1} \times S^{n-1} \to S^{n-1}$ is an H-space product, then the Hopf invariant of the Hopf construction $\hat{\mu} : S^{2n-1} \to S^n$ is ± 1.*

Proof. Let us consider the following commutative diagram:

$$\begin{tikzcd}
Cf \wedge Cf \arrow[d, "\simeq"] & Cf \arrow[l, "\Delta"'] \arrow[d, "j"] \\
Cf/D^n_- \wedge Cf/D^n_+ & Cf/S^n \arrow[l, "\Delta"'] \\
D^n{\times}D^n/\partial D^n{\times}D^n \wedge D^n{\times}D^n/D^n{\times}\partial D^n \arrow[u, "\Phi\wedge\Phi"] \arrow[d, "\pi_1\wedge\pi_2", "\simeq"'] & D^n{\times}D^n/\partial(D^n{\times}D^n) \arrow[l, "\Delta"'] \arrow[u, "\cong", "\Phi"'] \arrow[dl, "\cong"] \\
D^n{\times}\{e\}/\partial D^n{\times}\{e\} \wedge \{e\}{\times}D^n/\{e\}{\times}\partial D^n &
\end{tikzcd}$$

The map Φ is the attaching map $(D^n \times D^n, \partial(D^n \times D^n)) \to (Cf, S^n)$ of the $2n$-cell of Cf. The π_i are projections. We can verify that the diagram commutes by inspecting how the attaching map Φ is defined by the Hopf construction of $\hat{\mu}$.

The homotopy inverses of the projections π_1 and π_2 are the inclusions

$$\iota_1 : D^n{\times}\{e\}/\partial D^n{\times}\{e\} \longrightarrow D^n{\times}D^n/\partial D^n{\times}D^n$$

$$\iota_2 : \{e\}{\times}D^n/\{e\}{\times}\partial D^n \longrightarrow D^n{\times}D^n/D^n{\times}\partial D^n.$$

In the left column of the diagram, the restriction of Φ to $\partial(D^n \times D^n)$ factors as

$$\begin{tikzcd}
\partial(D^n{\times}D^n) \arrow[r, "\hat{\mu}"'] \arrow[rr, bend left, "\Phi"] & S^n \arrow[r, "i"'] & Cf
\end{tikzcd}$$

where i is the inclusion in the exact sequence defining the Hopf invariant. Therefore, by the fact that μ is an H-space product, the two composite maps $\Phi \circ \iota_1$ and $\Phi \circ \iota_2$ factor as homeomorphisms followed by inclusions:

$$\begin{tikzcd}
& D^n{\times}D^n/\partial D^n{\times}D^n \arrow[dr, "\Phi"] & \\
D^n{\times}\{e\}/\partial D^n{\times}\{e\} \arrow[ur, "\iota_1"] \arrow[dr, "\cong", "\hat{\mu}"'] & & Cf/D^n_- \\
& S^n/D^n_- \arrow[ur, "i"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
& D^n{\times}D^n/D^n{\times}\partial D^n \arrow[dr, "\Phi"] & \\
\{e\}{\times}D^n/\{e\}{\times}\partial D^n \arrow[ur, "\iota_2"] \arrow[dr, "\cong", "\hat{\mu}"'] & & Cf/D^n_+ \\
& S^n/D^n_+ \arrow[ur, "i"'] &
\end{tikzcd}$$

Let us pass to $\widetilde{K}^0$ in the diagram. We can compose the $\widetilde{K}^0$ of the left column on the left by the tensor product of reduced K-theory groups. By definition, β maps to the generator of $\widetilde{K}^0(S^n)$, α is the image of the generator of $\widetilde{K}^0(S^{2n})$, and $\beta^2 = h\alpha$ for Hopf invariant h. Consequently, because of the preceding factorizations, the element $\beta \otimes \beta \in \widetilde{K}^0(Cf) \otimes \widetilde{K}^0(Cf)$ is mapped vertically to a generator of $\widetilde{K}^0(D^n \times \{e\}, \partial D^n \times \{e\}) \otimes \widetilde{K}^0(\{e\} \times D^n, \{e\} \times \partial D^n)$. This generator then passes through the diagonal arrow to a generator of $\widetilde{K}^0(D^n \times D^n, \partial(D^n \times D^n)) \cong \widetilde{K}^0(Cf, S^n)$. This generator is finally sent upwards by j^* to a generator of $\widetilde{K}^0(Cf)$. By the definition of $\alpha \in \widetilde{K}^0(Cf)$, this must be $\pm\alpha$. Since the diagram commutes, $\beta^2 = \pm\alpha$. $\square$

It then follows immediately from Adams's Theorem that the statements in Theorem 1.2 cannot hold when $n > 1$ and $n \neq 2, 4, 8$. In the following section, we examine the cases $n = 1, 2, 4$, and 8.

5.1. **The Hopf fibrations.** The real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, quaternions $\mathbb{H}$, and octonions $\mathbb{O}$ form real division algebras of dimension $1, 2, 4$, and 8, respectively. Although the real and complex numbers are associative and commutative, the quaternions are associative but not commuative and the octonions are neither associative nor commutative. Nonetheless, all elements of these algebras possess multiplicative inverses. Moreover, in addition to being division algebras, these four algebras are also normed with a norm that coincides with the Euclidean norm. An introduction to the these algebras with a focus on the octonions is given in [4].

The division algebra structure on $\mathbb{R}^n$ for $n = 2, 4$, and 8 can be used to easily describe maps $S^{2n-1} \to S^n$ of Hopf invariant ± 1. Along with the case $n = 1$, these maps are the famous fiber bundles known as the *Hopf fibrations*:

$$S^{n-1} \to S^{2n-1} \to S^n \qquad\qquad n = 1, 2, 4, 8.$$

Let D be $\mathbb{R}, \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$ and n be the dimension of D. As for $D = \mathbb{R}, \mathbb{C}$, one can define the *projective line* $D\mathbb{P}^1$ of lines in D^2. Although this requires care since $\mathbb{H}$ is not commutative and $\mathbb{O}$ is not even associative, the resulting space will be a smooth manifold diffeomorphic to S^n. It possesses a *tautological line bundle*

$$L_D = \{(\ell, (x, y)) \in D\mathbb{P}^1 \times D^2 \mid (x, y) \in \ell\}.$$

The norm on D provides a norm on the fibers of L_D. The set $S(L_D)$ of vectors of unit norm will be a fiber bundle over $D\mathbb{P}^1$ with fiber S^{n-1}. Projecting $S(L_D)$ onto D^2 gives a homeomorphism with the set of vectors of unit norm in D^2, so $S(L_D) \cong S^{2n-1}$. The projection map of this fiber bundle is the Hopf fibration corresponding to D:

$$\begin{array}{ccc} S(L_D) \cong S^{2n-1} \subset D^2 & \longrightarrow & D\mathbb{P}^1 \cong S^n \\ (x, y) & \longmapsto & \ell(x, y), \end{array}$$

Here $\ell(x, y)$ means the line through $(x, y) \in D^2$. In terms of the familiar projective coordinates for $\mathbb{RP}^1$ and $\mathbb{CP}^1$, this takes the form

$$(x, y) \longmapsto [x, y].$$

It can be shown that these maps are of Hopf invariant ± 1.

Acknowledgements

I would like to thank my mentor Weinan Lin for his teaching and guidance over the eight weeks of the Chicago REU. I am also grateful to Professor Shmuel Weinberger for introducing me to vector bundles and K-theory. Finally, I would like to thank Professor Peter May for organizing the REU, teaching a series of fascinating lectures on algebraic topology, and providing advice on this paper.

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