Abstract. This paper is an introduction to certain topics in graph theory, spectral graph theory, and random walks. We begin with basic definitions in graph theory, moving then to topics in linear algebra that are necessary to study the spectra of graphs. Then, finally, to basic results of the graph’s matrices and an introduction to random walks.

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1. Introduction

While graphs are typically represented as a set of vertices and a set of edges, they can also be represented by several different matrices. This naturally motivates the inquiry as to the nature of a graph’s spectrum. The spectrum of a graph has proven itself to be surprisingly informative about that graph’s characteristics, such as the diameter of a graph and its connectivity. We will explore some of these results within this paper as well as discuss the basics of random walks on graphs.

We begin with basic but necessary definitions in graph theory that are important to both describe and prove results in spectral graph theory. Then, we move to topics in linear algebra that are required to understand spectral graph theory, such as the Rayleigh quotient and spectral theorem. Next, we discuss basic results of a graph’s matrices and characteristics of a graph that can be read from the matrices and their corresponding eigenvalues. Finally, we begin a very basic introduction to random walks on graphs with a discussion of the transition matrix.

2. Basic Definitions

Definition 2.1. A graph is a pair \( G = (V, E) \), where \( E \) is a multiset whose elements are 2-subsets of \( V \).
Elements of $V$ are called the vertices of a graph $G$ and the elements of $E$ are called the edges of a graph $G$. The above definition is sometimes referred to as an undirected graph. Edges can be thought of as lines connecting two vertices or as loops starting and ending at the same vertex.

**Definition 2.2.** An edge $e = (v_i, v_j)$ is *incident to* a vertex $v$ if $v = v_i$ or $v = v_j$.

**Definition 2.3.** An edge $e = (v_i, v_j)$ joins the two vertices $v_i$ and $v_j$.

An edge that joins a vertex $v$ to itself is called a *loop*, and if there exists more than one edge that is incident to a pair of vertices $u$ and $v$, then there are *multiple edges* joining those $u$ and $v$.

**Definition 2.4.** Two vertices $v$ and $u$ are *adjacent* if there exists an edge $e \in E$ where $e$ joins $v$ and $u$. If $v$ and $u$ are adjacent, then $v \sim u$.

**Definition 2.5.** A *simple* graph $G$ is a graph that contains no loops or multiple edges.

Unless stated otherwise, in this paper graphs will be *simple* and finite. This means that the set of vertices is finite.

**Definition 2.6.** The *degree* of a vertex $v$, denoted $d_v$, is the number of edges incident to $v$.

**Example 2.7.** The diagram above on the left is a simple graph $G$ with vertex set $V = \{v_1, v_2, v_3, v_4\}$ and edge set $E = \{(v_1, v_2), (v_1, v_3), (v_1, v_4), (v_2, v_3), (v_3, v_4), (v_2, v_4)\}$. A special property of the above graph is that every pair of vertices is adjacent, forming a *complete graph*. Complete graphs are denoted by $K_n$, with $n$ being the number of vertices in the graph, meaning the above graph is a $K_4$. It should also be noted that all vertices are incident to the same number of edges. Equivalently, for all $v \in V$, $d_v = 3$. We call a graph where $d_v$ is constant a *regular* graph. Therefore, all complete graphs are regular but not all regular graphs are complete.

The graph on the right, $H$, is the simplest example of a multigraph: a graph with one vertex and a loop.

**Definition 2.8.** A *walk* on a graph $G = (V, E)$ is a sequence of vertices $(v_0, \ldots, v_{n-1})$ where $\{v_{i-1}, v_i\} \in E$ for $1 \leq i \leq n - 1$. The *length* of the walk is $n - 1$.

**Definition 2.9.** A *path* on a graph $G = (V, E)$ is a walk where all vertices and edges are distinct.

**Definition 2.10.** If there exists a path in $G$ from vertex $v_1$ to vertex $v_2$, then $v_1$ and $v_2$ are *connected*. A graph $G$ is *connected* if every pair of vertices in $G$ is connected.
Remark 2.11. We will assume in this paper that graphs are connected unless stated otherwise.

Other than representing graphs visually with vertices and edges, one can also represent them in terms of matrices. Three matrices that can be used to study graphs are the adjacency matrix, the Laplacian, and the normalized Laplacian. While all three matrices have different constructions and eigenvalues, they all can indicate important features of a graph, including regularity, connectedness, and completeness.

Definition 2.12. The adjacency matrix $A$ of a graph $G$ is an $n \times n$ matrix indexed by $V$, where $n = |V|$, defined as:

$$A_{i,j} = \begin{cases} 1 & v_i \sim v_j \\ 0 & \text{otherwise} \end{cases}$$

Definition 2.13. The Laplacian matrix $L$ of a graph $G$ is also an $n \times n$ matrix indexed by $V$, where $n = |V|$, defined as:

$$L_{i,j} = \begin{cases} d_{v_i} & v_i = v_j \\ -1 & v_i \sim v_j \\ 0 & \text{otherwise} \end{cases}$$

Remark 2.14. An equivalent definition of the Laplacian would be $L = D - A$, where $D$ is a diagonal matrix with the degrees of the vertices down the diagonal and $A$ is the adjacency matrix.

Definition 2.15. The normalized Laplacian $\mathcal{L}$ is an $n \times n$ matrix defined as:

$$\mathcal{L}_{i,j} = \begin{cases} 1 & v_i = v_j \\ \frac{1}{\sqrt{d_{v_i}d_{v_j}}} & v_i \sim v_j \\ 0 & \text{otherwise} \end{cases}$$

Remark 2.16. Again, the normalized Laplacian can be defined with another, equivalent definition. It can be seen as the product of both the diagonal matrix $D$ and the Laplacian $L$. More specifically, it can be defined as $\mathcal{L} = D^{-1/2}LD^{-1/2}$.

All of the matrices that were defined depend on a numbering of the vertices in $V$. However, different numberings can be thought of as simple permutations of the vertices. This means that the different matrices are unique up to conjugation by permutation matrices. So then, no matter the numbering placed on the vertices of a graph $G$, the spectral properties of all three matrices defined will be consistent.
Example 2.17. The vertices of the above graph have been assigned an order, so the adjacency, Laplacian, and normalize Laplacian matrices can be expressed as:

\[
A = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

\[
L = \begin{pmatrix}
1 & -1 & 0 & 0 \\
-1 & 2 & -1 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & -1 & 1
\end{pmatrix}
\]

\[
\mathcal{L} = \begin{pmatrix}
1 & -\frac{1}{\sqrt{2}} & 0 & 0 \\
-\frac{1}{\sqrt{2}} & 1 & -\frac{1}{2} & 0 \\
0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\
0 & 0 & -\frac{1}{\sqrt{2}} & 1
\end{pmatrix}
\]

3. Relevant Linear Algebra

Spectral graph theory, as implied by the name, relies heavily on linear algebra. Before the results of the eigenvalues of the various matrices can be discussed, certain features of linear algebra must be defined and proven. By definition and seen in Example 2.14, all three matrices are symmetric, and therefore all of their respective eigenvalues, are real.

Definition 3.1. The standard inner product \( \langle \cdot, \cdot \rangle \) of vectors \( a, b \in \mathbb{C}^n \) is defined to be

\[
\langle a, b \rangle = a^T b
\]

Theorem 3.2. If \( A \) is a symmetric matrix with all real entries, then all eigenvalues of \( A \) are real. There also exists a basis for \( \mathbb{R}^n \) comprised of orthonormal eigenvectors of \( A \).

Proof. Let \( \lambda \in \mathbb{C} \) be an eigenvalue of \( A \) with corresponding eigenvector \( v \). Note that since \( A \) is a symmetric matrix, we know that \( A = A^T \). Then:

\[
\lambda \langle v, v \rangle = \langle Av, v \rangle \\
= \langle v, Av \rangle \\
= \langle v, \lambda v \rangle \\
= \lambda \langle v, v \rangle
\]

Because \( \lambda = \lambda \), it must be true that \( \lambda \in \mathbb{R} \). So if \( \lambda \) is an eigenvalue of \( A \), then \( \lambda \) must be real.

A proof for the basis of orthonormal eigenvectors can be found [4].

Definition 3.3. A matrix \( A \) is called Hermitian if \( A = A^\dagger \).

Because the adjacency matrix, Laplacian matrix, and normalized Laplacian matrix are all real, symmetric matrices, it is clear that all three of them are Hermitian matrices. A useful property of Hermitian matrices is that the Rayleigh quotient
can be used to find the eigenvalues, and by extension the eigenvectors, of Hermitian matrices.

**Definition 3.4.** Let $A$ be an $n \times n$, Hermitian matrix. The **Rayleigh quotient** $R_A$ is a function $\mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}$ where

$$R_A(x) = \frac{x^T A x}{x^T x}$$

Quickly note that if $x$ is an eigenvector with corresponding eigenvalue $\lambda$, then $R_A(x) = \lambda$ because

$$R_A(x) = \frac{x^T A x}{x^T x} = \frac{x^T \lambda x}{x^T x} = \frac{\lambda x^T x}{x^T x} = \lambda$$

The Rayleigh quotient can be used to find the minimum and maximum eigenvalues of a Hermitian matrix. Using those corresponding eigenvectors, all of the intermediate eigenvalues can be found.

So starting with a symmetric, $n \times n$ matrix $A$, there is a basis for $\mathbb{R}^n$ of orthonormal eigenvectors $v_1, \ldots, v_n$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$ where $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$. Then, because $v_1, \ldots, v_n$ forms a basis for $\mathbb{R}^n$, for any vector $x \in \mathbb{R}^n$, we can express $x$ as a linear combination of those eigenvectors

$$x = \sum_i \alpha_i v_i \quad (\alpha_i \in \mathbb{R}).$$

First looking at the denominator of the Rayleigh quotient:

$$x^T x = \sum_{i,j=1}^{n} \alpha_i \alpha_j v_i^T v_j = \sum_i \alpha_i^2$$

Note that the $j$ term is eliminated because when $i \neq j$, $v_i^T v_j = 0$ because they are orthogonal. Now, the numerator of Rayleigh’s quotient:

$$x^T A x = x^T \left( \sum_{i=1}^{n} \alpha_i A v_i \right) = \sum_{i=1}^{n} \alpha_i x^T (\lambda_i v_i) = \sum_{i=1}^{n} \lambda_i \alpha_i^2$$

The entirety of Rayleigh’s quotient can then be expressed as:

$$R_A(x) = \frac{\sum_{i=1}^{n} \alpha_i^2 \lambda_i}{\sum_{i=1}^{n} \alpha_i^2}$$

Then, because of the ordering placed on the eigenvalues:

$$R_A(x) = \frac{\sum_{i=1}^{n} \alpha_i^2 \lambda_i}{\sum_{i=1}^{n} \alpha_i^2} \leq \frac{\sum_{i=1}^{n} \alpha_i^2 \lambda_n}{\sum_{i=1}^{n} \alpha_i^2} = \lambda_n$$

Because $R_A(x)$ has a maximum value of $\lambda_n$ and we know that $R_A(x)$ achieves its maximum when $x = v_n$, we have that $\max_{x \neq 0} R_A(x) = \lambda_n$. A similar argument follows for $\min_{x \neq 0} R_A(x) = \lambda_1$.

We can expand this method to include all eigenvalues rather than just $\lambda_1$ and $\lambda_n$. For any $k$ where $1 \leq k \leq n$, let $W = \text{span}(v_1, \ldots, v_k)$. Then, for any $x \in W$, the Rayleigh quotient can be used to find the eigenvalues of $A$. If $x$ is an eigenvector of $A$ with corresponding eigenvalue $\lambda_k$, then $R_A(x) = \lambda_k$.
we can represent $x$ as $x = \sum_{i=1}^{k} \alpha_i v_i$. Therefore, the Rayleigh quotient for $x \in W$ will have similar bounds as the Rayleigh quotient for $x \in \mathbb{R}^n$:

$$R_A(x) = \frac{\sum_{i=1}^{k} \alpha_i^2 \lambda_i}{\sum_{i=1}^{k} \alpha_i^2} \leq \frac{\sum_{i=1}^{k} \alpha_i^2 \lambda_k}{\sum_{i=1}^{k} \alpha_i^2} = \lambda_k$$

So we have that $\lambda_k = \max_{x \in W, x \neq 0} R_A(x)$

For finding the $k$th eigenvalue with the minimum, we let $W = \text{span}(v_k, \ldots, v_n)$ and the bounds on the Rayleigh quotient are reversed

$$R_A(x) = \frac{\sum_{i=1}^{k} \alpha_i^2 \lambda_i}{\sum_{i=1}^{k} \alpha_i^2} \geq \frac{\sum_{i=1}^{k} \alpha_i^2 \lambda_k}{\sum_{i=1}^{k} \alpha_i^2} = \lambda_k$$

So we have that $\lambda_k = \min_{x \in W, x \neq 0} R_A(x)$

4. Results of the Laplacian and Adjacency Matrices

As stated previously, different properties of a graph can be identified through the adjacency matrix, Laplacian matrix, and normalized Laplacian matrix. All three can display the regularity of a graph, whether a graph is connected, and if a graph is complete.

**Theorem 4.1.** Let $G = (V, E)$ be a graph, where $|V| = n$, and let $A$ be the adjacency matrix of $G$. If $A$ is raised to the $m$-th power, then the entry $A^m_{i,j}$ is the number of walks from $v_i$ to $v_j$ of length $m$.

**Proof.** We induct on $m$.

Base Case: Let $m = 1$. The $A_{i,j}$ entry of the adjacency matrix is by definition 1 if $v_i \sim v_j$ and 0 otherwise. Therefore, the $A_{i,j}$ entry is the number of walks from $v_i$ to $v_j$.

Inductive Step: We assume the theorem is true for $k \geq 1$. Now let $m = k + 1$. We begin with $A^k$, where $A^k_{i,j}$ is the number of length $k$ walks from $v_i$ to $v_j$. Then, the number of walks from $v_i$ to $v_j$ is the sum of the number of walks from $v_i$ to $v_l$ multiplied by the number of walks from $v_l$ to $v_j$ for all $v_l \sim v_j$. So the desired number of walks can be represented as

$$\sum_{l=1}^{n} A^{k}_{i,l} \cdot A_{l,j}$$

This is the formula for matrix multiplication so $A^{k+1}_{i,j} = \sum_{l=1}^{n} A^{k}_{i,l} \cdot A_{l,j}$. Therefore, $A^{k+1}_{i,j}$ is the number of walks of length $k + 1$ from $v_i$ to $v_j$. \qed

**Remark 4.2.** Notice that the diagonal entries of $A^{k+1}$ will be the number of closed walks, or the number of walks of length $k + 1$ from a vertex $w$ to $w$, which leads to an interesting property of the trace of $A^{k+1}$.

**Lemma 4.3.** Let $M$ be an $n \times n$ matrix. If $\beta$ is an eigenvalue of $M$, then $\beta^n$ is an eigenvalue of $M^n$. 

Proof. We induct on $n$.

Base case: Let $n = 1$. Then $\beta^1 = \beta$ is the definition of eigenvalue of $M$.

Now we assume that the statement holds for $k \geq 1$. Let $n = k + 1$. Because $\beta$ is an eigenvalue of $M$, there is a corresponding eigenvector $v$. So then $\beta^{k+1}v = \beta \cdot \beta^kv = \beta A^kv$ by the Inductive hypothesis. Then

$$\beta A^kv = A^k \beta v
= A^k Av
= A^{k+1}v$$

Therefore, if $\beta$ is an eigenvalue of $M$, then $\beta^n$ is an eigenvalue of $M^n$. \hfill $\square$

We denote the eigenvalues of $A$ as $a_1 \leq \ldots \leq a_n$, the eigenvalues of $L$ as $\lambda_1 \leq \ldots \leq \lambda_n$, and the eigenvalues of $L^2$ as $\gamma_1 \leq \ldots \leq \gamma_n$.

**Theorem 4.4.** Let $G = (V, E)$ be a graph with adjacency matrix $A$. Then $\sum_{i=1}^n \alpha_i = \text{tr}(A^k)$ is the number of closed paths on $G$.

*Proof. Let $k \in \mathbb{N}$. By Theorem 4.1, $\sum_{i=1}^n A_{i,i}^k$ is the total number of closed walks of length $k$ on $G$. By Lemma 4.3, $A^k$ has eigenvalues $\alpha_1^k, \ldots, \alpha_n^k$. The trace of $A^k$ is $\sum_{i=1}^n \alpha_i^k$, so $\sum_{i=1}^n A_{i,i}^k = \sum_{i=1}^n \alpha_i^k$. Therefore, $\sum_{i=1}^n \alpha_i^k$ is the number of closed paths on $G$. \hfill $\square$

Now, shifting focus to the Laplacian, we can use the Rayleigh quotient to observe an interesting property of the Laplacian’s eigenvalues. Let $G = (V, E)$ be a graph with vertices $u_1, \ldots, u_n$. Looking just at the numerator of the quotient $\frac{f^TLf}{f^Tf}$, for any $f \in \mathbb{R}^n$:

$$f^T Lf = f^T(D - A)f$$

$$= (f_1, \ldots, f_n) \cdot \begin{pmatrix}
    d_1 f_1 - \sum_{u_i \sim u_1} f_i \\
    \vdots \\
    d_n f_n - \sum_{u_i \sim u_n} f_i
\end{pmatrix}$$

$$= \sum_{i=1}^n f_i \left( d_i f_i - \sum_{u_j \sim u_i} f_j \right)
= \sum_{i=1}^n \left( d_i^2 f_i^2 - \sum_{u_j \sim u_i} f_j \cdot f_i \right)$$
\[
\sum_{i=1}^{n} \left( d_i f_i^2 - \frac{1}{2} \sum_{u_j \sim u_i} (f_j - f_i)^2 - \frac{1}{2} \sum_{u_j \sim u_i} f_j^2 - \frac{1}{2} d_i f_i^2 \right)
\]

\[
= \sum_{i=1}^{n} d_i f_i^2 + \frac{1}{2} \left( 2 \cdot \sum_{u_i \sim u_j} (f_i - f_j)^2 \right) - \frac{1}{2} \sum_{u_j \sim u_i} f_j^2 - \frac{1}{2} \sum_{i=1}^{n} d_i f_i^2
\]

\[
= \sum_{u_i \sim u_j} (f_i - f_j)^2
\]

Therefore, \( f^T L f \geq 0 \) for all \( f \in \mathbb{R}^n \), which means that \( \lambda_i \geq 0 \) for all \( 1 \leq i \leq n \).

Also, this means that \( \gamma_i \geq 0 \) for all \( 1 \leq i \leq n \).

**Theorem 4.5.** Let \( G = (V,E) \) be a graph where \( d \) is the maximal degree of the vertices in \( G \). The interval \([-d,d]\) contains the adjacency eigenvalues. The interval \([0,2d]\) contains the Laplacian eigenvalues. The interval \([0,2]\) contains the normalized Laplacian eigenvalues.

**Proof.** Let \( \alpha \) be an adjacency eigenvalue with eigenvector \( a \) having entries \( a_i \) for \( 1 \leq i \leq n \). Then for vertices \( v_i \) for \( 1 \leq i \leq n \):

\[
A \cdot a = \alpha a = \begin{pmatrix}
\sum_{v_i \sim v_1} a_i \\
\vdots \\
\sum_{v_i \sim v_n} a_i
\end{pmatrix}
\]

Then, \( \alpha a_i = \sum_{v_j \sim v_i} a_j \). Let \( v_k \in V \) be a vertex in \( G \) where \( |a_k| \) is maximal. Then:

\[
|\alpha| |a_k| \leq \sum_{v_i \sim v_k} |a_i| \leq \sum_{v_i \sim v_k} |a_k| = d_v \cdot |a_k| \leq d \cdot |a_k|
\]

Therefore, \( |\alpha| \leq d \), implying that \(-d \leq \alpha \leq d\).

Next, let \( \lambda \) be a Laplacian eigenvalue with eigenvector \( b \). Then for vertices \( v_i \) for \( 1 \leq i \leq n \):

\[
L \cdot b = \lambda b = \begin{pmatrix}
d_{v_1} b_1 - \sum_{v_i \sim v_1} b_i \\
\vdots \\
d_{v_n} b_n - \sum_{v_i \sim v_n} b_i
\end{pmatrix}
\]

Then, \( \lambda b_i = d_{v_i} b_i - \sum_{v_j \sim v_i} b_j \). Let \( v_k \in V \) be a vertex in \( G \) where \( |b_k| \) is maximal. Then:

\[
|d_{v_k} - \lambda||b_k| \leq \sum_{v_i \sim v_k} |b_i| \leq |b_k|d_{v_k}
\]
This implies that \(|d_{v_k} - \lambda| \leq d_{v_k}\). Therefore, \(0 \leq \lambda \leq 2d_{v_k} \leq 2d\).

Lastly, let \(\gamma\) be a normalized Laplacian eigenvalue with eigenvector \(c\). Then for vertices \(v_i\) for \(1 \leq i \leq n\):

\[
\mathcal{L} \cdot c = \gamma c = \begin{pmatrix}
  c_1 - \sum_{v_i \sim v_1} \frac{c_i}{\sqrt{d_{v_i} d_{v_1}}} \\
  \vdots \\
  c_n - \sum_{v_i \sim v_1} \frac{c_i}{\sqrt{d_{v_n} d_{v_1}}}
\end{pmatrix}
\]

Then, \(\gamma c_i = c_i - \sum_{v_j \sim v_k} \frac{c_j}{\sqrt{d_{v_j} d_{v_k}}}\). Let \(v_k \in V\) be a vertex in \(G\) where \(|c_k|\) is maximal. Then:

\[
|1 - \gamma| |c_k| \leq \sum_{v_j \sim v_k} \frac{c_j}{\sqrt{d_{v_j} d_{v_k}}} \leq \sum_{v_j \sim v_k} \frac{c_j}{d_{v_j}} \leq c_k
\]

This implies that \(|1 - \gamma| \leq 1\). Therefore, \(0 \leq \gamma \leq 2\).

**Definition 4.6.** A graph \(G = (V,E)\) is called *bipartite* if \(V\) is the union of two disjoint, independent sets \(V_1\) and \(V_2\) where every edge connects a vertex in \(V_1\) to a vertex in \(V_2\). We call \(V_1\) and \(V_2\) partite sets.

**Lemma 4.7.** A graph is bipartite if and only if it only contains even closed walks.

**Proof.** First, let us prove the forward direction. Let \(G\) be a graph with partite sets \(V_1\) and \(V_2\). Let \(v \in V\), and without loss of generality let \(v \in V_1\). A closed walk from \(v\) to \(v\) has to alternate between \(V_1\) and \(V_2\), starting and ending in \(V_1\). Therefore, the closed walk must have an even length, or it would never return to \(v\).

Now we prove the reverse. Assume that \(G\) has no odd closed walks. Then let \(u \in V\) and divide the remaining vertices into two sets:

\[
A = \{v \in V | \text{the shortest path between } u \text{ and } v \text{ has an odd length}\}
\]

\[
B = \{v \in V | \text{the shortest path between } u \text{ and } v \text{ has an even length}\}
\]

Then let \(u \in B\). Suppose that there exists \(a, b \in A\) such that \(a \sim b\). Then there is an odd path from \(u\) to \(a\), an odd path from \(u\) to \(b\), and a path of length 1 from \(a\) to \(b\). Combined, this would form an odd closed walk, so no such \(a, b\) exist. A similar argument follows for adjacent vertices in \(B\). Therefore, \(G\) is bipartite.

**Theorem 4.8.** Consider a graph \(G = (V,E)\). The following statements are equivalent:

1. \(G\) is bipartite.
2. If \(\alpha\) is an eigenvalue of \(A\), then \(-\alpha\) is also an eigenvalue.
3. \(\sum_{i=1}^{n} \alpha_i^r = \text{tr}(A^r) = 0\) for all odd \(r\).

**Proof.** First we prove that (1) implies (2): Let \(G\) be a bipartite graph. Then, if \(G\) is bipartite, disconnected vertices can be added to both partite sets in order to make them the same size. Each disconnected vertex will add a row of zeroes to
the adjacency matrix, meaning the rank of the matrix will not change. Then, the matrix can be permuted to achieve a matrix that resembles this form:

\[ A = \begin{pmatrix} 0 & M \\ M^T & 0 \end{pmatrix} \]

Now, consider \( \alpha \) to be an eigenvalue of \( A \) with a corresponding eigenvector \( x = \begin{pmatrix} u \\ v \end{pmatrix} \). Then \( Ax = \alpha x = \begin{pmatrix} Mv \\ Mu \end{pmatrix} \). This implies that \( Mv = \alpha u \) and \( Mu = \alpha v \).

Consider then vector \( y = \begin{pmatrix} u \\ -v \end{pmatrix} \). So then \( Ay = \begin{pmatrix} -Mv \\ Mu \end{pmatrix} = \begin{pmatrix} -\alpha u \\ \alpha v \end{pmatrix} \), which implies that \(-\alpha\) is an eigenvalue of \( A \).

Now we prove that (2) implies (3): For each eigenvalue \( \alpha_i \) there is a corresponding eigenvalue \( \alpha_j \) where \( \alpha_j = -\alpha_i \). Then, for all odd \( r \), \( \alpha_j^r = -\alpha_i^r \). Therefore,

\[ \sum_{i=1}^{n} \alpha_i^r = 0 \] for all odd \( r \).

Finally, we prove that (3) implies (1): This implies that there are no closed walks of odd length in \( G \). So by Lemma 4.7, \( G \) is bipartite. \( \square \)

5. Basics of Random Walks

Random walks on a graph can be thought of as choosing, at random, a vertex of the graph. Then taking a step away from the chosen vertex simply entails moving randomly to one if its neighbors. The sequence of randomly chosen vertices can then be referred to as a random walk. When discussion random walks on graphs, the graphs discussed will typically be connected.

There are several types of graphs random walks can take place on, namely unweighted, undirected graphs and weighted, undirected graphs. A common feature of both is that the transition probabilities, which are the probabilities from one vertex to another, are described in a matrix \( P \) called the transition matrix. It should be noted that unlike other matrices discussed in this paper, the transition matrix is not symmetric.

**Definition 5.1.** For an unweighted graph, the \( i,j \)th entry of the transition matrix \( P \) is described by

\[
P_{i,j} = \begin{cases} \frac{1}{d_i} & \text{if } v_i \sim v_j \\ 0 & \text{else} \end{cases}
\]

**Definition 5.2.** An edge weight \( w(u,v) \) is a weight on moving from vertex \( u \) to vertex \( v \) where \( w(u,v) = w(v,u) \) and \( d_u = \sum_x w(u,x) \).

**Definition 5.3.** The \( i,j \)th entry for the transition matrix for a weighted graph is then described by

\[
P_{i,j} = \begin{cases} \frac{w(v_i, v_j)}{d_i} & \text{if } v_i \sim v_j \\ 0 & \text{else} \end{cases}
\]

For an unweighted graph, it is fairly clear that for a vertex \( u \), \( \sum_v P_{u,v} = 1 \), showing that it is a probability distribution. For a weighted graph, it is still true that \( \sum_v P_{u,v} = 1 \) by how the edge weights are defined, so it is also a probability
distribution. For a graph $G = (V, E)$ where $|V| = n$, the initial distribution can be determined by a $1 \times n$ vector $f$ where $\sum_{i=1}^{n} f_i = 1$.

**Lemma 5.4.** Let $G$ be a graph and $P$ be the corresponding transition matrix. Then the rows of $P^k$ are distributions.

**Proof.** We induct on $k$.
Base case: Let $k = 1$. By definition the rows of $P$ are distributions.
Now we suppose the statement is true for $l \geq 1$. Let $k = l + 1$. We have that $P^{l+1} = P^l P$ and by the inductive hypothesis the rows of $P^l$ are distributions. Let $P^l_{i,j}$ denote the $i,j$th element of $P^l$. The product $P^l P$ gives us

$$
\begin{pmatrix}
P^l_{1,1} & \cdots & P^l_{1,n} \\
\vdots & \ddots & \vdots \\
P^l_{n,1} & \cdots & P^l_{n,n}
\end{pmatrix}
\begin{pmatrix}
P^{i,1} & \cdots & P^{i,n} \\
\vdots & \ddots & \vdots \\
P^{n,1} & \cdots & P^{n,n}
\end{pmatrix}
= 
\begin{pmatrix}
\sum_i P^l_{i,1}P^{i,1} & \cdots & \sum_i P^l_{i,1}P^{i,n} \\
\vdots & \ddots & \vdots \\
\sum_i P^l_{n,1}P^{n,1} & \cdots & \sum_i P^l_{n,1}P^{n,n}
\end{pmatrix}
$$

Without loss of generality, let us look at the first row. The sum of all the elements in the first row can be expressed as $\sum_j \sum_i P^l_{i,j} P_{i,j}$. However this can be rearranged to be:

$$
\sum_j \sum_i P^l_{i,j} P_{i,j} = \sum_i P^k_{i,i} \sum_j P_{i,j}
= \sum_i P^l_{i,i} \cdot 1
= \sum_i P^l_{i,i}
= 1
$$

Therefore, the rows of $P^k$ are distributions.

**Theorem 5.5.** Let $G = (V, E)$ be a graph where $|V| = n$, $P$ is the transition matrix, and $f$ is the initial distribution. Then for any $k \in \mathbb{N}$, $f P^k$ is the distribution after $k$ steps.

**Proof.** We induct on $k$.
Base case: Let $k = 1$. By definition, $P$ is the transition matrix containing the probabilities of taking one step from one vertex to another. The product of $f$ and $P$ is then

$$
f P = 
\begin{pmatrix}
\sum_i f_i P_{i,1} \\
\vdots \\
\sum_i f_i P_{i,n}
\end{pmatrix}
$$
To prove this is a distribution we take the sum $\sum \sum f_i P_{i,j}$. This sum can be reordered so that

$$\sum \sum f_i P_{i,j} = \sum f_j \sum P_{j,i}$$

$$= \sum f_j \cdot 1$$

$$= \sum f_j$$

$$= 1$$

Therefore, $fP$ is the distribution after one step.

Now we suppose the statement is true for $l \geq 1$. Let $k = l + 1$. We begin with $P^l$, where $P^l_{i,j}$ is the probability of moving from $v_i$ to $v_j$ in $l$ steps. Then the probability of moving from $v_i$ to $v_m$ in $l + 1$ steps is the sum of the probability of moving from $v_i$ to $v_j$ in $l$ steps multiplied by the probability of moving from $v_j$ to $v_m$ in one step. This can be represented as

$$\sum_j P^k_{i,j} P^l_{j,m}$$

This is the formula for matrix multiplication so $P^{l+1}_{i,m} = \sum_j P^k_{i,j} P^l_{j,m}$. Therefore $P^{l+1}_{i,m}$ is the probability of moving from $v_i$ to $v_m$ in $l + 1$ steps. The proof of $fP^{l+1}$ being a distribution is similar to the proof in the base case. \qed

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References