THE TOPOLOGY OF CAYLEY GRAPHS

STEPHANIE OH

Abstract. The aim of this paper is to discuss the topological properties of Cayley graphs as a means of demonstrating connections between group theory and topology. We first briefly introduce Cayley graphs. We then establish some basic notions in algebraic topology, namely fundamental groups and related structures including covering spaces, lifts, and deck transformations. Finally, we use this foundation to explore a few illustrative examples of Cayley complexes.

Contents

1. Cayley Graphs

We begin by giving a very brief introduction to the topic of graphs with an emphasis on Cayley graphs, which will be the focus of all of our examples in section 3. We assume some familiarity with groups.

Definition 1.1. A **graph** is a pair $\Gamma = (V, E)$, where V is a set of points called **vertices** and E is a collection of vertex pairs called **edges**.

A **loop** is an edge whose associated vertices are the same. In Figure 1, the first graph contains one loop and the second and third graphs contain none.

Graphs can also have an explicit orientation to their edges. A graph is **directed** if its edges consist of ordered (rather than unordered) pairs of vertices. Figure 2 shows some examples.

One very important aspect of graphs is that they can depict relations between the elements of a group.

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FIGURE 1. Three different graphs.

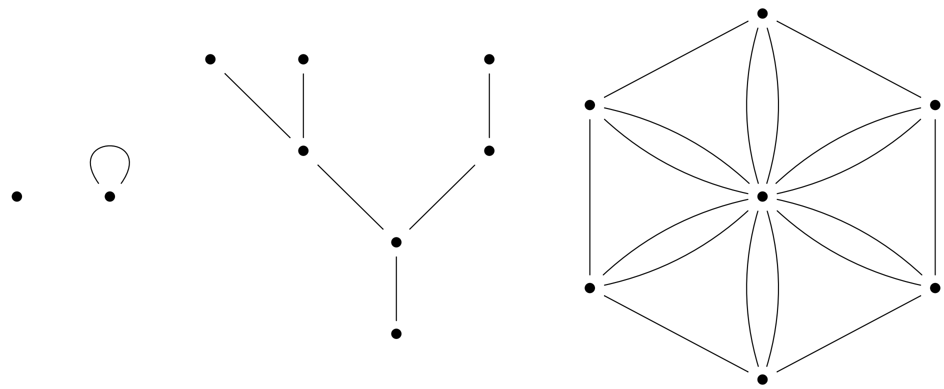
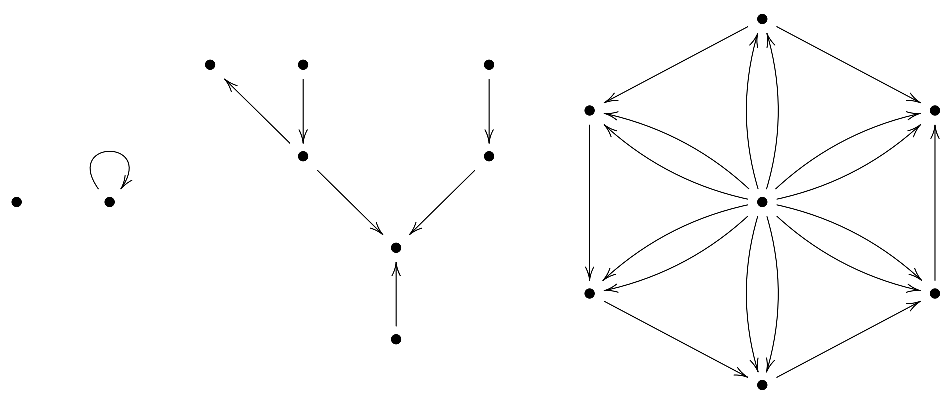


FIGURE 2. Directed versions of the graphs from Figure 1. Note that the directions chosen here are completely arbitrary.



Definition 1.2. Let G be a group and let $S \subseteq G$ be a set of generators. A **Cayley graph** is a graph where the following hold:

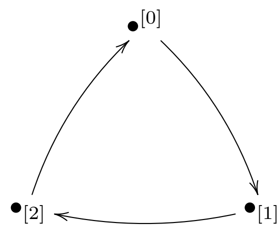
(1) $V = G$ and
(2) given any two vertices $v_1, v_2 \in V$, there is an edge from v_1 to v_2 if and only if $v_1 \cdot s = v_2$ for some $s \in S$.

In other words, the vertices of a Cayley graph are group elements and the edges between them are multiplication by group generators. The Cayley graph of a group is not necessarily unique, but depends on the choice of generating set. If a group has multiple generating sets, we will specify the one we are using. Figure 3 contains examples of Cayley graphs that we will use later in this paper.

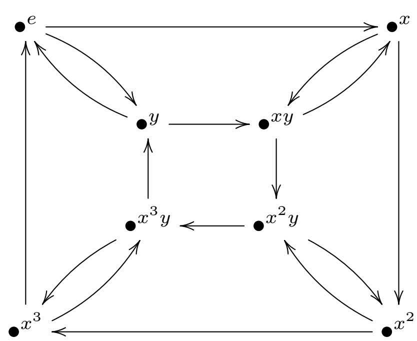
2. An Introduction to Algebraic Topology

Topology is concerned with the continuous deformations, such as shrinking, stretching, or twisting, of objects. This section establishes some elementary concepts in algebraic topology, which uses algebraic structures, such as groups, to

FIGURE 3. Here are some examples of Cayley graphs that we will encounter again in section 3.

(1) 

$\mathbb{Z}/3\mathbb{Z}$.

(2) 

The dihedral group of order 8, D_4.

study topological spaces (or vice versa). We first define the fundamental group of a space. We then present covering spaces, lifts, the Galois correspondence, and deck transformations as tools to assist with the computation of the fundamental group.

Throughout this section, X denotes a topological space. We assume some familiarity with basic properties of topological spaces. Intermediate propositions are mostly stated without proof or with proof outline; full proofs can be found in [1].

2.1. **Homotopies and the Fundamental Group.**

Definition 2.1. A **path** is a continuous map $a : [0,1] \to X$. A **loop** is a path where $a(0) = a(1) = x_0 \in X$. We call x_0 the **basepoint** of the loop a.

Note the resemblance of the definition given here to the definition of a loop of a graph in section 1. This will become important in section 3, where we will discuss Cayley graphs in the context of topology. Moving forward, we will often write (X, x_0) to refer to a space X with designated basepoint x_0.

If $a, b : [0,1] \to X$ are paths such that $a(1) = b(0)$, we define their **composition** to be

$$a \cdot b(s) = \begin{cases} a(2s) & 0 \le s \le \frac{1}{2} \\ b(2s-1) & \frac{1}{2} \le s \le 1 \end{cases}$$

In other words, $a \cdot b(s)$ is the path that first traverses a twice as fast as usual, then traverses b twice as fast as usual.

We now provide an exact definition for the notion of "continuous deformation."

Let a and b be paths on X. A **homotopy** between a and b is a function $F : [0,1] \times [0,1] \to X$ such that $F(0,t) = a(t)$ and $F(1,t) = b(t)$. We say that a and b are **homotopic**.

One can check that homotopy is an equivalence relation on paths. Using this fact, we can now define a very important group.

Definition 2.2. The **fundamental group**, denoted $\pi_1(X, x_0)$, is the set of all homotopy classes $[a]$ of loops a on X with basepoint x_0 under the operation of homotopy class composition $[a][b] = [a \cdot b]$.

In other words, the fundamental group contains all the different kinds of loops on a space with the same basepoint. We will see that there exist loops that cannot be "shrunk down" to points. Hence, the fundamental group describes the different "holes" in a space and therefore encodes its general structure.

Example 2.3. The fundamental group of $\mathbb{R}^2$ with basepoint x_0 is trivial, i.e. all loops on the real plane are homotopic to the point x_0. To understand this very informally, picture any loop in $\mathbb{R}^2$ with fixed basepoint. The path runs through the plane for some distance in some directions, then arrives back at the basepoint. Since $\mathbb{R}^2$ has no "holes," this loop can be shrunk back down to the basepoint. If we were considering, say, $\mathbb{R}^2/0$, we would not be able to shrink down those loops that wrapped around 0.

2.2. **Covering Spaces and Lifts.** This section will continue our discussion of the fundamental group by introducing some important related structures.

Definition 2.4. A **covering space** of X is a topological space $\tilde{X}$ along with a map $p : \tilde{X} \to X$ where:

(1) each $x \in X$ has an open neighborhood U in X such that $p^{-1}(U)$ is a union of disjoint open sets in $\tilde{X}$, and
(2) each open set in $p^{-1}(U)$ is mapped homeomorphically onto U by p.

We say that U is **evenly covered**. The disjoint open sets that make up $p^{-1}(U)$ are called **sheets**.

The term "covering space" gives good intuition into its meaning. Every open neighborhood around a point in X is "blanketed" by a collection of sheets in $\tilde{X}$, which are applied by the map p. As we will see, covering spaces are important because they give us key information about the spaces they cover.

Henceforth, p_* will denote the group homomorphism induced from p on the fundamental group of a covering space. When we use this notation, we will not explicitly prove that f is a homomorphism, but the interested reader can find justification in [1].

Definition 2.5. A **lift** of a map $a : Y \to X$ is a map $\tilde{a} : Y \to \tilde{X}$ such that $p\tilde{a} = a$. We say that $\tilde{a}$ **lifts** a.

As shown in Figure 4, a lift of a maps Y to the same image in X as a does, but passes through the covering space of X first. We can lift many different kinds of maps by the following four lifting properties.

Proposition 2.6. (Path-Lifting Property). Let $a : [0,1] \to X$ be a path starting at $a(0) = x_0$. If $p : \tilde{X} \to X$ is a covering space with some $\tilde{x_0} \in \tilde{X}$ such that

FIGURE 4. A diagram of a lift $\tilde{a}$ of a.

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{a}"] \arrow[r, "a"'] & X \end{tikzcd}$$

$p(\tilde{x_0}) = x_0$, then there exists a unique path $\tilde{a} : [0,1] \to \tilde{X}$ starting at $\tilde{a}(0) = \tilde{x_0}$ that lifts a (i.e., $p\tilde{a} = a$).

Proposition 2.7. (Homotopy-Lifting Property) Let $a, b : [0,1] \to X$ be homotopic paths starting at $a(0) = b(0) = x_0$ with homotopy $F : [0,1] \times [0,1] \to X$. If $p : \tilde{X} \to X$ is a covering space, then there is a unique homotopy $\tilde{F}$ from the lifts $\tilde{a}$ of a and $\tilde{b}$ of b to $\tilde{X}$ starting at some $\tilde{x_0} \in \tilde{X}$ with $p(\tilde{x_0}) = x_0$.

These lifting properties give us the useful fact that if $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ is a covering space of X, the induced map $p_*(\pi_1(\tilde{X}, \tilde{x_0}))$ is injective. To see this, consider a loop $\tilde{c} \in \pi_1(\tilde{X}, \tilde{x_0})$ for which $p\tilde{c}$ is homotopic to the constant loop 0 in $\pi_1(X, x_0)$ via a homotopy F. By the above properties, we can uniquely lift F to a homotopy starting at the constant loop $\tilde{0}$ in $\pi_1(\tilde{X}, \tilde{x_0})$ and ending at $\tilde{c}$. This implies that the kernel of p_* is $[0]$, the homotopy class of the constant loop in $\pi_1(\tilde{X}, \tilde{x_0})$. Since p_* is a homomorphism, this is enough to conclude that it is injective.

The following lifting property generalizes the types of maps we can lift. It is important enough that we give a full proof.

Proposition 2.8. (Lifting Criterion) Let $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ be a covering space with $p(\tilde{x_0}) = x_0$. Let $a : Y \to X$ be a map where Y is path-connected and $a(y_0) = x_0$ is the starting point of a. A lift $\tilde{a} : (Y, y_0) \to (\tilde{X}, \tilde{x_0})$ of a exists if and only if $a_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x_0}))$.

Proof. Let γ be a path from y_0 to some $y_1 \in Y$. By the path-lifting property, the path $a\gamma$ starting at x_0 can be lifted uniquely by $\tilde{a}$ to a path $\widetilde{a\gamma}$ in $\tilde{X}$ starting at $\tilde{x_0}$. Define $\tilde{a}(y_1) = \widetilde{a\gamma}(1)$, i.e. the new path $\widetilde{a\gamma}$ starts at $\tilde{x_0}$ and ends at $\tilde{a}(y_1)$. We need to check that this equation holds regardless of what path γ we choose, fixing endpoints. Let γ' be a different path in Y from y_0 to y_1. We can define a loop h with basepoint x_0 that first traverses $a\gamma'$, then traverses $a\gamma$ backwards. Since $[h] \in a_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x_0}))$, there is a homotopy H from h_0 to h_1, where h_1 is a loop that lifts to a loop $\tilde{h_1}$ in $\tilde{X}$ with basepoint $\tilde{x_0}$. By the homotopy lifting property, there is a lift $\tilde{H}$ of H from loops $\tilde{h_0}$ to $\tilde{h_1}$. Since paths are uniquely lifted by the path-lifting property, $\tilde{h_0}$ is the loop that first traverses $\widetilde{a\gamma'}$ up to $\widetilde{a\gamma}(1) = \widetilde{a\gamma'}(1)$, then traverses $\widetilde{f\gamma}$ backwards. Thus, our desired equation holds.

To finish, we must check that $\tilde{a}$ is continuous at y. We need to show that for every open neighborhood $\tilde{U}$ of $\tilde{a}(y)$, there is an open neighborhood V of y such that $\tilde{a}(V) \subseteq \tilde{U}$. Let $U \subseteq X$ be an open neighborhood of $a(y)$. Suppose U has a lift $\tilde{U} \subseteq \tilde{X}$ containing $\tilde{a}(y)$ such that $p : \tilde{U} \to U$ is a homeomorphism. Let V be a path-connected open neighborhood of y such that $a(V) \subseteq U$. Consider an arbitrary path in Y from y_0 to $y' \in V$. Define γ to be a path from y_0 to y and η to be a path from y to y'. By the path-lifting property, the paths $(a\gamma) \cdot (a\eta)$ lift to

$(\tilde{a\gamma}) \cdot (\tilde{a\eta})$ where $\tilde{a\eta} = p^{-1}a\eta$. Hence, $\tilde{a}(V) \subseteq \tilde{U}$ and $\tilde{a}|V = p^{-1}a$, implying that $\tilde{a}$ is continuous at y. $\square$

The final lifting property tells us that the lift of a path that runs through a specified point is unique.

Proposition 2.9. (Unique Lifting Property) Let $p : \tilde{X} \to X$ be a covering space and let $a : Y \to X$ be a map with Y connected. If $\tilde{a_1}$ and $\tilde{a_2}$ are lifts with $\tilde{a_1}(y) = \tilde{a_2}(y)$ for some $y \in Y$, then $\tilde{a_1} = \tilde{a_2}$ on all of Y.

Proof. Let $y \in Y$ be arbitrary. Let U be an open neighborhood of $a(y)$ evenly covered by p. Consider the sheets $\tilde{U_1}$ and $\tilde{U_2}$ of $p^{-1}(U)$ where $\tilde{a_1}(y) \in \tilde{U_1}$ and $\tilde{a_2}(y) \in \tilde{U_2}$. Since $\tilde{a_1}$ and $\tilde{a_2}$ are continuous, there exists some neighborhood N of y such that $\tilde{a_1}(N) \subseteq \tilde{U_1}$ and $\tilde{a_2}(N) \subseteq \tilde{U_2}$. Thus, if $\tilde{a_1}(y) \neq \tilde{a_2}(y)$, then $\tilde{U_1} \neq \tilde{U_2}$ and $\tilde{a_1} \neq \tilde{a_2}$. If $\tilde{a_1}(y) = \tilde{a_2}(y)$, then $\tilde{U_1} = \tilde{U_2}$. Since p is injective, this implies that $p\tilde{a_1} = p\tilde{a_2}$ and $\tilde{a_1} = \tilde{a_2}$. $\square$

Thus, lifts allow us to establish an association between paths on a space and a covering space.

2.3. **The Galois Correspondence and the Universal Covering Space.**

A space can have many—or even infinite—different covering spaces. This section is devoted to developing an important bijection, the Galois correspondence, which associates each subgroup of the fundamental group of a space with a covering space so long as the covered space meets some conditions. The following describes one of these conditions.

Definition 2.10. Let X be a space such that every $x \in X$ has an open neighborhood U around it in which the map $\pi_1(U, x) \to \pi_1(X, x)$ is trivial. We say that X is a **semilocally simply connected** space.

Theorem 2.11. (The Galois Correspondence) Let X be path-connected, locally path connected, and semilocally simply connected. For every subgroup $H \subseteq \pi_1(X, x_0)$, there exists a covering space $p : X_H \to X$ unique up to isomorphism such that $p_*(\pi_1(X_H, x_H)) = H$ for some $x_H \in X_H$.

Proof. We will give a broad outline of the proof of this key result. First, we want to show that every subgroup H of $\pi_1(X, x_0)$ can be identified with a covering space of X through $p_*(\pi_1(X_H, x_H)) = H$. We start with the case where H is the trivial subgroup. Equivalently, we want to find a covering space where all loops in its fundamental group are homotopic to the constant loop, i.e. can be shrunk down to their basepoint. The following definition describes this type of space.

Definition 2.12. A space is **simply connected** if it is path-connected and if it has a trivial fundamental group.

We propose that our simply connected covering space is $\tilde{X} = \{[\gamma]\}$, where γ is a path in X with starting point x_0. Take $p : \tilde{X} \to X$ to be the function that maps $[\gamma]$ to $\gamma(1)$, the fixed endpoint of $[\gamma]$ in X. By path-connectedness of X, we can take any point of X to be $\gamma(1)$. This implies that p is surjective. Let $\mathcal{U}$ be the topology on X consisting of the path-connected open sets U whose fundamental groups map trivially to the fundamental groups of X. If $U \in \mathcal{U}$ and γ is a path in X starting at x_0 and ending in U, define $[\gamma] = \{[\gamma \cdot \eta]\}$ where η is a path in U that starts at the endpoint of γ. We will take the collection of $U_{[\gamma]}$ to be our topology on $\tilde{X}$. Informally, this mirrors the notion of closeness given by the topology on X.

Now that we have described what we want our covering space to look like, we need to check that it is actually simply connected. To see that $\tilde{X}$ is path-connected, pick any $[\gamma] \in \tilde{X}$ and let γ_t be the path in X that first traverses γ on the interval $[0, t]$ and is stationary on $[t, 1]$. There is a lift of this path to $\tilde{X}$ that is a path starting at $[x_0]$ and ending at $[\gamma]$. This tells us that we can draw a path from $[x_0]$ to any other point in $\tilde{X}$, so $\tilde{X}$ is path-connected. Next, we have to prove that the fundamental group of $\tilde{X}$ at basepoint $[x_0]$ is trivial. Since p_* is injective, this is equivalent to proving that $p_*(\pi_1(\tilde{X}, [x_0])) = 0 \in X$. Consider an arbitrary element of the image of p_*, the loop γ with basepoint x_0. As before, we can lift γ to the path $[\gamma_t]$. Since this lift is a loop, we have that its basepoint is $[\gamma_1] = n[x_0] = [\gamma]$. Therefore, γ is homotopic to the point x_0. Since we chose γ arbitrarily, $p_*(\pi_1(\tilde{X}, [x_0])) = 0 \in X$ as desired.

We have completed the construction of our simply connected space $\tilde{X}$. Even though it was only meant to give us the covering space corresponding to the trivial subgroup, we can actually derive the covering spaces for all the other subgroups from $\tilde{X}$ as well. Let $[\gamma], [\gamma'] \in \tilde{X}$ be arbitrary. Define $[\gamma] \sim [\gamma']$ if $\gamma(1) = \gamma'(1)$ and if $[\gamma \cdot (\gamma')^{-1}]$, where $(\gamma')^{-1}$ is γ' traversed backwards. One can check that this is an equivalence relation. Let $X_H = \tilde{X}/\sim$. If $[\gamma] \sim [\gamma']$, then $[\gamma \cdot \eta] \sim [\gamma' \cdot \eta]$. In other words, if two paths are equivalent and another path is attached to each of their endpoints, the resulting new paths are also equivalent. This implies that X_H with a mapping from $[\gamma]$ to $\gamma(1)$ is a covering space. Associate $\tilde{x_0}$ with $[c] \in (X, x_0)$ where c is the constant loop at x_0. A loop $\gamma \in X$ with basepoint x_0 then lifts to a loop in $\tilde{X}$ with basepoint $[\gamma] \sim [c]$. Thus, $[\gamma] \in H$ and we have $p_*(\pi_1(X_H, \tilde{x}_0)) = H$.

We have demonstrated the existence of an association between subgroups of the fundamental group of X and covering spaces of X via p_*. All that is left to verify is the uniqueness of this association. First we must define what it means for covering spaces to be equivalent, or isomorphic, in this context.

Definition 2.13. Two covering spaces, $p_1 : \tilde{X_1} \to X$ and $p : \tilde{X_2} \to X$, are **isomorphic** if there exists a homeomorphism $\tau : \tilde{X_1} \to \tilde{X_2}$ such that $p_1 = p_2\tau$.

Moreover, $p_1 : \tilde{X_1} \to X$ and $p_2 : \tilde{X_2} \to X$ are isomorphic if and only if the images of their fundamental groups are the same. To see the forward direction of this statement, let $p_{1*}(\pi_1(\tilde{X_1}, \tilde{x_1})) = p_{2*}(\pi_1(\tilde{X_2}, \tilde{x_2}))$. By the lifting criterion, p_1 lifts to a map $\tilde{p_1} : (\tilde{X_1}, \tilde{x_1}) \to (\tilde{X_2}, \tilde{x_2})$ where $p_2\tilde{p_1} = p_1$ and p_2 lifts to a map $\tilde{p_2} : (\tilde{X_2}, \tilde{x_2}) \to (\tilde{X_1}, \tilde{x_1})$ where $p_1\tilde{p_2} = p_2$. By the unique lifting property, $\tilde{p_1}\tilde{p_2} = \tilde{p_2}\tilde{p_1}$ are the identity functions of their respective spaces. Therefore, $\tilde{p_1}$ and $\tilde{p_2}$ are inverses and isomorphic. For the converse, suppose $f : (\tilde{X_1}, \tilde{x_1}) \to (\tilde{X_2}, \tilde{x_2})$ is an isomorphism between the two covering spaces. Since $p_1 = p_2 f$ and $p_2 = p_1 f^{-1}$, we immediately get that $p_{1*}(\pi(\tilde{X_1}, \tilde{x_1})) = p(\pi(\tilde{X_2}, \tilde{x_2}))$. Thus, if two covering spaces are isomorphic, they correspond to the same subgroup of $\pi(X, x_0)$. This concludes our proof of the existence of the Galois correspondence. □

Given how we obtained the covering spaces X_H by quotienting the simply connected covering space $\tilde{X}$ by $\sim$, it is perhaps a not-so-surprising fact that the simply connected covering space of X is a covering space for every other covering space of X. Therefore, $\tilde{X}$ is called the **universal cover** of X. It is unique up to isomorphism.

FIGURE 5. A diagram of a deck transformation τ.

$$\begin{tikzcd}
\tilde{X} \arrow[rr, "\tau"] \arrow[dr, "p"'] & & \tilde{X} \arrow[dl, "q"] \\
& X &
\end{tikzcd}$$

2.4. **Deck Transformations.** Building on the covering space theory we have been developing, we now present a final technique for computing fundamental groups of spaces. Once again, we assume some knowledge of groups.

Definition 2.14. Let $p : \tilde{X} \to X$ be a covering space. An isomorphism $\tilde{X} \to \tilde{X}$ is called a **deck transformation**.

Deck transformations can be thought of as "shufflings" of different covering spaces $\tilde{X}$, where the spaces are analogous to cards in a deck. This general idea is depicted in Figure 5. One can check that the set of deck transformations of a covering space under composition is a group, which we will denote by $G(\tilde{X})$. We have actually already dealt with deck transformations in the previous section, when we defined what it meant for two covering spaces to be isomorphic.

The following definition describes covering spaces that have maximal symmetry, that is,

Definition 2.15. A covering space $p : \tilde{X} \to X$ is **regular** if for each $x \in X$ and each pair of lifts $\tilde{x}, \tilde{x}'$ of x, there is a deck transformation $\tau : \tilde{X} \to \tilde{X}$ such that $\tau(\tilde{x}) = \tilde{x}'$.

In other words, the lifts of an evenly covered point to a regular covering space differ by deck transformations. This next proposition gives another way of precisely identifying regular covering spaces.

Proposition 2.16. *Let X be path-connected and locally path-connected with covering space $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$. Suppose $H = p_*(\pi_1(\tilde{X}, \tilde{x_0}))$ is the subgroup of $\pi_1(X, x_0)$ corresponding to $\tilde{X}$. $\tilde{X}$ is a regular covering space if and only if $H \subseteq \pi_1(X, x_0)$ is a normal subgroup of X.*

Proof. To start, suppose H is a normal subgroup of X. Let $\tilde{x_1}$ be another element of $p^{-1}(x_0)$. We want to show that there is a deck transformation τ such that $\tau(\tilde{x_1}) = \tilde{x_0}$. We propose that the lift $\tau : (\tilde{X}, \tilde{x_1}) \to (\tilde{X}, \tilde{x_1})$ satisfies this requirement. First, we must show that this lift actually exists. Let γ be a loop in (X, x_0) whose lift to $\tilde{X}_0$ is the path $\tilde{\gamma}$ from $\tilde{x_0}$ to $\tilde{x_1}$. Let $\gamma' \in \tilde{X}$ be any loop with basepoint $\tilde{x_0}'$. We have that $\tilde{\gamma}\gamma'\tilde{\gamma}^{-1}$ is a loop with basepoint $\tilde{x_0}$, so $[\gamma][p\gamma'][\gamma^{-1}]$ is in H. Because H is normal, it also contains $[\gamma]^{-1}[\gamma][p\gamma'][\gamma]^{-1}[\gamma]$. Thus, $[p\gamma']$ is in H and $p_*\pi_1(\tilde{X}, \tilde{x_1}$ is a subset of H. This means that we can apply the lifting criterion, which tells us that τ exists. Without loss of generality, we also have $\tau' : (\tilde{X}, \tilde{x_0}) \to (\tilde{X}, \tilde{x_1})$, a lift of $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$. We have that $\tau\tilde{\tau} = \tilde{\tau}\tau$ is the identity map of covering spaces, so τ is a homeomorphism and is the deck transformation we desire.

For the other direction, assume p is regular. Let $h \in H$ and $g \in \pi_1(X, x_0)$ be arbitrary. We want to show that H is invariant under conjugation by G, or that $ghg^{-1} \in p_*\pi_1(\tilde{x}, \tilde{x_0})$. Consider the loop $\gamma \in X$ with basepoint x_0 such that $[\gamma] = g$. By the path-lifting property, there exists a path $\tilde{\gamma}$ starting at x_0 that lifts

γ. Furthermore, there exists a loop γ' in $\tilde{X}$ with basepoint $\tilde{x_0}$ such that $[p\gamma'] = h$. Since p is regular, there exists a deck transformation τ that takes $\tilde{x_0}$ to $\tilde{\gamma}(1)$, which in turn takes $\gamma]$ to the loop $\tau\gamma'$. Thus, $\gamma(p\gamma')\gamma^{-1}$ lifts to the loop $\gamma(\tau\gamma')\tilde{\gamma}^{-1}$ and $ghg^{-1} = [\gamma(p\gamma')\gamma^{-1}]$ is an element of H. □

The next theorem gives us a important relation between the fundamental group of a space and the deck transformation group of its regular covers.

Theorem 2.17. *Let p, X, $\tilde{X}$, and H be defined as in Proposition 2.17. If $\tilde{X}$ is a regular covering space of (X, x_0), then $G(\tilde{X})$ is isomorphic to $\pi_1(X, x_0)/H$.*

It follows immediately that if $\tilde{X}$ is the universal cover of X, then $G(\tilde{X})$ is isomorphic to $\pi_1(X)$.

As G is a group, we can discuss the group actions of G on $\tilde{X}$.

Definition 2.18. Let Y be a space acted on by a group G. Actions of G are called **covering space actions** if each $y \in Y$ is contained by a neighborhood U for which $g_1(U) \cap g_2(U) \neq \emptyset$ implies $g_1 = g_2$.

Proposition 2.19. *Let G be a group whose elements are covering space actions on a space Y. If Y is path-connected and locally path-connected, then G is isomorphic to $\pi_1(Y/G)/p_*(\pi_1(Y))$.*

We will use these findings in the next and final section.

3. Cayley Graphs as Topological Spaces

We are ready to perform some computations. We will take the Cayley graphs we saw in Figure 5 and apply the topological theory we developed in Section 2 through the construction of Cayley complexes. Our emphasis here will not be on rigor but on gaining insight into the algebraic properties of the groups encoded by these Cayley graphs. We begin with some new terminology.

Definition 3.1. An n-dimensional open cell, or **n-cell**, is a topological space homeomorphic to the n-dimensional open ball.

For our purposes, all that is necessary to know about n-cells is that a 0-cell is a point, a 1-cell is a line segment, and a 2-cell is a polygon. We can "glue" together these cells to build a new topological space:

A **cell complex** X is a space constructed by taking a set of 0-cells, attaching 1-cells to the 0-cells, then attaching 2-cells to the 1-cells.

In algebraic topology, graphs are considered cell complexes, where the vertices are 0-cells and the attached edges are 1-cells. Thus, the Cayley graph of a group is still drawn in the same way. The main structure we will be working with in this section is related to the Cayley graph. It will allow us to visualize the relationships between a space and its covering spaces.

Definition 3.2. The **Cayley complex** $\tilde{X}$ of a group X is the Cayley graph of X that has a 2-cell attached by its boundary to each loop at each vertex.

We use the notation $\tilde{X}$ in this definition because the Cayley complex is a simply connected covering space, and is thus the universal cover of X. We know from our previous discussion that if $\tilde{X}$ is the universal cover of X, then the deck transformation group $G(\tilde{X})$ is isomorphic to $\pi_1(X)$. Moreover, if H is a subgroup of $\pi_1(X)$

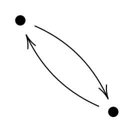
corresponding to some covering space $X_H = \tilde{X}/H$, then $\pi_1(\tilde{X}/H)$ is isomorphic to H. With this, we can now look at a couple of concrete examples.

(1) $X = \mathbb{Z}/3\mathbb{Z}$

The graph of X in Figure 5 has three vertices and one loop that can be based at each of them. Thus, we see that the Cayley complex $\tilde{X}$ is three disks (two-cells) glued to each other on top of the graph. The deck transformation group of $\tilde{X}$ is $\mathbb{Z}/3\mathbb{Z}$. It acts on the disks in $\tilde{X}$ by rotations of $\frac{2\pi n}{3}$.

(2) $X = D_4$

Consider the subgroup of D_4 generated by 90 degree rotations of the square, or $\{1, x, x^2, x^3\}$. This subgroup is isomorphic to $\mathbb{Z}/4\mathbb{Z}$, and we will refer to it as such. Looking at the graph of X in Figure 5, quotienting $\tilde{X}$ by $\mathbb{Z}/4\mathbb{Z}$ shrinks down the inner and outer squares of the graph whose sides correspond to multiplications by x and deletes the three "petals" corresponding to multiplications by y that are not attached to e:



Thus, $\tilde{X}/\mathbb{Z}/4\mathbb{Z}$ is the above graph $\mathbb{Z}/2\mathbb{Z}$ with four 2-cells attached. This is the covering space of $\mathbb{Z}/4\mathbb{Z}$. The fundamental group of this new complex is exactly $\mathbb{Z}/4\mathbb{Z}$.

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