MORSE THEORY AND BOTT PERIODICITY

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Abstract. In this paper, we will aim to prove the celebrated Bott periodicity theorem, which calculates the homotopy groups of a unitary group in arbitrary dimension. We will go about doing this through a study of Morse theory. Morse theory allows us to study the structure of a manifold based on a function defined on it. We then turn our attention to Riemannian geometry and the loop space of a manifold, and focus on topologizing it. We can then study the structure of the path space of a manifold by using similar methods to Morse theory on manifolds. We can then connect these ideas about the path space of a manifold to homotopy groups of various spaces and then use these facts to prove the Bott Periodicity theorem.

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1. Basic Morse Theory

Morse theory studies the critical points of real-valued functions on manifolds. In this section, we aim to use the properties of critical points to present a nice decomposition of a smooth manifold M. Consider a function $f : M \to \mathbb{R}$. At each point p on M, f induces a map $f_* : TM_p \to T\mathbb{R}_{f(p)}$ between the tangent spaces of M and $\mathbb{R}$.

Definition 1.1. $p \in M$ is a critical point of f if the induced map f_* is zero. More specifically, p satisfies $\frac{df}{dx^1}(p) = \frac{df}{dx^2}(p) = ... = 0$ if $(x^1, ..., x^n)$ is the local coordinate system at p.

An important characteristic of critical points is whether they're degenerate. Regarding this concept, we shall make a few more definitions.

Definition 1.2. The Hessian $H_f(p)$ of a function $f : M \to \mathbb{R}$ at p is the $n \times n$ matrix whose ij-th entry is $\frac{\partial^2 f}{\partial x^i \partial x^j}$, where $(x^1, ..., x^n)$ is the local coordinate system in a neighborhood of p.

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Definition 1.3. A critical point p is nondegenerate if the matrix $H_f(p)$ is nonsingular.

To further study $H_f(p)$, we define two quantities regarding functionals.

Definition 1.4. The index of a functional H on a vector space V is the maximal dimension of subspaces of V on which H is negative definite. In particular, the index of $H_f(p)$ is called the index of f at p.

Definition 1.5. The nullity of a functional H is the dimension of the null space of H.

By our definition, p is a nondegenerate critical point iff $H_f(p)$ has nullity 0.

Definition 1.6. $f : M \to \mathbf{R}$ is a Morse function if f is continuous and all of its critical points are nondegenerate.

We are now ready to prove the Morse Lemma, which is a very important tool in Morse theory because it gives us a nice coordinate system to work with near a critical point. Furthermore, we only need information about the index of f at p to apply this theorem. It will allows us to more easily prove theorems that involve working in a neighborhood of a critical point.

Theorem 1.7. *(Morse Lemma) If p is a nondegenerate critical point of $f : V \to R$ and the index of f at p is λ, then there exists neighborhood U of p and local coordinate $(y^1, y^2, ...y^n)$ such that there is a map $\psi : U \to R^n$ satisfying*

$$f^*\psi^{-1} = f(p) - (y^1)^2 - ... - (y^\lambda)^2 + ... + (y^n)^2.$$

Proof. First, we can assume $V = R^n$ as we can map V by a chart to R^n and it does not change what U can map into. We can furthermore translate the critical point so that it is located at 0. Finally, we can assume that the hessian of f at 0 is a diagonal matrix. We can make this assumption as f is smooth, so the hessian at 0 is a symmetric matrix and is therefore diagonalizable. We will now prove this theorem by induction on the dimension of V.

We start with the base case V=R. In this space,

$$f(x) = f(0) + \frac{1}{2}f''(0)x^2 + \epsilon(x)x^2.$$

Where ϵ(x)=$\frac{1}{2}$ $\int_0^x f(t)^3(x-t)^2dt$. We get this equality by Taylors theorem. We can then rewrite f as $f(0) + ax^2(1 + \epsilon(x))$. Then let $X_1 = \phi(x) = x\sqrt{a(1 + \epsilon(x))}$, where ϕ(x) is a diffeomorphism, and $\phi'(0)$=$\sqrt{a}$, which is nonzero. We can now use the implicit function theorem to see that in some neighborhood of 0 we have $f * \phi^{-1}(X_1) = f(X) = f(0) \pm x^2$. So we have proven our base case.

In order to show that the lemma holds for V=$\mathbb{R}^{n+1}$ if it is true for V=$\mathbb{R}^n$, it is useful to rewrite $\mathbb{R}^{n+1}$ as $\mathbb{R}^n \times \mathbb{R}$. We denote a point in $\mathbb{R}^n \times \mathbb{R}$ as (x,y) with x$\,\epsilon\,\mathbb{R}^n$ and y$\,\epsilon\,\mathbb{R}$ let f(x,y)=$f_x(y)$. We can then taylor expand this function with respect to the one variable y. The taylor expansion is

$$f(x, y) = f_y(0) + f'_y(0)x + \frac{1}{2}f''_y(0)(x^2) + x^2 + \epsilon(x, y).$$

If $f_y'(0)$=0, then this proof is identical to the first case. Then to prove this theorem, we must study the equation $\frac{df}{dx}$(x,y)=0. As we put the hessian of f at 0 in diagonal form, we know that it is nonzero. Then we can apply the implicit function theorem to see that the solutions of our equation are given by smooth functions in a neighborhood of (0,0). Also, note that the derivative with respect to y of $\frac{df}{dx}$ is 0 at (0,0) by our observations about the second derivative.

Now we can make a change of variable to complete the theorem. let ϕ be the function given by the implicit function theorem,then as $\frac{df}{dx}$ is 0, we must also have dϕ is 0. Now consider $g = f(x + \phi(y), y)$. We have $\frac{dg}{dx}$=0 as dϕ=0. We also have $d^2g_{0,0}$=$d^2\mathrm{f}_{0,0} = 0$. We are now in the desired case and the theorem is proven.

Finally, we show that if there is a λ so that f takes the form

$f = f(p) - (y^1)^2 - ... - (y^\lambda)^2 + ... + (y^n)^2$, then λ must be the index of the f at p. If f did take this form, then we can take derivatives directly to see that $\frac{d^2f}{dy^idy^j}$ is -2 for i=j λ, 2 for i=j $> \lambda$ and is zero for all other values of i and j. Then the matrix corresponding to f_{**} has -2 as the first λ entries of the diagonal, and 2 as the rest of the diagonal entries, All nondiagonal entries are 0. Then the largest dimension subspace where H_fp is negative definite must have dimension λ. □

Now that we have proved the Morse lemma, we can prove two theorems that together classify the homotopy type of M^a, for any value of a. First, we have to build up two more tools:

Definition 1.9. A 1-parameter group of diffeomorphisms on a manifold M is a set of diffeomorphisms ϕ: R × M → M, that satisfy $\phi_t(m)\phi_s(m)$=$\phi_{t+s}(m)$.

We will use this in the following theorem to get a vector field X. For a function f and at a point p, It is defined as

$$X_p(f) = lim_{x\to 0}\frac{f(\phi_s(m)) - f(m)}{m}$$

X is said to generate the group. We will need one more fact about 1-parameter groups, which we will not prove.

Theorem 1.10. *A smooth vector field on a manifold M that vanishes outside of some compact subset of M generates a 1-parameter group of diffeomorphisms.*

Definition 1.11. Let $f : M \to \mathbb{R}$ be a Morse function on a manifold M, a pseudo gradient field is a vector field X on M with the following properties.

1.$(df)_x(X_x) \leq$0 and there is equality iff x is a critical point.

2. In a neighborhood given by the Morse lemma, X coincides with the gradient of the euclidean metric given by $< grad_f, Y >$.

We can then use the flow of a pseudo gradient field to prove the following statement. The theorem is significant as it shows us the homotopy type of M^a can only change if a moves past a critical point, we will investigate the effects of when a moves past a critical point after this theorem.

Theorem 1.12. *If f is a smooth real valued function on M, $a \leq b$ and $f^{-1}[a,b]$ is compact and contains no critical points of f, then M^a is diffeomorphic to M^b. In fact, M^a is a deformation retract of M^b.*

Proof. The idea is to use the gradient of f to construct vector field that gives us our desired diffeomorphism. To do this, we define a function $p : M \to \mathbb{R}$ as $\frac{1}{<grad(f),grad(f)>}$ on $f^{-1}[a,b]$, and 0 outside some compact set containing $f^{-1}[a,b]$. Define X as $X_p = p(q)(grad(f))_q$. Then by theorem 1.10, X generates a 1 parameter group of diffeomorphisms by $X_p(f) = \lim_{x\to 0} \frac{f(\phi_h(q))-f(q)}{x}$. By this definition, we also have $\frac{d\phi_t(q)}{dt} = X_{\phi_t(q)}$.

We now use the flow of this gradient field to create this retraction. For q in M, consider the function taking t to $f(\phi_t(q))$. If $\phi_t(q)$ is in $f^{-1}[a,b]$, then

$$\frac{df(\phi_t(q))}{dt} =< \frac{d\phi_t(q)}{dt}, gradf >=< X, gradf >= 1$$

by the definition of X.

So the map that takes t to $f(\phi_t(q))$ always has derivative 1.

So $f(\phi_t(q)) = f(q) + t$. Then ϕ_{b-a} is the diffeomorphism we are looking for. We now focus on the diffeomorphism. Let $r_t(q)$ be q if $f(q) \leq a$, and is $\phi_{t(a-f(q))}(q)$ if $a \leq f(q) \leq b$.

This is the desired deformation retract as for $r_0(q)$, $\phi_0(q)$=q. Then r_0 is the identity map. For r_1, $\phi_{a-f(q)}(q)$ only takes values in $f^{-1}(a)$ so that we have a retraction. □

With the next theorem, we will have completely characterized the homotopy type of M^a based on a Morse function f defined on it.

Theorem 1.13. *Let $f : M \to \mathbb{R}$ be a smooth function, and let p be a nondegenerate critical point of f with index λ. if $f(p) = c$, then for some $\epsilon > 0$, $f^{-1}[c-\epsilon, c+\epsilon]$ is compact. It also contains no critical points of f other then p. Then for all sufficiently small ϵ, $M^{c+\epsilon}$ has the homotopy type of $M^{c-\epsilon}$ with a λ cell attached.*

Proof. We first modify f to a new function, F, that agrees with f except for in a small neighborhood of c. Then, when we look at those x such that $F(x) \leq a$ there will be an extra portion that M^a will not have. Studying this extra portion will allow us to prove the theorem.

We now construct F. By the Morse lemma, we can set

$f = c - (x^1)^2 - ... - (x^\lambda)^2 + ... + (x^n)^2$, where the x^i are local coordinates in a neighborhood U of p. We first construct a function g so that:
1. $g(0) > \epsilon$
2. $g(r) = 0$ for $r > 2\epsilon$
3. $-1 < g'(r) \leq 0$ for all r.

In a small neighborhood U around the critical point, let

$$F = f - g((u^1)^2 + ... + (u^\lambda)^2) + 2((u^{\lambda+1})^2 + ... + (u^n)^2).$$

To make notation easier, set $\xi = (u^1)^2 + ... + (u^\lambda)^2$ and $\eta = (u^{\lambda+1})^2 + ... + (u^n)^2$. In the neighborhood U, $f = c - \xi + \eta$ and $F = c - \xi(q) + \eta(q) - g(\xi(q) + 2\eta(q))$. We still need to show that outside some neighborhood, F and f are equal. This happens outside the region $\xi + 2\eta \leq 2\epsilon$, because g is zero for $r > 2\epsilon$.

Now we compare the functions inside this region. Note that in this case, g is at least 0 as g' is between -1 and 0, and if g were less than 0, then g could not be 0 outside the region. Then $F = c - \epsilon(q) + \eta(q) - g(\xi(q) + 2\eta(q)) \leq c - \epsilon + \eta = f$ so $F \leq f$.

The main reason for defining F in this way is because it has the same critical points as f. We will now show this by computing

$dF = (-1 - g'(\xi + 2\eta))d\xi + (1 - 2g'(\xi + 2\eta))d\eta$ as obtained by taking derivatives of F with respect to ξ and η.

Note that $-1 - g'(\xi + 2\eta) < 0$, as $-1 < g' \leq 0$ and $1 - 2g'(\xi + 2\eta)$ is at least 1 for the same reason. Then dF is 0 in the region $\xi + 2\eta \leq 2\epsilon$ iff $d\xi$ and $d\eta$ are both 0. Then F has no critical points in U other than the origin, which was the only critical point of f in U. Then f and F have the same critical points. Note also that $F(p) = c - g(0) < c - \epsilon$, so that F has no critical points in the region $[c-\epsilon, c+\epsilon]$.

We can now use the properties of F to show a nice deformation retract. Let e^λ be the region with $\xi \leq \epsilon$ and for any $i > \lambda$, u^i=0. Note that by the definition of e^λ, $e^\lambda \cap M^{c-\epsilon}$ is the boundary of e^λ. We will now show that $F^{-1}[-\infty, c-\epsilon]$ is a deformation retract of $M^{c+\epsilon}$.

As $\frac{df}{d\xi} < 0$, then $F(q) < F(p) < c - \epsilon$. But $f(q)$ is at least $c - \epsilon$ in this region. Then by our previous lemmas, since $F^{-1}(-\infty, c-\epsilon) \supseteq M^{c+\epsilon}$ and contains the critical point p, then we have the desired deformation retract.

We are now in a position to prove the original theorem. All that is left to show is that $M^{c-\epsilon} \cup e^\lambda$ is a deformation retract of $M^{c-\epsilon} \cup H$, where H is the closure of $F^{-1}(-\infty, c-\epsilon] - M^{c-\epsilon}$

We now define the retraction. It will take different values on three regions.

We will map the region $\xi \leq \epsilon$ into e^λ.

We will map the region $\epsilon \leq \xi \leq \epsilon\eta$ into $M^{c-\epsilon}$.

We will map the region $\eta + \epsilon \leq \xi$ into $M^{c-\epsilon}$.

Note that if $\eta + \epsilon \leq \xi$ then $F = c - \xi(q) + \eta(q) - g(\xi(q) + 2\eta(q)) \leq c - \epsilon - g(\xi(q) + 2\eta(q)$, but as g(r) is at least 0, then F(q)$< c - \epsilon$ in the region $\eta + \epsilon \leq \xi$.

Then this region is a subset of $M^{c-\epsilon}$.

When $\xi \leq \epsilon$, then let $r_t(\epsilon, \eta)$=$(\epsilon, t\eta)$. Then r_1 is the identity and r_0 carries the region into e^λ.

If $\epsilon \leq \xi \leq \epsilon\ \eta$, let $r_t(\xi,\eta)$=(ξ,t+(1-t)$\frac{\sqrt{\xi-\epsilon}}{\eta}$).
Finally, If $\eta+\epsilon \leq \xi$ then we can let r_t be the identity.
All that remains to show is that this retraction is continuous. Note that for the region $\epsilon \leq \xi \leq \eta+\epsilon$, if we let ξ approach ϵ then s_t becomes t and we are in the first case. If we let ξ approach $\eta+\epsilon$, then s_t becomes 1 and we are in the 3rd case so our defined function is continuous. Our deformation retract is now complete.

□

2. Preliminaries from Riemannian Geometry

We will now aim to develop some ideas about the loop space of a manifold. We will first look at some useful ideas from Riemannian geometry to help us in this task.

Definition 2.1. An affine connection, denoted ∇ takes each pair of vector field Y and tangent vector $X_p\ \epsilon\ TM_p$ and maps it to a new tangent vector in TM_p. We now give some properties of ∇:
∇ is bilinear and for any function f: M→R, ∇ also satisfies $X_p\nabla(fY) = (X_pf)Y_p + f(p)X_p\nabla Y$.

Definition 2.2. A connection on M assigns an affine connection ∇_p to every point p.
Formally, it has four properties:
1. $X\nabla Y$ is bilinear with respect to X and Y.
2. $(fX)\nabla Y = f(X\nabla Y)$
3. $X\nabla fY = (Xf)Y + f(X\nabla Y)$
4. If X and Y are smooth vector fields on M then $(X\nabla Y)_p = X_p\nabla_p Y$ is smooth

A great advantage of affine connections is that they allow us to define a derivative of vector fields along curves. Later, this idea will be useful in studying the acceleration of a curve. We now show how we can get this derivative.

Definition 2.3. From a vector field V along c, we can get a new vector field $\frac{DV}{dt}$ called the covariant derivative.
The map from V to $\frac{DV}{dt}$ satisfies the following properties:
1.$\frac{D(V+W)}{dt} = \frac{DV}{dt} + \frac{DW}{dt}$
2. If f is a smooth real valued function on R then $\frac{DfV}{dt} = \frac{Df}{dt}V + f\frac{DV}{dt}$
3. If V is induced by a vector field Y on M, $V_t = Y_{c(t)}$. Then $\frac{Dv}{dt}$ is equal to $c^*\frac{d}{dt}\nabla$ Y, where c^*: $T\mathbb{R}_t \to TM_{c(t)}$ is a mapping between the tangent spaces induced by c.

We assert without proof that there is a unique map with these properties.

Definition 2.4. A parallel vector field is a vector field V along γ so that $\frac{DV}{dt}$ is 0.

It turns out that the differential equation for a parallel vector field is linear. Then for any solution V_0, we can find a corresponding solution V_t. We say that the V_t is obtained from V_0 by parallel translation.

We will now present one useful formula:

Theorem 2.5. *If parallel translation preserves inner products, and if V and W are two vectors fields along c, then*

$$\frac{d}{dt} < V, W >=< \frac{DV}{dt}, W > + < V, \frac{DW}{dt} >$$

Proof. Choose parallel vector fields $P_1(0), ..., P_n(0)$ along c. Note that at a single point we can choose these vector fields to be orthonormal and we can extend them by parallel transport so that $P_1(t), ..., P_n(t)$ are an orthonormal basis of $T_{c(t)}(M)$ Then $V = \Sigma(v^i P_i)$, $W = \Sigma(w^i P_i)$. Where v^i, w^i are the inner products of W and V with P_i. Then $< V, W >= \Sigma(v^i w^i)$. We can now calculate

$$\frac{d}{dt} < V, W >= \Sigma(\frac{dv^i}{dt} w^i + \frac{dw^i}{dt} v^i) =< \frac{dV}{dt}, W > + < V, \frac{dW}{dt} > .$$

□

Definition 2.6. We now define the Energy of a path as $E_a^b = \int_a^b |\frac{d\omega}{dt}|^2$ and let E=E_0^1.

This quantity gives us a way of comparing curves and will be the primary function we consider when doing Morse theory on the path space of a manifold. We will now focus on applying the ideas we have developed in this section to finding the critical points of this function.

The answer to this question is that a path is a critical point of E if and only if it is a geodesic. We will aim to prove this result with the remainder of the section.

Definition 2.7. A path γ: I→M (I is an interval in $\mathbb{R}$) is a geodesic if $\frac{D}{dt}\frac{d\gamma}{dt}$ is always 0.

Note that for a geodesic, $\frac{d}{dt} < \frac{d\gamma}{dt}, \frac{d\gamma}{dt} >=< \frac{D}{dt}\frac{d\gamma}{dt}, \frac{d\gamma}{dt} >$=0. Then $|\frac{d\gamma}{dt}|$ is constant along γ.

We can again use this condition to create a differential equation in local coordinates. It turns out that for every point, there is a compact neighborhood in which there is a unique solution to the differential equation. From this, we can find a unique geodesic $\gamma : I \to M$ (I is any finite interval in $\mathbb{R}$), satisfying the conditions: $\gamma_v(0) = p$, and $\frac{d(\gamma)_v}{dt}(0) = v$. we will denote the solution to this equation as $\gamma(t) = exp_q(tv)$.

Definition 2.8. A manifold M is geodesically complete if $exp_q(v)$ is defined for all $q \,\epsilon\, M$ and all vectors in the tangent space.

There is one more theorem we must state without proof in order to get back to Morse theory more quickly:

Theorem 2.9. *If M is geodesically complete, then any 2 points can be joined by a minimal geodesic. A minimal geodesic is a geodesic which has length less than or equal to the length of any other smooth path joining the endpoints.*

Definition 2.10. A variation of w is a function $\alpha : (-\epsilon, \epsilon) \times [0,1] \to \Omega$ with
α(0,t)=ω(t) and for a subdivision $(t_0, , t_n)$ of [0,1], α is smooth on each $(-\epsilon, \epsilon) \times [t_{i-1}, t_i]$.

In α(s,t), if we fix s and let t vary, then we have a 'neighboring curve' to ω(t). The variation allows us to study these neighboring curves of some specified curve α(0,t)=ω(t), which will be useful when we determine the critical points of E. Often, we will fix the t coordinate of α(s,t). In this case, we may suppress the t coordinate from the notation if no confusion can occur.

Definition 2.11. We now prove some formulas that will be helpful in topologizing the path space of a manifold. To do this, we need to define the curvature tensor.

Definition 2.12. The curvature tensor R takes vector fields X,Y, and Z, and forms a new vector field

$$X\nabla(Y\nabla Z) + Y\nabla(X\nabla Z) + [X, Y\nabla Z]$$

We can gain more information about the curvature tensor by considering a coordinate system $(x^1, ..., x^n)$. Note that $[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}]$=0, so

$$R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})\frac{\partial}{\partial x^j} = (\frac{\partial}{\partial x^j}\nabla\frac{\partial}{\partial x^i} - \frac{\partial}{\partial x^i}\nabla\frac{\partial}{\partial x^j})\nabla\frac{\partial}{\partial x^k}.$$

Then the curvature tensor measures the non-commutativity of the affine connections defined on the manifold.

We now present a formula for the derivative of the energy function, which we will use to show that the a curve, γ is a critical point of the E iff γ is a geodesic.

Theorem 2.13. *(The First Variation Formula):*

$$\frac{1}{2}\frac{dE(\alpha(u))}{du}|_0 = -\Sigma < W_t, \Delta_t V > - \int_0^1 < W_t, A_t > dt$$

where $V_t = \frac{d\omega}{dt}$,
$A_t = \frac{D}{dt}\ \frac{d\omega}{dt}$,
$\Delta_t = V_{t_+} - V_{t_-}$.

Proof. This proof is a computation. We know

$$\frac{d}{du} < \frac{d\alpha}{dt}, \frac{d\alpha}{dt} >= 2 < \frac{D}{du}\frac{d\alpha}{dt}, \frac{d\alpha}{dt} > .$$

Applying this to $\frac{dE(\alpha(u))}{du}$ we get:

$$\frac{dE(\alpha(u))}{du} = \frac{d}{du}\int_0^1 < \frac{d\alpha}{dt}, < \frac{d\alpha}{dt} > dt = 2\int_0^1 < \frac{D}{du}\frac{d\alpha}{dt}, \frac{d\alpha}{dt} > dt.$$

Also,

$$< \frac{D}{du}\frac{d\alpha}{dt}, \frac{d\alpha}{dt} >=< \frac{D}{dt}\frac{d\alpha}{du}, \frac{d\alpha}{dt} > .$$

So we find that

$$\frac{dE(\alpha(u))}{du} = \frac{d}{du}\int_0^1 < \frac{d\alpha}{dt}, < \frac{d\alpha}{dt} > dt = 2\int_0^1 < \frac{D}{dt}\frac{d\alpha}{du}, \frac{d\alpha}{dt} > dt.$$

The theorem now follows by applying the identity

$$\frac{D}{dt} < \frac{d\alpha}{du}, \frac{d\alpha}{dt} >=< \frac{D}{dt}\frac{d\alpha}{du}, \frac{d\alpha}{dt} > + < \frac{d\alpha}{du}, \frac{D}{dt}\frac{d\alpha}{dt} >$$

in a manner similar to integration by parts.

□

Theorem 2.14. *A path ω is a critical point of E iff ω is a geodesic*

Proof. If ω is a geodesic, then the first variation formula will clearly be 0 as A_t is 0 and ω has no discontinuities by the definition of a geodesic. So a geodesic is always a critical point.

To prove the converse, the idea is to pick some variation that is always 0 for one term of the first variation formula, then we study the other term to gain information about the function.

If a path is a critical point, and we pick a variation $W(t) = f(t)A(t)$, where $f(t)$ vanishes at each t_i and is positive elsewhere, then the first term of first variation formula vanishes and we get $-\int_0^1 f(t) < A(t), A(t) > dt = 0$. This implies that $A(t)$ is zero, so that ω is a geodesic on each interval t_i, t_{i+1}.

We now pick a variation $W(t_i) = \Delta_{t_i}V$ then $A_t = 0$ and the first variation formula gives -$\Sigma < \Delta_{t_i}V, \Delta_{t_i}V >$=0. So there must be no points of discontinuity and thus ω is a geodesic. □

We will show one more formula before moving to topics closer to Morse theory:

We define the Hessian of E as $E_{**} =< \frac{d^2E(\alpha(u_1,u_2))}{du_1du_2}|_{0,0}\frac{d\alpha}{dt}, \frac{d\alpha}{dt} >$

Theorem 2.15. *(The Second Variation Formula): Let $\alpha : U \to \Omega$ be a variation of γ with $W_i = \frac{d\alpha}{du_i}$ (0,0) then we can write E_{**} as*

$$-\Sigma < W_2(t), \Delta_t \frac{DW_1}{dt} > - \int_0^1 < W_2, \frac{D^2 W_1}{dt^2} + R(V, W_1)V > dt$$

We will not show a proof but the idea is to take the derivative of the first variation formula and the rest is just rearranging terms. A proof can be found in [1].

3. Jacobi Vector Fields

We now introduce Jacobi vector fields:

Definition 3.1. Jacobi vector fields are vector fields that satisfy the equation $\frac{D^2 J}{dt^2} + R(V, J)V = 0$. Where $V = \frac{d\gamma}{dt}$.

Jacobi vector fields will allow us to further characterize the critical points of E_{**}, we can then use these results directly on the loop space of a manifold in order to develop more theorems from Morse theory.

We want to make another definition that helps in the use of Jacobi fields on the loop space.

Definition 3.2. p and q are conjugate along $\gamma(t)$ if there is a non-zero Jacobi field J along $\gamma(t)$ which vanishes at $t = a$ and $t = b$. The multiplicity of the conjugate points is the dimension of the vector space of all the Jacobi fields satisfying this condition.

Theorem 3.3. *A vector field $W_1 \in T\Omega_\gamma$ is in the null space of E_{**} iff W_1 is a Jacobi field.*

Proof. By the definition of a Jacobi field, the second variation formula must be 0 as Jacobi fields are smooth and $\frac{D^2 J}{dt^2} + R(V, J)V = 0$.

If the second variation formula is zero, then we can apply the same reasoning we used for the first variation case. Namely, we consider variations that isolate one of the terms of the formula.

First, let W_1 be smooth on each $[t_i, t_{i+1}]$. Now choose $W_2 = f(t)(\frac{D^2 W_1}{dt^2} + R(V, W_1)V)_t$, where f is a smooth function that vanishes only at t_1, t_n. This choice causes the first term of the second variation formula to vanish.

Then -$\frac{1}{2} E_{**}(W_1, W_2) = \int_0^1 f(t) |\frac{D^2 W_1}{dt^2} + R(V, W_1)V|^2 dt$.

For this to be zero, $W_1|_{t_i, t_{i+1}}$ must be a Jacobi field.

Now we set $W_2(t_i) = \Delta_{t_i}\frac{DW_1}{dt}$ Then the second variation formula gives $\Sigma|\Delta_{t_i}, \frac{DW_1}{dt}|^2$=0 So there are no discontinuities and we have a smooth Jacobi field.

□

Now we can get to the main significance of the Jacobi field.

Theorem 3.4. *If a is a variation of γ by geodesics, then W(t)= $\frac{da}{du}(0,t)$ is a Jacobi field along γ.*

Proof. As a is a variation by geodesics, we must have $\frac{D}{dt}\frac{da}{dt}(0,t) = 0$. In order to bring in the Jacobi field condition, we compute the curvature:

$$R(\frac{da}{dt}, \frac{da}{du})\frac{da}{dt} = \frac{D}{du}\frac{D}{dt}\frac{da}{dt} - \frac{D}{dt}\frac{D}{du}\frac{da}{dt}.$$

So that

$$0 = \frac{D}{du}\frac{D}{dt}\frac{da}{dt} = R(\frac{da}{dt}, \frac{da}{du})\frac{da}{dt} + \frac{D}{dt}\frac{D}{du}\frac{da}{dt}$$

.

Also, as we are on a geodesic,

$$\frac{D}{dt}\frac{D}{du}\frac{da}{dt} = \frac{D^2}{dt^2}\frac{da}{du}.$$

Then

$$R(\frac{da}{dt}, \frac{da}{du})\frac{da}{dt} + \frac{D^2}{dt^2}\frac{da}{du} = 0$$

and so $W(t)$ is a Jacobi field along γ. □

We can now start proving theorems that we can use directly to obtain results from Morse theory. We do this by characterizing the index of E_{**}.

Theorem 3.5. *the index λ of E_{**} is equal to the number of points $\gamma(t)$ that are conjugate to $\gamma(0)$ along γ.*

Proof. We show this by first describing a decomposition of $T\Omega$ and then using this decomposition to prove the theorem.

Let $T\Omega_\gamma(t_0, , t_k)$ be a vector field that restricts to a Jacobi field on each $[t_i, t_{i+1}]$ and vanishes at t=0,1.

T' is the vector space of W where $W(t_i)$=0. We will first show

$$T\Omega_\gamma = T\Omega_\gamma(t_0, ..., t_k) \oplus T'.$$

In addition, we also show E_{**} is positive definite on T'.

Let W be an arbitrary vector field in $T\Omega_\gamma$ and let W_1 be the field $T\Omega(t_0, , t_n)$. Clearly, $W - W_1$ is zero at each t_i and is therefore in T'.

Now we use the second variation formula to show that $T\Omega_\gamma$ and T' are mutually perpendicular.

The formula gives

$$\Sigma < W_2(t), \Delta_t \frac{DW_1}{dt} > - \int_0^1 < W_2, 0 > dt = 0,$$

as $W_2(t)$ is zero at every t_i.

Finally, we show $E_{**} \geq 0$ for $W \epsilon T'$. As $\gamma|_{[t_i, t_{i-1}]}$ is a minimal geodesic, it has smaller energy than any other path. Therefore, $E(\gamma) \leq E(\alpha(u))$ and by the second derivative test, the second derivative must be at least 0 for W in T'

Now we show the derivative is greater than 0.

We have

$$0 \leq E_{**}(W + cW_2, W + cW_2) = 2cE_{**}(W_2, W) + c^2 E_{**}(W_2, W_2),$$

but as this is true for positive as well as negative c, then $E_{**}(W_2, W)$ must be 0.

Then W is in the null space of E_{**} , which only has Jacobi fields. As T' has no Jacobi fields other than 0, then W=0. So T is positive definite on T'

□

Now we complete the proof with the following lemma:

Theorem 3.6. *The index of E_{**} is equal to the index of E_{**} restricted to $T\Omega_\gamma(t_0, , t_k)$, which is equal to the number of points that are conjugate to $\gamma(0)$ along γ.*

Proof. We prove this statement by letting i(t) be the index of $(E_0^t)_{**}$ and $\gamma(t)$ be the restriction of γ to [0,t]. We will study $i(t)$ to prove the statement. We then complete the proof by computing the Jacobian.

Note first that $i(t)$ is a nondecreasing function of t. To see this, note that any vector field that is 0 at some value $k < t$ can be extended to be 0 for all values greater than k, and will therefore be 0 at t.

$i(t)$ is 0 at t for small t because if t is small enough, then $\gamma(t)$ is a minimal geodesic.

We also have $i(t - \epsilon) = i(t)$ and $i(t + \epsilon) = i(t) + d$, if there is a conjugate point with multiplicity d between t and $t + \epsilon$.

To see that $i(t - \epsilon) = i(t)$, note that we already have $i(t - \epsilon) \leq i(t)$ so we just need to show $i(t - \epsilon) \geq i(t)$. To see this, note that if $(E_0^t)_{**}$ is negative definite on a subspace with dimension $i(t)$, then by the continuity of the hessian, there is some ϵ so that $(E_0^{t-\epsilon})_{**}$ is still negative definite in that subspace.

To see that $i(t + \epsilon) \leq i(t) + d$. Note that $TM_{\gamma(t_1)} \oplus ... \oplus TM_{\gamma(t_i)}$ has dimension ni for some n. Our hessian will be positive definite on $ni - i(t) - d$ dimension space so for small ϵ, $i(t + \epsilon) \leq i(t) + d$.

All that is left is to show $i(t + \epsilon) \geq i(t) + d$. We prove this by computing the Jacobian. We first choose $i(t)$ vector fields $W_1, ..., W_{i(t)}$ along γ_t so that the matrix formed by plugging these vector fields into the hessian is negative definite. We now choose d linearly independent Jacobi fields $J_1, ..., J_d$ along γ_t. We choose all of these vector fields so that they vanish at the endpoints of γ_t. Again we choose d vector fields so that $< \frac{DJ_h}{dt}(t), X_k(t) >$ is the $d \times d$ identity matrix.

Now we choose vector fields $W_1, ..., W_{i(t)}, \frac{1}{c}J_1 - cX_1, ..., \frac{1}{c}J_d - cX_d$.

If we compute the Jacobian with respect to this basis then we would find that the result is negative definite. To see this, note that the upper left corner is the matrix formed by plugging in the W_k into the hessian. The lower right corner of the matrix can be calculated to be -4I+c^2B where B is some matrix. Then if we pick c small enough, the matrix is negative definite and the theorem is proved.

□

4. The Morse Decomposition of the Loop Space

Now we will show that there is an analogous decomposition for functions on path spaces as there is for functions on manifolds. First we must define a metric on our manifold. Let p be a Riemannian metric on M, then

$$d(\omega, \omega') = Max(p(\omega(t)), \omega'(t)) + \int_0^1 (\frac{ds}{dt} - \frac{ds'}{dt})^2 dt,$$

where s and s' are the arc-lengths of ω and ω' respectively.

The second term ensures that the energy function is continuous. Now, let $\Omega^c = E^{-1}([0, c])$

We now define finite dimension approximations: $\Omega(t_0, , t_k)^c = \Omega^c \cap \Omega(t_0, , t_k)$

We will show one Lemma before proving our desired decomposition of the path space holds:

Theorem 4.1. *Let M be a complete Riemannian manifold, and choose c so that Ω^c is nonempty, then for a fine enough partition $(t_0, , t_k)$, $Int\Omega(t_0, , t_k)^c$ can be given the structure of a smooth manifold.*

Proof. The idea is to show that the path length between points is small enough so that there is a unique geodesic joining two points of the partition. We can then use this to define a homeomorphism that will give us the desired result.

For all paths ω, we have $L^2 \leq E \leq c$, where L is the arc-length of ω. Where the first inequality comes from Cauchy Schwarz by taking f to be 1 and g to be $|\frac{d\omega}{dt}|$, so that for all paths in Ω^c, we have $p(x, x_0) \leq \sqrt{c}$.

Now we need to use the condition that M is complete to allow the set $p(x, x_0) \leq \sqrt{c}$ to be compact. Then if $p(x, y) < \epsilon$, then there is a unique minimal geodesic from x to y with length less than ϵ.

Now for $\omega \, \epsilon \, \Omega(t_0, , t_k)^c$, we can pick a partition so that $t_i - t_{i-1} < \frac{\epsilon^2}{c}$.

Then

$$(L_{t_{i-1}}^{t_i}(\omega))^2 = (t_i - t_{i-1})(E_{t_{i-1}}^{t_i}(\omega)) \leq (t_i - t_{i-1})E(\omega) \leq (t_i - t_{i-1})c < (\epsilon)^2$$

and so each geodesic $\omega|[t_i, t_{i+1}]$ is determined by its endpoints. Then ω is determined by its values at each t_i, so there is a homeomorphism between ω and $(\omega(t_1), ..., \omega(t_{k-1}))$ which gives a homeomorphism between the finite path space and an open subset of $M \times M ... \times M$. The manifold formed by these open sets will be denoted as B, and E' will denote the energy functional on B.

□

We now wish to investigate some of the properties of E and B^a in order to get a nice characterization of the homotopy type of Ω^a.

Theorem 4.2. *E' is smooth and B^a is compact and is a deformation retract of Ω^a. The critical points of E are geodesics from p to q of length less than $\sqrt{c}$. The index of E'_{**} is the index of E_{**} at γ.*

From this theorem, we immediately get the following decomposition:

Let M be a complete Riemannian manifold and let p,q in M be two points which are not conjugate on any geodesic of length at most $\sqrt{a}$. Then Ω^a has the homotopy type of a finite CW-complex with a cell of dimension λ for each geodesic where E_{**} has index λ. The proof of this comes from our Morse decomposition of manifolds once we prove the previous theorem.

Proof. as ω depends on its values on each t_i, then $E'(\omega)$ must also depend on these values.

By the formula for E, $E'(\omega) = \Sigma \frac{p(\omega(t_{i-1}),\omega(t_i))^2}{t_i - t_{i-1}}$.

So B^a is those c=$(c_1, , c_k)$ with

$\Sigma \frac{p(c_{t_{i-1}}, c_{t_i})^2}{t_i - t_{i-1}} \leq a$.

Now we create a retraction from $Int\Omega^c$ to B. Note again that $p(\omega(t_{i-1}, \omega(t_i))^2 \leq L(\omega)^2 \leq E(w) < c$

Then if we pick the partition so that $t_i - t_{i-1} < \frac{\epsilon^2}{c}$, then

$$p(\omega(t_{i-1}), \omega(t_i))^2 \leq (t_i - t_{i-1}) E_{t_{i-1}}^{t_i} \omega \leq \epsilon^2.$$

So we can define a retraction $r(\omega)|t_i, t_{i+1}$ as a geodesic of length less than ϵ from $w(t_i)$ to $w(t_{i+1})$. Note that this geodesic is in B.

then we define a family r_t as

$r_t(\omega)|_{[0,t_{i-1}]} = r(\omega)|[0, t_{i-1}]$

$r_t(\omega)|_{[t_{i-1},u]}$= minimal geodesic from $\omega(t_{i-1})$ to $\omega(u)$

$r_t(\omega)|_{[u,1]}$=$\omega|[u, 1]$

r_0 is the identity map as it is is equal to ω everywhere and r_1 is the retraction r we defined above, so B is a deformation retract of Int(Ω^c). As $E(r_u(\omega)) \leq$ E(w), then B^a is a deformation retract of Ω^a.

Note that the critical points of E' are the geodesics as the process of finding critical points is near identical for E and E'. We have now finished the proof of the theorem. Then the index of E'_{**} is equal to the index of E_{**} restricted to $T\Omega_\gamma(t_0, , t_k)$ □

In order to use our previous results about the index of Jacobi fields, we first need to find an easier way to find conjugate points. To do this, we would like to find a condition so that finding solutions of the Jacobi equations is easier. Note that if $R(V, U_1)V = e_i U_i$, then the Jacobi equation $\frac{D^2 W}{dt^2} + R(V, W)V$ becomes

$$\Sigma \frac{d^2 w_i}{dt^2} U_i + \Sigma e_i w_i U_i = 0.$$

This equation is clearly the equation for simple harmonic motion in each variable. So if we can find some condition that gives the desired form of the curvature tensor, then we can easily find the conjugate points along a curve γ.

The space in which the curvature tensor takes the desired form is known as a symmetric space. We will now investigate the properties of a symmetric space

Definition 4.3. A symmetric space is a connected Riemannian manifold, M, so that for all p ϵ M there is an isometry $I_p : M \to M$ defined on a geodesic γ as $I_p(\gamma(t)) = \gamma(-t)$ and such that $\gamma(0) = p$.

We will now prove that that the curvature tensor takes the desired form in a symmetric space. We will do this by showing the following

Theorem 4.4. *If U,V, and W are parallel vector fields along γ, then $R(U, V)W$ is also a parallel vector field.*

Proof. Let X, U,V, and W be parallel vector fields. By the definition of R(U,V)W, we have $< R(U, V)W, X >$ is constant along γ. Now we would like to use a composition of 2 of our previous isometries to change this expression at one parallel vector field X to any other parallel vector field. We do this by showing that $I_q I_p(\gamma(t)) = \gamma(t + 2c)$ and that this composition preserves parallel vector fields.

The first claim is clear as

$$I_q(I_p(\gamma(t))) = I_q(\gamma(-t) = I_q(\gamma(-t - c + c) = (\gamma(t + 2c)).$$

The second claim is also true as $I_{p_*}(V)$ is parallel because I_p is an isometry. We also have $I_{p_*}(V_t) = -V(-t)$, as both the point we are considering and the direction have been reflected. Then $I_{q*}(I_{p_*}(V(t)) = V(t + 2c)$ by similar logic to the first claim.

With this isometry, we can now translate the identity $< R(U_p, V_p > W_p, X_p >$ to other values by defining T as $I_{\gamma(c/2)} I_p$, so that

$$< R(U_p, V_p)W_p, X_p >=< R(T_* U_{\gamma(c)}, T_* V_{\gamma(c)})T_* W_{\gamma(c)}, T_* X_{\gamma(c)} > .$$

We have this equality because T is an isometry. We have now shown that $R(U, V)W$ is constant along other parallel vector fields and it is therefore parallel. □

Note that from this proposition and the fact that $R(V, W)V$ is self adjoint, which we will not show, we can choose an orthogonal basis $U_1, ..., U_n$ so that $R(V, U_i)V =$

e_iU_i, where $e_1, ..., e_n$ are eigenvalues. Then we have the desired simple form of the Jacobi equation. Clearly, the solution to $\frac{d^2w_i}{dt^2} + e_iw_i = 0$ is $w_i(t) = c_i sin(\sqrt{e_i})t$ for some c_i. So the zeroes of $w_i(t)$ are the multiples of $\frac{\pi}{\sqrt{e_i}}$, where e_i is positive. If e_i is negative, then there are no real roots. So we have found all the conjugate points to p along γ.

For our purposes, we will apply this theorem for the specific case of a Lie group. In this case, we can also get a nicer formula to work with for the curvature tensor to make applying this theorem easier. However, we will not prove that a Lie group is a symmetric space.

Theorem 4.5. *If G is a Lie group with a left and right invariant Riemannian metric. If X, Y, Z, W are left invariant vector fields, then $R(X,Y)Z = \frac{1}{4}[[X,Y],Z]$.*

Proof. For a left invariant vector field, $X\nabla X = 0$, which we will not show.

So $(X+Y)\nabla(X+Y) = (X\nabla X) + (X\nabla Y) + (Y\nabla X) + (Y\nabla Y) = 0$

Then, $X\nabla Y + Y\nabla X = 0$.

But $X\nabla Y - Y\nabla X = [X,Y]$, So we can then add these equations to get $2X\nabla Y = [X,Y]$.

But by definition, $R(X,Y)Z$ is $X\nabla(Y\nabla Z) + Y\nabla(X\nabla Z) + [X,Y]\nabla Z$, so we can apply the identity $2X\nabla Y = [X,Y]$ to get

$$R(X,Y)Z = -\frac{1}{4}[X,[Y,Z]] + \frac{1}{4}[Y,[X,Z]] + \frac{1}{2}[[X,Y],Z].$$

To make this even nicer, note that by the Jacobi Bracket condition,

$-\frac{1}{4}[X,[Y,Z]] + \frac{1}{4}[Y,[X,Z]] + \frac{1}{4}[[X,Y],Z] = 0$. So that we have the desired equality $R(X,Y)Z = \frac{1}{4}[[X,Y],Z]$. ☐

We now study the space of minimal geodesics directly, with the goal to prove the following theorem. This theorem will be very valuable in proving Bott periodicity because it allows us to get information about homotopy groups from our results on the path space.

Theorem 4.6. *If the space of minimal geodesics from p to q is a topological manifold, and if every non-minimal geodesic from p to q has index at least λ_0, then the relative homotopy group $\pi_i(\Omega, \Omega^d)$ is zero for $0 \le i \le \lambda_0$.*

To do this proof, we use our previous result about conjugate points. We first show that for any smooth function g, we can make a good approximation to this function with a much nicer function, f. More formally:

Theorem 4.7. *If $g : U \to \mathbb{R}$ is a smooth function such that all the critical points of g have index $\ge \lambda$, then we can choose some function f so that $|\frac{dg}{dx^i} - \frac{df}{dx^i}| < \epsilon$ and $|\frac{d^2g}{dx^i dx^j} - \frac{d^2f}{dx^i dx^j}| \le \epsilon$ for all i,j uniformly throughout K, for some small ϵ. Furthermore, all the critical points of f have index at least λ_0 and they are all nondegenerate.*

Proof. let $K_f = \Sigma|\frac{df}{dx^i}|$. Let $e^1(x) \leq ... \leq e^n(x)$ be the eigenvalues of the matrix that has ij-th entry $\frac{d^2f}{dx^i dx^j}$. So we see a critical point of x is of index at least lambda iff $e^\lambda(x)$ is negative. Note that these functions are continuous as the eigenvalues of a matrix depend continuously on the entries of the matrix.

Now, consider $h_g(x) = max(K_g(x), -(e_g)^{\lambda_0})$ and define $h_f(x)$ similarly. As the critical points of f have index at most λ_0, we must have $(-e_f)^{\lambda_0}(x) > 0$ if $k_f(x) > 0$. So $m_f(x) > 0$. Now, we consider an f so that $|K_g(x) - K_f(x)|$ and $|(e_g)^{\lambda_0}(x) - (e_f)^{\lambda_0}(x)|$ are less than the minimum of m_g. As m_f is always positive, every critical point of f will have index at least λ_0. We can now easily show the desired inequalities $|\frac{dg}{dx^i} - \frac{df}{dx^i}| < \epsilon$ and $|\frac{d^2g}{dx^i dx^j} - \frac{d^2f}{dx^i dx^j}| \leq \epsilon$ hold. □

We can now show a special case of the desired theorem.

Theorem 4.8. *If M^0 is a manifold, and the critical points of $M - M^0$ has index at least λ_0, then $\pi_r(M, M^0)$=0 for $0 \leq r < \lambda_0$.*

Proof. We can choose a neighborhood around each point of M^0 so that M^0 is a retract of the open set formed by the unions of the neighborhood. We can also assume that each point of U is joined to the point of M^0 of which it is in a neighborhood of(we can shrink the neighborhoods so that each neighborhood contains only one point of M^0 if necessary).

Let I^r be the unit cube of dimension $r < \lambda_0$. consider a function
$h : (I^r, S^r) \to (M, M^0)$. We will show that h is homotopic to a map h' where $h'(I^r) \subset M^0$.

First, We choose a g that approximates f on M^c, where c is the maximum of f on $h(I^r)$.By the previous lemma, we can choose g so that it has no degenerate critical points and each critical point has index at least λ.

Let δ be the minimum of f on $M - U$, then $g^{-1}(M^c)$ has the homotopy type of the union of $g^{-1}(-\infty, \delta]$ and cells of dimenstion λ. Then consider h: $(I^r, S^r) \to (M^c, M^0) \subset (g^{-1}(-\infty, c + \epsilon), M^0)$.

Since $r < \lambda$, then h is homotopic to some h' that maps into $(g^{-1}(-\infty, \delta], M^0)$. This is true because all the critical points of g have index $> \lambda$ However, $g^{-1}(-\infty, 2\epsilon]$ is contained in U and U can be deformed into M^0 so we have $\pi_r(M, M^0)$=0. □

We are now able to prove Theorem 4.6.

Proof. We use the energy function restricted to $Int\Omega^c(t_0, ..., t_k)$ to relate the previous theorem to geodesics. Note that the energy function satisfies all the hypotheses of the previous theorem except that it does not range over [0,∞). We can fix this by just applying some diffeomorphism that takes the range of E into [0,∞). Call such a diffeomorphism f, then applying the previous theorem to the function $f * E$ gives $\pi_i(Int\Omega^c(t_0, ..., t_k), \Omega^d) = 0$ as desired. □

We now show one useful application

Theorem 4.9. *If the space of minimal geodesics is a topological manifold, and if every non minimal geodesic has index at least λ_0 then $\pi_i(\Omega^d)$ is isomorphic to $\pi_{i+1}(M)$ for i at most $\lambda_0 - 2$*

Proof. $\pi_i(\Omega^d)$ is isomorphic to $\pi_i(\Omega)$ for i less than $\lambda_0 - 1$ because the relative homotopy group is 0, and $\pi_i(\Omega)$ is isomorphic to $\pi_{i+1}(M)$. □

5. Proof of The Bott Periodicity Theorem

Now we can finally turn our attention to the Bott Periodicity Theorem.

Definition 5.1. We first define the map exp that acts on matrices.

$$exp(A) = I + A + \frac{A^2}{2!} + ...$$

Note that $exp(A^*) = (exp(A))^*$ by the definition of exp(A).

Then we have $exp(TAT^{-1}) = I + TAT^{-1} + \frac{TA^2T^{-1}}{2!} + ... = Texp(A)T^{-1}$. We also have $exp(A + B) = exp(A)exp(B)$ as long as A and B commute. This is true by the binomial theorem. Also, exp maps a neighborhood of 0 diffeomorphically onto a neighborhood of I.

Theorem 5.2. *From these properties, we can see that A is skew-Hermitian iff $exp(A)$ is unitary for A in a neighborhood of 0.*

Proof. Note that if A is skew-hermitian, then $A + A^* = 0$, so that $A = -A^*$ and $exp(A + A^*) = I = exp(A)exp(A)^*$. Then $exp(A)$ is unitary.

If $exp(A)$ is unitary, then $exp(A + A^*) = I$. So as A is in the neighborhood of 0, we have that $A + A^* = 0$. □

It follows from this proposition that $U(n)$ is a smooth submanifold of the space of $n \times n$ matrices and the tangent space of $TU(n)_I$ can be identified with the space of $n \times n$ skew-Hermitian matrices.

Then the Lie algebra of $U(n)$ can be identified with the space of skew-Hermitian matrices as a tangent vector at I extends uniquely to a left invariant vector field on $U(n)$. The Lie bracket corresponding to this Lie algebra is simply $[A, B] = AB - BA$ and we can show this by a direct computation of the Jacobi identity.

As exp: $TU(n)_I \to U(n)$ defines a 1-parameter subgroup of $U(n)$ as

exp(A+B)=exp(A)exp(B). Then it must define a geodesic. We can now define an inner product $< A, B >$=Re(trace(AB^*).

It is clearly positive definite, 0 iff A or B=0, conjugate symmetric and linear. It also determines a left invariant Riemannian metric on $U(n)$. We now show it is also right invariant.

Definition 5.3. an action is an adjoint action if each $S \epsilon U(n)$ determines an automorphism $X \rightarrow SXS^{-1} = (L_s R_s^{-1})X$. The induced mapping $(L_s R_s^{-1})_*$ is denoted Ad(s). as $exp(TAT^{-1}) = Texp(a)T^{-1}$, we then have $Ad(s)A = SAS^{-1}$.

The inner product we have defined is invariant under $Ad(s)$. To see this, we do the computation

$$Ad(S)A(Ad(S)B)^{-1} = SAS^{-1}(SBS^{-1})^* = SAB^*S^{-1}.$$

But $trace(AB) = trace(\mathrm{SABS}^{-1})$, so the inner product is invariant under SBS^{-1}.

For $A \epsilon TU(n)$, we can choose some $T \epsilon U(n)$ so that TAT^{-1} is a diagonal matrix with imaginary entries. Also, for $S \epsilon U(n)$, there is a $T \epsilon U(n)$ so that TST^{-1} is a diagonal matrix with entries on the complex unit circle. Then $exp : g \rightarrow U(n)$ is onto because we can form elements of $U(n)$ by applying the exp operator to elements of g.

Note that $det(exp(A)) = e^{traceA}$ as we can see from the diagonal form of matrices in $U(n)$. Then the Lie algebra of $SU(n)$ are those matrices A such that $A + A^* = 0$ and $traceA = 0$.

To do Morse theory on these topologies, we now need to consider the geodesics. We first consider those geodesics from I to $-I$. In this case, we must have $exp(A) = -I$. To make this question easier, we can put the matrix in diagonal form.

$exp(TAT^{-1}) = T(expA)T^{-1} = -I$ so that $exp(A) = T(-I)T^{-1} = -I$. As we can assume that A is in the diagonal form explained above, and we know that we can obtain $exp(A)$ by exponentiating all the elements of A, then every diagonal entry of A must have the form $i(\pi)k$, where k is an odd integer.

We can further characterize this matrix by looking at the length of the geodesic. $|A|=\sqrt{< A, A >}=\sqrt{tr(AA^*)}$. Because of the form of A that we have, $|A|=\sqrt{(k_1)^2 + ... + (k_n)^2}$. So for A to be a minimal geodesic, we must have $k_i = \pm 1$. To make this space easier to work with, we replace $U(n)$ by $SU(n)$, as $SU(n)$ must have the same number of positive and negative eigenvalues.

Then $eigen(i\pi)$ is an m dimensional subspace of $\mathbb{C}^{2m}$ so that the space of minimal geodesics from I to $-I$ is the special unitary group $SU(2m)$ which is known to be homeomorphic to the Grassmannian $G_m(\mathbb{C}^{2m})$.

We now have two final lemmas before proving the Bott Periodicity theorem.

Theorem 5.4. *Every non-minimal geodesic from I to $-I$ in $SU(2m)$ has index at least $2m + 2$.*

Proof. To prove this, we compute the indices of the geodesics from I to $-I$ on $SU(n)$. This is where our result about symmetric spaces comes in handy. As we are in a symmetric space, we just need to find the positive eigenvalues of $R(A, W)A$. By our result on Lie algebras, we have $R(A, W)A = \frac{1}{4}[[A, W], A]$. To make this easier, we assume that A has the earlier specified diagonal form.

What remains is just computation, $[A, W] = [A, W] = AW - WA$. If the ij-th entry of W is $w_{i,j}$, then the ij-entry of $AW = i\pi kjw_{i,j}$ and the ij-th entry of $WA = i\pi k_i w_{j,i}$. So that $[A, W] = i\pi(k_i - k_j)w_{i,j}$.

We now compute

$$[A, [A, W]] = A[A, W] - [A, W]A = -\pi^2 k_i k_j - (k_j)^2 - \pi^2(k_i)^2 - (k_j)(k_i) = -\pi^2(k_j - k_l)^2 w_j l.$$

So $R(A, W)A = \frac{1}{4}(-\pi^2(k_j - k_l)^2 w_j l)$.

Now we are concerned with the eigenvalues of this matrix. Let $e_{j,l}$ with $j < l$ have 1 in the jl-th position and -1 in the lj-th place. This is an eigenvector corresponding to the eigenvalue $\pi^2(k_j - k_l)^2 w_{jl}$.

let $i_{j,l}$ be the matrix with i in the jl-th place and the lj-th place then this is also an eigenvector with the same eigenvalue.

Also, any diagonal matrix will be an eigenvector of 0.

So if $k_j > k_l$ then the eigenvalue is positive. Now, by our earlier proposition, for the geodesic $\gamma(t) = exp(tA)$, we get conjugate points $t = \frac{k\pi}{\sqrt{e}}$, where k is an integer. Now we can use our earlier formula for e to get $t = k\frac{\pi}{\frac{\pi}{2}(k_j - k_l)} = \frac{2k}{k_j - k_l}$.

To see how many conjugate points are in the interval (0,1), we note that t must be greater than 0 and strictly less than $\frac{k_j - k_l}{2}$, so there are $\frac{k_j - k_l}{2} - 1$ values possible. Note we get 2 copies of each eigenvalue so that the index $\lambda = \Sigma(k_j - k_l - 2)$. For a minimal geodesic, this formula is clearly 0 as k_j and k_i are $\pm$ 1. For a nonminimal geodesic, with $n = 2m$, we have either at least $m + 1$ of the k_i are negative or at least m are positive. If at least $m + 1$ of the k_i are negative, then some k_i must be at least 3 so that

$$\lambda \leq \Sigma(3 - (-1) - 2) = 2(m + 1).$$

For the other case,

$$\lambda \leq \Sigma(3 - (-1) - 2) + \Sigma(1 - (-3) - 2) + (3 - (-3) - 2) = 4m.$$

So that $\lambda \leq 2m + 2$. □

We can now prove the Bott Periodicity theorem after one lemma, which helps us with one of the group isomorphisms in the statement of the Bott periodicity theorem.

Theorem 5.5. *The inclusion $G_m(C^{2m}) \to \Omega(SU(2m; I, -I)$ induces an isomorphism of homotopy groups in dimension $< 2m + 1$. $\pi_i G_M(C^{2m})$ is isomorphic to $\pi_{i+1} SU(2m)$ for $i \leq 2m$.*

This comes directly from the fact that the minimal geodesics of $SU(2m)$ is homeomorphic to $G_m(C^{2m})$ and that the relative homotopy group $(\pi_i(\omega, \omega^d)) = 0$.

We now use this to show Bott periodicity:

Theorem 5.6. *$\pi_i G_m(C^{2m})$ is isomorphic to $\pi_{i-1}U(m)$ which is isomorphic to $\pi_{i-1}U(m+k)$ and also $\pi_j(U(m))$ is isomorphic to $\pi_j(SU(m))$.*

Proof. We prove this by using homotopy exact sequences and the preceding theorem.

We can choose fibrations

$$U(m) \to U(m+1) \to S^{2m+1}$$

and

$$U(m) \to U(2m) \to \frac{U(2m)}{U(m)}.$$

From the first fibration, we get

$$\pi_i(S^{2m+1}) \to \pi_{i-1}(U(m)) \to \pi_{i-1}(U(m)).$$

This becomes

$$0 \to \pi_i(U(m+1)) \to \pi_i(U(m)) \to 0$$

.

So that $\pi_i U(m+1)$ is isomorphic to $\pi_i U(m)$ for I at most 2m.

We denote this value as $\pi_i(U)$.

From the other fibration we get:

$$\pi_i U(m) \to \pi_i U(2m) \to \frac{U(2m)}{U(m}.$$

Then $\pi_i(\frac{U(2m)}{U(m)}) = 0$.

So we can use the fibration

$$U(m) \to \frac{U(2m)}{U(m)} \to G_m(C^{2m})$$

to get that

$\pi_i(G_m(C^{2m})$ is isomorphic to $\pi_{i-1}U(m)$.

We then use the fibration

$$SU(m) \to U(m) \to S^1$$

to get that π_i SU(m) is isomorphic to $\pi_i U(m)$.

Note that we have $\pi_{i-1}(U)$ is isomorphic to $\pi_{i-1}(U(m))$ is isomorphic to $\pi_i(G_m(C^{2m})$ which is isomorphic to $\pi_{i+1}(SU(2m))$ which is isomorphic to $\pi_{i+1}(U)$ Then we have proved the theorem. □

From this, we get that $\pi_i U$ is isomorphic to $\pi_{i+2}U$ We know π_0 U is 0 and as U(1) is topologically a circle, $\pi_1 U$ is isomorphic to the group Z. So for odd i, $\pi_i U$ is isomorphic to Z and for even i, $\pi_i U$ is 0. We have now calculated the homotopy groups of unitary martrices and completed the proof of the Bott Periodicity theorem.

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