AN INTRODUCTION TO THE ZARISKI TOPOLOGY

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ABSTRACT. We give an introduction to the spectrum of a ring and its Zariski topology, a fundamental tool in algebraic geometry. In addition, we cover the ring theory and topology necessary for defining and proving basic properties of the Zariski topology. Finally, we give examples of various ring spectra.

CONTENTS

1. INTRODUCTION

Algebraic geometry is the branch of math that studies problems in geometry that can be solved with algebra, and vice versa. Modern algebraic geometry unfortunately has a reputation for being very difficult and inaccessible to learn. Many standard algebraic geometry textbooks are written at a graduate level or higher. The idea that one needs advanced techniques from category theory and commutative algebra to gain an appreciation for algebraic geometry is far from the truth, however. It is still possible for the undergraduate student to engage with aspects of the theory, and it is the goal of this paper to introduce an essential tool of modern algebraic geometry using only undergraduate ring theory and topology.

In this paper we will study *the spectrum of a ring*, which gives a way to define a topological space that can be created from any ring. This topological space, called *the Zariski topology*, gives a geometric way to interpret the algebra of a ring using the language of topology. A quick Google search of "the Zariski topology" is enough to see its relevance in the theory of modern algebraic geometry, but many sources will still be saturated with graduate level material. The good news is that there is still a lot one can learn about the spectrum of a ring without having to know what a sheaf or a scheme is. We have tried to combine the material that only relies on basic ring theory and topology into a single source.

This paper should be accessible to second or third year undergraduate math majors. The paper is divided into three main sections so that readers familiar with ring theory or topology may skip ahead. Readers who have had a first course in group theory should have no trouble reading this paper. We will begin with an overview of ideals in rings, so readers who are unfamiliar with the definition of a ring, a subring, or a product of rings may see [1]. We will assume that all rings are commutative and with unity. In addition, we assume ring homomorphisms send 1 to 1. No topology background is necessary for reading this paper.

2. Rings and Ideals

We begin our study of ring theory with the definition of an ideal. An ideal is very similar to a normal subgroup in group theory. This is because ideals allow one to construct quotient rings similar to the way quotient groups are constructed using a normal subgroup. As we will see, many theorems of quotient groups reappear in the form of quotient rings.

Definition 2.1. Let R be a ring and I be a subgroup of R under addition. For $r \in R$, consider the set $rI = \{ri \mid i \in I\}$. If $rI \subseteq I$ for all $r \in R$, then I is said to be an *ideal* of R.

Before we continue our study of ideals, we will pause to introduce some notation for constructing ideals out of ring elements. If A is some subset of a ring R, then (A) will be the smallest ideal of R containing A. Such an ideal always exists because A is contained in the ideal R. We note that if A is finite, $A = \{a_1, a_2, \ldots, a_n\}$, then the ideal (A) is also the set of all R-linear combinations of the a_i. Sometimes we will write the ideal (0) simply as 0.

One way to understand the definition of an ideal is to consider the ring of integers $\mathbb{Z}$. For any $n \in \mathbb{Z}$, the set $n\mathbb{Z} = \{nx \mid x \in \mathbb{Z}\}$ is an ideal of $\mathbb{Z}$. The ideal $2\mathbb{Z}$ is exactly the set of even integers. Adding two even numbers together will always give an even number, and multiplying an even number by *any* integer will result in an even number. The key difference between the additive and multiplicative structure of an ideal is that multiplication by elements outside the ideal must always stay inside the ideal. This is not true for addition. Going back to our example of the even integers, an even number will not remain even if an odd number is added to it. As we will see next, the multiplicative structure of an ideal allows for a well-defined construction of a quotient ring.

Proposition 2.2. *Let R be a ring and let I be an ideal of R. Then the additive quotient group R/I is a ring under addition and multiplication defined by:*

$$(r + I) + (s + I) = (r + s) + I$$

$$(r + I) \times (s + I) = (rs) + I.$$

Proof. Since R is an abelian group and I is a normal subgroup, R/I is automatically an abelian quotient group under addition. We will prove that multiplication in the quotient group is well defined. The remaining ring axioms should be verified by the reader. If we chose representatives $r, s \in R$ and $i, j \in I$ then,

$$(r + i)(s + j) = rs + rj + is + ij.$$

Since I is closed under multiplication by elements of R, each of rj, is, and ij is in I. Furthermore, their sum $rj + is + ij$ is in I. Writing this in terms of cosets we have the desired result:

$$(r + I) \times (s + I) = (rs) + I.$$

□

In group theory one can understand the structure of a group through group homomorphisms. The isomorphism theorems for groups establish a relationship between groups, normal subgroups, quotient groups, and group homomorphisms. This perspective is very useful for studying rings, too. The following theorems will prepare us for proving two isomorphism theorems for rings.

Theorem 2.3. *Let R and S be rings and let $\varphi\colon R \to S$ be a ring homomorphism. Then, the image of φ is a subring of S, and $\ker\varphi$ is an ideal of R.*

Proof. If $s_1, s_2 \in \operatorname{im}(\varphi)$, then there are $r_1, r_2 \in R$ such that $s_1 = \varphi(r_1)$ and $s_2 = \varphi(r_2)$. From the homomorphism property, we know $s_1 + s_2 = \varphi(r_1) + \varphi(r_2) = \varphi(r_1 + r_2)$ and $s_1 s_2 = \varphi(r_1)\varphi(r_2) = \varphi(r_1 r_2)$. Hence, $s_1 + s_2 \in \operatorname{im}(\varphi)$ and $s_1 s_2 \in \operatorname{im}(\varphi)$. Finally, $1 \in \operatorname{im}(\varphi)$ because $\varphi(1) = 1$, and this proves that $\operatorname{im}(\varphi)$ is a subring of S.

Next, suppose $r_1, r_2 \in \ker\varphi$. Since $\varphi(r_1) = \varphi(r_2) = 0$, it follows again from the homomorphism property that $\varphi(r_1 + r_2) = 0$ which proves $r_1 + r_2 \in \ker\varphi$. Now let a be any element of R, and let $r \in \ker\varphi$. Multiplying $\varphi(a)$ and $\varphi(r)$, we see $\varphi(ar) = \varphi(a)\varphi(r) = \varphi(a)0 = 0$, and $ar \in \ker\varphi$. □

Theorem 2.4. *Let $\varphi\colon R \to S$ be a ring homomorphism. If J is an ideal of S, then $\varphi^{-1}(J)$ is an ideal of S.*

Proof. Suppose $r_1, r_2 \in \varphi^{-1}(J)$. By definition, $\varphi(r_1), \varphi(r_2) \in J$, and $\varphi(r_1) - \varphi(r_2) = \varphi(r_1 - r_2) \in J$ because J is an ideal of S. It follows that $r_1 + r_2 \in \varphi^{-1}(J)$ which proves $\varphi^{-1}(J)$ is closed under addition. Next, suppose $a \in \varphi^{-1}(J)$ and $r \in R$. Since J is an ideal of S, $\varphi(a)\varphi(r) \in J$. This implies $ar \in \varphi^{-1}(J)$ because $\varphi(ar) = \varphi(a)\varphi(r) \in J$. □

Remark 2.5. It is not true in general that if $\varphi\colon R \to S$ is a ring homomorphism, then $\varphi(J)$ is an ideal if J is an ideal. However, if φ is a surjective homomorphism, then $\varphi(J)$ is an ideal in S.

We are now ready to prove two ismomorphism theorems for rings. In total, there are four standard isomorphism theorems for rings, but only two of them will be presented here. The first isomrophism theorem is a useful tool to prove two rings are ismorphic, and establishes a relationship between ring homomorphisms and quotient rings.

Theorem 2.6 (The First Isomorphism Theorem for Rings)**.** *If $\varphi\colon R \to S$ is a ring homomorphism, then $R/\ker\varphi$ is isomorphic to the image of φ. In particular, if φ is surjective, then $R/\ker\varphi \cong S$.*

Proof. Let $I = \ker\varphi$. First we note that R/I is a valid ring because $\ker\varphi$ is an ideal by Theorem 2.3. Consider the following map $\pi\colon R/I \to \operatorname{im}(\varphi)$ where $r + I \mapsto \varphi(r)$. First we will prove that this map is well defined. We will use the notation $\overline{r}$ to denote the coset $r + I$. Suppose for some $r_1, r_2 \in R$, $\overline{r_1} = \overline{r_2}$. Then $r_1 - r_2 \in I = \ker\varphi$, which means

$$\pi(\overline{r_1}) = \varphi(r_1) = \varphi(r_1 + (r_2 - r_2)) = \varphi(r_1 - r_2) + \varphi(r_2) = 0 + \varphi(r_2) = \pi(\overline{r_2}).$$

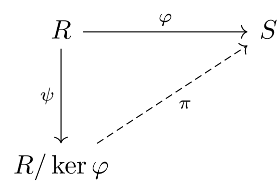
Next we will prove π is an isomorphism between rings $R/\ker\varphi$ and $\operatorname{im}(\varphi)$. First note that π is a homomorphism.

$$\pi(\overline{r_1 r_2})) = \pi(\overline{r_1 r_2}) = \varphi(r_1 r_2) = \varphi(r_1)\varphi(r_2) = \pi(\overline{r_1})\pi(\overline{r_2})$$

$$\pi(\overline{r_1} + \overline{r_2}) = \pi(\overline{r_1 + r_2}) = \varphi(r_1 + r_2) = \varphi(r_1) + \varphi(r_2) = \pi(\overline{r_1}) + \pi(\overline{r_2})$$

The map π is surjective. For every $\varphi(r) \in \operatorname{im}(\pi)$ we have $\pi(\overline{r}) = \varphi(r)$. Finally suppose $\pi(\overline{r_1}) = \pi(\overline{r_2})$. Then $\pi(\overline{r_1}) - \pi(\overline{r_2}) = 0$ and we get $\pi(\overline{r_1}) - \pi(\overline{r_2}) = \varphi(r_1) - \varphi(r_2) = \varphi(r_1 - r_2) = 0$. This means $r_1 - r_2 \in \ker\varphi = I$, so $\overline{r_1} = \overline{r_2}$. This proves the map is injective, and hence an isomorphism. □

If $\psi\colon R \to R/\ker\varphi$ is the projection map from R to the quotient ring $R/\ker\varphi$, then the diagram below illustrates the proof of Theorem 2.6.



To see how the first isomorphism is useful, we will give an example. Recall the ring of Gaussian integers: $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$. We will prove that $\mathbb{Z}[x]/(x^2+1) \cong \mathbb{Z}[i]$. Consider the homomorphism $\varphi\colon \mathbb{Z}[x] \to \mathbb{Z}[i]$ given by $p(x) \mapsto p(i)$. This map is surjective because every Gaussian integer $a + bi$ is mapped to by its corresponding linear polynomial $a + bx$. Furthermore, the kernel of φ is the ideal $(x^2 + 1)$, thereby proving $\mathbb{Z}[x]/(x^2+1) \cong \mathbb{Z}[i]$ by Theorem 2.6.

When constructing quotient rings, the ideal structure of the original ring is preserved. For example, in the ring of integers there are three ideals containing $4\mathbb{Z}$: $4\mathbb{Z} \subset 2\mathbb{Z} \subset \mathbb{Z}$. In the ring $\mathbb{Z}/4\mathbb{Z}$, there are three ideals total: (0), $(\overline{2})$, and $\mathbb{Z}/4\mathbb{Z}$. It in not a coincidence that the number of ideals in the quotient ring is the same as the number of ideals that contain $4\mathbb{Z}$. This relationship will be made precise in the next theorem.

Theorem 2.7 (Lattice Isomorphism Theorem)**.** *Let I be an ideal of a ring R. There is an inclusion preserving bijection between the set of ideals of R containing I and the set of ideals of R/I.*

Proof. Let $\phi\colon R \to R/I$ be the projection map from R to the quotient ring R/I. For each ideal $J \supseteq I$, there is a correspond ideal $\phi(J) \subseteq R/I$ since ϕ is surjective. Similarly, if K is an ideal of R/I, then $\phi^{-1}(K)$ is an ideal of R. To prove there is a bijection, we will show $\phi^{-1}(\phi(J)) = J$ for $J \supseteq I$. Written explicitly as a set, $\phi^{-1}(\phi(J)) = \{a \in R \mid \phi(a) \in \phi(J)\}$. When written in this way, it is clear that $J \subseteq \phi^{-1}(\phi(J))$. If $\phi(a) \in \phi(J)$, then there is a $b \in J$ such that

$$\begin{aligned}
&\phi(a) = \phi(b) \\
&\Rightarrow \phi(a - b) = 0 \\
&\Rightarrow a - b \in I \\
&\Rightarrow a - b = c, \text{ for some } c \in I \\
&\Rightarrow a = b + c
\end{aligned}$$

Since $b \in J$ and $c \in I \subseteq J$, we have shown $a \in J$, and thus $\phi^{-1}(\phi(J)) \subseteq J$. □

The lattice isomorphishm theorem is a fundamental result related to the ideal structure of a ring. Oftentimes ideals can best be understood by looking at their containment relationship to other ideals. Next, we will define two special classes of ideals: prime ideals and maximal ideals.

Definition 2.8. An ideal M of a ring R is a *maximal ideal* if M is a proper ideal of R, and the only ideals of R containing M are R and M.

We will state the following theorem, called Krull's Theorem, without proof. Krull's Theorem addresses the existence of maximal ideals in a ring, and it's proof uses Zorn's Lemma which is equivalent to the axiom of choice. The reader can find a proof of Krull's Theorem in [1].

Theorem 2.9 (Krull's Theorem)**.** *Every proper ideal of a ring is contained in some maximal ideal*

Maximal ideals are "maximal" in the sense that they are a maximum element in a chain of ideal inclusions. It is important to note that a ring can contain many maximal ideals. The ring of integers contains infinitely many of them. The maximal ideals of $\mathbb{Z}$ are the ideals $p\mathbb{Z}$ for a prime number p. However, the question of finding the maximal ideals for any general ring R is not always as straightforward as finding the maximal ideals of $\mathbb{Z}$. Fortunately for us, there is a way to determine whether some ideal M is maximal by looking at the quotient ring R/M.

Recall that an element $u \in R$ is called a unit if there exists an element v of R such that $uv = 1$.

Lemma 2.10. *If I is an ideal of a ring R then $I = R$ if and only if I contains a unit.*

Proof. If $I = R$, then I contains the unit 1. Conversely, if u is a unit in I with inverse v, then $1 = uv \in I$. If I contains 1, then $I = R$ because $r = 1 \cdot r \in I$ for every $r \in R$. □

Lemma 2.11. *A ring R is a field if and only if its only ideals are* 0 *and R.*

Proof. If R is a field, then every nonzero ideal of R contains some unit, and so the only nonzero ideal of R must be R itself by Lemma 2.10. Conversely, if R is the only nonzero ideal of R, then for any nonzero $u \in R$ we have $(u) = R$. Thus $1 \in (u)$, and so there must be some $v \in R$ such that $uv = 1$. Since every nonzero element of R is a unit, R is a field. □

Theorem 2.12. *If M is an ideal of a ring R, then M is a maximal ideal if and only if the ring R/M is a field.*

Proof. By definition, M is maximal if and only if there are no proper ideals I that contain M. By Theorem 2.7, the ideals of R that contain M correspond bijectively with the ideals of R/M. If R/M is a field, then the only ideals of R/M are (0) and R/M by Lemma 2.11. Hence, M is maximal exactly when R/M is a field. □

Corollary 2.13. *A ring R is a field if and only if* 0 *is a maximal ideal of R.*

The reader may be familiar with the fact that the ring $\mathbb{Z}/p\mathbb{Z}$ is a field whenever p is prime. This should be true intuitively because if n is composite, $n = pq$, then the ring $\mathbb{Z}/n\mathbb{Z}$ contains zero divisors with $\overline{p} \cdot \overline{q} = \overline{pq} = 0$. However, the statement can be proven in a simpler way using Theorem 2.12. If we accept the fact that the maximal ideals of of $\mathbb{Z}$ are $p\mathbb{Z}$ for prime p, then the claim immediately follows from Theorem 2.12.

The second class of ideals we will be studying are the *prime ideals*. As one would expect, the prime ideals of $\mathbb{Z}$ are $p\mathbb{Z}$ for a prime p. The set of maximal ideals and nonzero prime ideals in $\mathbb{Z}$ are the same, but this is not true for every ring. Prime ideals are based on a certain notion of primality in the integers. That is, if p is prime and p divides ab, then p must divide at least one of a or b. With this in mind, we now present the definition of a prime ideal.

Definition 2.14. Let R be a ring and P be a proper ideal of R. The ideal P is a *prime ideal* if for every $a, b \in R$, whenever the product ab is an element of P, at least one of a or b is an element of P.

Just as in the case for maximal ideals, there is a way to test whether an ideal is prime by analyzing its quotient ring. Once we have this result, it will follow that every maximal ideal is a prime ideal. We remind the reader that an integral domain is a nonzero ring where the product of any two nonzero elements is nonzero.

Theorem 2.15. *An ideal P of a ring R is prime if and only if R/P is an integral domain.*

Proof. An element $r \in R$ will be an element of P if and only if $\overline{r} = 0$ in the quotient ring R/P. Therefore, a product $ab \in R$ will be in P if and only if $\overline{ab} = \overline{a}\overline{b} = 0$ in R/P, and this will be true whenever R/P is an integral domain. □

Corollary 2.16. *Every maximal ideal of a ring is a prime ideal.*

Proof. This follows directly from theorems 2.12 and 2.15 since every field is an integral domain. □

We now turn to studying the behavior of prime ideals under ring homomorphisms. If one has a homomorphism $\varphi\colon R \to S$ between rings R and S, then the prime ideal structure of S is preserved under the inverse image of φ. Later, we will see that this fact is essential in order to show the existence of continuous maps between ring spectra.

Theorem 2.17. *Let $\varphi\colon R \to S$ be a ring homomorphism between rings R and S. If P is a prime ideal of S, then $\varphi^{-1}(P)$ is a prime ideal of R.*

Proof. Denote $\varphi^{-1}(P)$ by Q. We know by Theorem 2.4 that Q is an ideal of R, so what is left to prove is that Q is a prime ideal. Consider the canonical homomorphism $\psi\colon S \to S/P$. Composing ψ with φ gives us the following ring homomorphism from R to S/P:

$$R \xrightarrow{\varphi} S \xrightarrow{\psi} S/P.$$

We will refer to this map as $\pi = \psi \circ \varphi$. First we claim that $\ker \pi = Q$. This is because any element in $\ker \pi$ must map to $P = \ker \psi$ in S, and therefore must be in $\varphi^{-1}(P) = Q$. By The First Isomorphism Theorem, R/Q is isomorphic to $\operatorname{im}(\pi)$ which is a subring of S/P. Since P is a prime ideal, S/P is an integral domain. Any subring of an integral domain is also an integral domain, so Q is a prime ideal since R/Q is an integral domain. □

It should be noted that Theorem 2.17 can be proven in a more straightforward manner using the standard definition of a prime ideal. However, we prefer our proof because it emphasizes the viewpoint of using quotient rings and ring homomorphisms to solve problems in ring theory.

We now present two important constructions between ideals of a ring. These constructions will appear later in our discussion of the Zariski topology. Let R be a ring, and let I and J be ideals of R.

Definition 2.18. The *sum* of I and J is the ideal $I + J = \{i + j \mid i \in I, j \in J\}$. If $I + J = R$, then I and J are said to be *comaximal*.

Definition 2.19. The *product* of I and J is the ideal consisting of all finite sums of products of the form ij where $i \in I$ and $j \in J$.

Theorem 2.20. *Let I and J be ideals of a ring R. Then the following are true:*

(1) $I + J$ is the smallest ideal of R containing both I and J

(2) The ideal IJ is contained in $I \cap J$. Furthermore, if $I + J = R$, then $IJ = I \cap J$.

Proof.

(1) We want to show that if K is some ideal containing both I and J then $I + J \subseteq K$. Take some $x \in I + J$ and write $x = i + j$ for $i \in I$ and $j \in J$. Since K is an ideal that contains both I and J, we know $i, j \in K$ and $i + j \in K$. Since every element of $I + J$ is in K, we have $I + J \subseteq K$.

(2) Suppose for some $x \in IJ$ we have $x = \sum_{k=1}^{n} i_k j_k$. Each term of the sum is in I, and similarly each term is in J. Therefore each term is in $I \cap J$, and so their sum x must also be in $I \cap J$. If $I + J = R$, then $1 \in I + J$. This means that $1 = i + j$ for some $i \in I$ and $j \in J$. We have already proved that $IJ \subseteq I \cap J$, so what needs to be proven is that $I \cap J \subseteq IJ$. Let x be an element of $I \cap J$. Note that $x = x \cdot 1 = (i + j)x = ix + jx$. This proves $x \in IJ$, because x is an element of *both* I and J, and we have written x as a finite sum of elements of I multiplied by elements of J.

□

Theorem 2.21. *Let I and J be ideals of a ring R. If P is some prime ideal of R that contains IJ, the P contains either Ior J.*

Proof. If I is contained in P we are done, so suppose I is not contained in P. This means there is some $i \in I$ such that $i \notin P$. For every $j \in J$, $ij \in IJ \subseteq P$. Since P is a prime ideal, j must be in P, as i was assumed to not be in P. Since this is true for all $j \in J$, we can conclude $J \subseteq P$. □

Definition 2.22. If I is an ideal of a ring R, the radical of I, denoted $rad(I)$ or $\sqrt{I}$, is defined to be the set

$$rad(I) = \{r \mid r^n \in I, n \in \mathrm{N}\}$$

Proposition 2.23. *For any ideal I, $rad(I)$ is an ideal of R.*

Proof. If $r \in rad(I)$ and $a \in R$, then $(ar)^n = a^n r^n \in I$. Hence $ar \in rad(I)$ by definition. If $a, b \in rad(I)$, then there is some n large enough such that $a^n \in I$ and $b^n \in I$. By the Binomial Theorem,

$$(a + b)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} a^{2n-k} b^k.$$

Since every term in the sum has a or b with degree at least n, each term is in I, and $a + b \in rad(I)$. □

An element $x \in R$ is said to be *nilpotent* if $x^n = 0$ for some n. The ideal $\sqrt{0}$, sometimes called the *nilradical* of R, is the set of all nilpotent elements in R.

Theorem 2.24. *If P is a prime ideal of a ring R, then $\sqrt{0} \subset P$.*

Proof. If $x \in \sqrt{0}$, then $x^n = 0$ for some n. Since $0 \in P$, we have $0 = x^n \in P$. We prove that x is in P by induction. If $n = 2$, then $x^2 = x \cdot x \in P$ implies that $x \in P$ since P is prime. Now assume $x \in P$ if $x^{n-1} \in P$. Since $x^n = x^{n-1}x \in P$, it follows that $x \in P$ by the inductive hypothesis and the fact that P is prime. □

We now present a few theorems about ring products. The Chinese Remainder Theorem gives a way to know when a ring is really a product of rings "in disguise". The proof of the Chinese Remainder Theorem is not difficult, but it is on the lengthier side and is not important to know for our purposes. For these reasons, we have omitted the proof, but the curious reader may find a full proof in [1].

Theorem 2.25 (Chinese Remainder Theorem)**.** *Let $A_1, A_2, \ldots, A_k$ be ideals of a ring R. If for every $i \neq j$ A_i and A_J are comaximal ($A_i + A_j = R$), then*

$$R/(A_1 A_2 \ldots A_k) = R/(A_1 \cap A_2 \cap \ldots \cap A_k) \cong R/A_1 \times R/A_2 \times \ldots \times R/A_k$$

As an example, let's apply the Chinese Remainder Theorem to the ring $\mathbb{Z}/6\mathbb{Z}$ to prove that $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Since $3 - 2 = 1$, we have $1 \in 3\mathbb{Z} + 2\mathbb{Z}$, and $3\mathbb{Z} + 2\mathbb{Z} = \mathbb{Z}$. The intersection of $3\mathbb{Z}$ and $2\mathbb{Z}$ will correspond to integers that are divisible by 3 and 2 which is exactly the ideal $6\mathbb{Z}$. Applying the Chinese Remainder Theorem proves $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Definition 2.26. An element e of a ring R is idempotent if $e^2 = e$.

Remark 2.27. Every ring contains the trivial idempotents 1 and 0. If e is idempotent, then so is $1 - e$.

Theorem 2.28. *If a ring R has nontrivial idempotents if and only if R is a product of rings.*

Proof. If $R = S \times T$, then $(1, 0)^2 = (1^2, 0^2) = (1, 0)$. Thus R contains the nontrivial idempotent $(1, 0)$. Conversely, suppose R contains a nontrivial idempotent e. We will apply Theorem 2.25 to the ideals (e) and $(1 - e)$. These ideals are comaximal because $e + (1 - e) = 1$, i.e. $(1) + (1 - e) = (1) = R$. The fact that $(e) \cdot (1 - e) = e - e^2 = 0$ proves the product of the ideals is (0). By the Chinese Remainder Theorem, $R \cong R/(0) \cong R/(e) \times R/(1 - e)$. Since e is nontrivial, (e) and $(1 - e)$ will never be equal to (0) or (1), and so the ring product $R/(e) \times R/(1 - e)$ will be nontrivial. □

3. Topology

In the section we give the background in topology necessary for defining the spectrum of a ring. We will cover the definition of a topological space and other basic notions in topology. This section is by no means a comprehensive introduction to topology, and we only include the theory of topology that is necessary for defining the Zariski topology. For a more complete introduction to topology, see [2]. With that being said, this section is self-contained, and we do not assume any background in topology. Readers who already have a basic understanding of topology should feel free to skip ahead to the next section and refer back to this one as needed.

Definition 3.1. A topological space $\langle X, \mathcal{T} \rangle$ includes a nonempty set X and $\mathcal{T}$, where $\mathcal{T}$ is a collection of subsets of X called open sets. Open sets must satisfy the following properties:

(1) The empty set $\emptyset$ and X are open, i.e. $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.
(2) The union of any collection of open sets is open, i.e. if $\{A_i\}_{i \in I}$ is a collection of open sets of $\mathcal{T}$, then $\bigcup_{i \in I} A_i \in \mathcal{T}$.
(3) The intersection of a finite collection of open sets is an open set, i.e. if $\{A_i\}_{i=1}^n$ is a collection of open set of $\mathcal{T}$, then $\bigcap_{i=1}^n A_i \in \mathcal{T}$.

From now on, we may refer to a topological space $\langle X, \mathcal{T} \rangle$ simply as X when it is clear. It is important to understand, however, that in order to clearly define a topological space both $\mathcal{T}$ and X need to be specified. Giving X alone is not enough, as there can be many choices for different topological spaces over a given base set.

Definition 3.2. Let X be a topological space. A subset A of X is said to be a closed set if its complement $X \setminus A$ is open.

When it is convenient, we may refer to the complement of an open set as $\overline{A}$ instead of $X \setminus A$. When using the notation $\overline{A}$, we will always be taking the complement of A relative to the base set X. Learning topology requires getting a good grasp on the relationship between open sets versus closed sets. As a word of advice to the reader, when learning a new idea in topology it is useful to reimagine the idea in terms of both open and closed sets. The duality between open and closed sets often times allows one to replace definitions and theorems stated in terms of open sets with equivalent ones stated in terms of closed sets. Instead of specifying a collection of open subsets when defining a topological space over a set X, it is sometimes more convenient to start with a collection of closed subsets of X. Using DeMorgan's Law, one can derive an equivalent definition of a topological space $\langle X, \mathcal{T} \rangle$ by specifying a collection of closed sets instead of open sets. This definition will be used in the next section when we define the Zariski topology.

Definition 3.3. An equivalent definition of a topological space $\langle X, \mathcal{T} \rangle$ can be defined where $\mathcal{T}$ is a collection of closed sets as follows:

(1) The empty set $\emptyset$ and X are closed.
(2) The union of a finite collection of closed sets is closed.
(3) The intersection of any collection of closed sets is closed.

Ring homomorphisms are useful in ring theory because they allow one to study one ring in terms of another. There is a similar kind of map in topology called a *continuous map.*

Definition 3.4. Let X and Y be topological spaces. A map $f \colon X \to Y$ is *continous* if the preimage of every open set of Y is an open set of X.

Remark 3.5. The reader should verify that a map $f \colon X \to Y$ between topological spaces X and Y is continuous if the preimage of every closed set of Y is a closed set of X. We will use this definition in the next section.

The δ, ϵ definition of continuity from calculus is a special case of the more general Definition 3.4. Since open sets remain open under the inverse image of a continuous map, the continuous map gives a way to measure similarity between topological spaces. In fact, two topological spaces are "isomorphic" if there is a continuous way to travel back and forth between the two spaces. This is made precise in the next definition.

Definition 3.6. Two topological spaces $X$and Y are *homeomorphic* if there is a there is a continuous bijection $f\colon X \to Y$ with a continuous inverse.

We will now present several examples of topological spaces.

Example 3.7. The *indiscrete topology* on a nonempty set X is a topological space with open sets X and $\emptyset$. The indiscrete topology is sometimes called the "trivial topology" because it is a relatively uninteresting topology that can be defined over any set.

Example 3.8. The *cofinite topology* on a nonempty set X is a topological space where open subsets of X include the empty set or sets with a finite complement. In other words,

$$\mathcal{T} = \{A \subseteq X \mid A = \emptyset \text{ or } \overline{A} \text{ is finite}\}$$

The cofinite topology is indeed a valid topological space because

(1) The empty set is an open set by definition and $\overline{X} = \emptyset$.
(2) If $\{X_\alpha\}_{\alpha\in\Lambda}$ is a collection of open sets then
$$\overline{\bigcup_{\alpha\in\Lambda} X_\alpha} = \bigcap_{\alpha\in\Lambda} \overline{X_\alpha}.$$
This is finite because for every X_α,
$$\bigcap_{\alpha\in\Lambda} \overline{X_\alpha} \subseteq \overline{X_\alpha},$$
and every $\overline{X_\alpha}$ is assumed to be finite.
(3) If X_1 and X_2 are open sets then $\overline{X_1 \cap X_2} = \overline{X_1} \cup \overline{X_2}$. Since $\overline{X_1}$ and $\overline{X_2}$ are both finite, then their union will be finite as well.

Example 3.9. Let $\langle X, \mathcal{T}_X\rangle$ be a topological space and Y be a subset of X. The *subspace topology* of X on Y, $\langle Y, \mathcal{T}_Y\rangle$, consists of open sets

$$\mathcal{T}_Y = \{Y \cap A \mid A \in \mathcal{T}_X\}$$

Finally we will give two more definitions which describe properties of a topological space. These will be used later, and will give us a way to study a ring in a geometric way.

Definition 3.10. A topological space is *disconnected* if it is the union of two nonempty disjoint open sets. A topological space is said to be *connected* if it is not disconnected.

Definition 3.11. A topological space X is *irreducible* if it cannot be written as the union of two nonempty proper closed subsets of X.

4. The Spectrum of a Ring

At this point, the reader should be familiar with the ring theory and topology presented in the previous two sections. With these tools, we are now at a point where we can define a topological space from a ring. First we will begin by defining a point set. This will serve as our underlying base set for the Zariski topology.

Definition 4.1. Let R be a commutative ring with 1. The spectrum of R, denoted Spec(R), is the set of all prime ideals of R. We will additionally define mSpec(R) to be the set of all maximal ideals of R . Since every maximal ideal is a prime ideal, it is always true that $\text{mSpec}(R) \subset \text{Spec}(R)$.

Example 4.2. The points of Spec($\mathbb{Z}$) are the ideals (0) and $p\mathbb{Z}$ where p is a prime number. The points of mSpec($\mathbb{Z}$) are almost the exact same, with one important exception: mSpec($\mathbb{Z}$) does not include (0).

At this point we will pause to make some important distinctions before defining the Zariski topology attached to Spec(R). Previously, we thought of ideals as a set which includes certain ring elements. However, now we will be constructing a topological space on the set of prime ideals of R. The points of this topological space are the *ideals themselves*, rather than the ring elements which make up the ideal. In other words, the ideal should be viewed as a kind of discrete object or point which encodes information about the ring. The set Spec(R) should be thought of as the collection of these points. On its own, Spec(R) is nothing but a set, and at this point we have yet to turn it into a topological space. To do this, we need to specify the closed sets of Spec(R).

Definition 4.3. If R is a ring, and I is an ideal of R, then define the set

$$V(I) = \{\mathfrak{p} \in \text{Spec}(R) \mid \mathfrak{p} \supset I\}$$

as the set of all prime ideals containing I. In the case of mSpec(R), we will use the notation $V_m(I)$ to denote the set of all maximal ideals containing I.

Lemma 4.4. *If R is a ring and I is an ideal of R, then $V(I) = \emptyset$ if and only if $I = R$.*

Proof. Since every prime ideal is proper, no prime ideal can contain R, hence $V(R) = \emptyset$. Conversely, suppose for some ideal I we have $V(I) = \emptyset$. If I were a proper ideal, then by Theorem 2.9 it must be contained in some maximal ideal M of R. We know that every maximal ideal is prime, and this means $M \in V(I)$. By hypothesis $V(I) = \emptyset$, so I cannot be a proper ideal of R, and thus $I = R$. ☐

Proposition 4.5. *Closed sets of the form V(I) for ideals I of a ring R form a topology on Spec(R). This topology is called the Zariski topology.*

Proof. Since every ideal contains (0), $V(0) = \text{Spec}(R)$, and by Lemma 4.4 we have $V(R) = \emptyset$. Therefore, Spec(R) and $\emptyset$ are closed sets of the Zariski topology. Next, we claim that if $\{V(I_\alpha)\}_{\alpha \in \Lambda}$ is any collection of closed sets, then

$$\bigcap_{\alpha \in \Lambda} V(I_\alpha) = V\left(\sum_{\alpha \in \Lambda} I_\alpha\right)$$

Recall that $\sum_{\alpha \in \Lambda} I_\alpha$ is the smallest ideal containing every I_α in R. If $p \in V(\sum I_\alpha)$, then $\sum I_\alpha \subseteq p$. For all $\alpha \in \Lambda$, $I_\alpha \subseteq \sum I_\alpha \subseteq p$. This is equivalent to saying

$$\bigcap_{\alpha \in \Lambda} I_\alpha \subseteq p$$

or,

$$V\left(\sum_{\alpha \in \Lambda} I_\alpha\right) \subseteq \bigcap_{\alpha \in \Lambda} V(I_\alpha).$$

To prove the other direction, suppose $p \in \bigcap V(I_\alpha)$. Then for any α, the ideal I_α is contained in p. However, since $\sum I_\alpha$ is the smallest ideal of R containing every I_α, it must be true that

$$I_\alpha \subseteq \sum_{\alpha \in \Lambda} I_\alpha \subseteq p.$$

Thus,

$$p \in V\left(\sum_{\alpha \in \Lambda} I_\alpha\right).$$

Finally, we will prove that $V(I) \cup V(J) = V(IJ)$. Suppose $p \in V(IJ)$. By definition, the prime ideal p contains IJ. By Theorem 2.21, p contains at least one of I or J, which is the same as saying $p \in V(I) \cap V(J)$. This proves $V(IJ) \subseteq V(I) \cap V(J)$. If $p \in V(I) \cap V(J)$, then $I \subseteq p$ or $J \subseteq p$. Without loss of generality assume $I \subseteq p$. From Theorem 2.20, it follows that $IJ \subseteq I \cap J \subseteq I \subseteq p$. This proves that $p \in V(IJ)$, and $V(I) \cap V(J) \subseteq V(IJ)$. □

Proposition 4.5 defines a topology on Spec(R). The difference between Spec(R) and "the Zariski topology" is that Spec(R) is nothing but a set, while the Zariski topology is the topological space whose base set is Spec(R). We should note that this distinction is not always carefully made. Oftentimes Spec(R) is used interchangeably with "the Zariski topology," but usually the meaning can be inferred from context.

Proposition 4.6. *If R is a ring, then the closed points of* Spec(R) *correspond to $V(M) = \{M\}$ where M is a maximal ideal of R.*

Proof. Since every maximal ideal is prime, $M \in V(M)$ because $M \subseteq M$. The only other ideal that contains M is R. However, R is not a prime ideal because it is not a proper ideal. Hence $V(M) = \{M\}$. If P is a prime ideal that is not maximal, then $\{P\}$ cannot be closed. Any closed set $V(I)$ that contains P will also contain some maximal ideal that contains P. □

Theorem 4.7. *If I is an ideal, then $V(I) = V(\sqrt{I})$.*

Proof. Since $I \subseteq \sqrt{I}$, it follows that $V(I) \subseteq \sqrt{I}$. To prove the other direction, we want to show that if p is a prime ideal such that $I \subseteq p$, then $\sqrt{I} \subseteq p$. For an ideal $J \supseteq I$, we will use the notation R/J to denote the ideal that is the image of the projection map from R to R/I. If p is prime, then the ideal R/p is prime in R/I. By Theorem 2.24, $R/p \supseteq \sqrt{0}$. Note that the ideal $R/\sqrt{I}$ is the nilradical of the quotient ring R/I. Therefore we have $R/p \supseteq R/\sqrt{I}$, and so $p \supseteq \sqrt{I}$ by the Lattice Isomorphism Theorem. □

Corollary 4.8. *If $V(I) = V(J)$, then $\sqrt{I} = \sqrt{J}$.*

Readers who are familiar with topological notion of compactness may find the following remark interesting.

Remark 4.9. If R is a ring, then Spec(R) is compact.

Proof. If $\{I_\alpha\}_{\alpha \in \Lambda}$ is a collection of ideals of a ring R where

$$\bigcap_{\alpha \in \Lambda} V(I_\alpha) = \emptyset = V(R)$$

then

$$\sum_{\alpha \in \Lambda} I_\alpha = R$$

Since $1 \in R$, then $1 \in \sum_{\alpha\in\Lambda} I_\alpha$ so $1 = i_{\alpha_1} + i_{\alpha_2} + \ldots + i_{\alpha_k}$ for $i_{\alpha_j} \in I_{\alpha_j}$. From this we can conclude $1 \in \sum_{j=1}^{k} I_{\alpha_j}$. Finally we have

$$V\left(\sum_{j=1}^{k} I_{\alpha_j}\right) = \bigcap_{j=1}^{k} V(I_{\alpha_j}) = V(R) = \emptyset$$

which concludes the proof. □

Compactness is usually a very important property of a topological space, but the fact that Spec(R) is compact is not very useful for studying the spectrum of a ring. For this reason, we chose not to cover compactness in the topology section. However, the next theorem we are about to prove is very important because it will connect several ideas in ring theory to topology. Specifically, we will be able to use ring homomorphisms to generate continuous maps between ring spectra.

Theorem 4.10. *Let $\phi\colon R \to S$ be a ring homomorphism for rings R and S. Then the map $\phi^\star\colon \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ defined by $\phi^\star(P) = \phi^{-1}(P)$ is continuous.*

Proof. To prove $\phi^\star$ is continuous, we must show that if $V(I)$ is some closed set of Spec(R), then $\phi^{\star-1}(V(I)) = V(J)$ for some ideal J of Spec(S). We will prove that $J = \phi(I)$.

$$\begin{aligned}
\phi^{\star-1}(V(I)) &= \{p \in \mathrm{Spec}(S) \mid \phi^\star(p) \in V(I)\} \\
&= \{p \in \mathrm{Spec}(S) \mid I \subseteq \phi^\star(p)\} \\
&= \{p \in \mathrm{Spec}(S) \mid \phi(I) \subseteq p\} \\
&= V((\phi(I)).
\end{aligned}$$

□

Notice that the direction of the maps in Theorem 4.10 reverse. The original ring homomorphism goes from R to S, but the continuous map goes from Spec(S) to Spec(R). Next, we will look at what happens when Theorem 4.10 is applied to the case of a quotient ring.

Theorem 4.11. *For a ring R and an ideal I, let $\phi\colon R \to R/I$ be the projection map sending $r \mapsto r + I$. If ϕ^* is the continuous map from* Spec(R/I) *to* Spec(R) *given by ϕ, then the following are true:*

(1) The image of ϕ^* is $V(I)$.
(2) The map is injective.
(3) The topology on Spec(R/I) *is homeomorphic to the topology on $V(I)$.*

Proof.

(1) Since the prime ideals containing I are the inverse images of the prime ideals of R/I, we have

$$\begin{aligned}\operatorname{im}(\phi^*) &= \{\phi^*(p) \mid p \in \operatorname{Spec}(R/I)\} \\ &= \{\phi^{-1}(p) \mid p \in \operatorname{Spec}(R/I)\} \\ &= \{p \mid p \supseteq I,\ p \text{ is a prime ideal}\} \\ &= V(I).\end{aligned}$$

(2) If $\varphi(p_1) = \varphi(p_2)$, then $\phi(p_1) = \phi(p_2)$. Since there is a bijection between the ideals of R/I and the ideals of R containing I, $p_1 = p_2$.

(3) The prime ideals of R/I are the ideals p/I for prime ideals $p \in V(I)$. This gives a continuous bijection from $V(I)$ to $\operatorname{Spec}(R/I)$.

□

The consequence of this theorem is that we can view the closed set $V(I)$ as a topological space itself by looking at $\operatorname{Spec}(R/I)$. Next, we will present two more theorems that will connect topological properties of the ring spectrum to the algebraic properties of a ring.

Theorem 4.12. *If R is a ring, then* $\operatorname{Spec}(R) = \operatorname{Spec}(R/\sqrt{0})$.

Proof. For every $p \in \operatorname{Spec}(R)$, we have $\sqrt{0} \subseteq p$ by Theorem 2.24. In other words, $V(\sqrt{0}) = \operatorname{Spec}(R)$, and Theorem 4.11 proves that $\operatorname{Spec}(R) = \operatorname{Spec}(R/\sqrt{0})$. □

Theorem 4.13. *Let R_1 and R_2 be rings, and $\sqcup$ denote the disjoint union of sets. Then* $\operatorname{Spec}(R_1 \times R_2) = \operatorname{Spec}(R_1) \sqcup \operatorname{Spec}(R_2)$.

Proof. The reader should verify that the an ideal of $R_1 \times R_2$ is of the form $I_1 \times I_2$, where I_1 is an ideal of R_1 and I_2 is an ideal of R_2. Let $P_1 \times P_2$ be a prime ideal of $R_1 \times R_2$. Then the quotient ring $R_1/P_1 \times R_2/P_1$ must be an integral domain, but the product of two integral domains is never an integral domain. Therefore one of P_1 or P_2 is a prime ideal, and the other is equal to its corresponding ring. □

The converse of Theorem 4.13 is also true. Namely, if $\operatorname{Spec}(R)$ is disconnected, then R is a product of rings. We will prove this fact in two parts. First we will prove the theorem is true for a ring with no nilpotents, and then we will prove the general case.

Theorem 4.14. *Suppose for a ring R,* $\operatorname{Spec}(R)$ *is disconnected, i.e.* $\operatorname{Spec}(R) = X \sqcup Y$ *for closed sets X and Y. If R contains no nilpotent elements, then $R \cong S \times T$ where* $X = \operatorname{Spec}(S)$ *and* $Y = \operatorname{Spec}(T)$.

Proof. Since $\operatorname{Spec}(R) = X \sqcup Y$, we have $X \cap Y = \emptyset$ and $X \cup Y = \operatorname{Spec}(R)$. With X and Y being closed, we write $X = V(I)$ and $Y = V(J)$ for ideals I and J of R. Therefore

$$X \cap Y = V(I) \cap V(J) = V(I + J) = \emptyset = V(R) \tag{4.15}$$

and

$$X \cup Y = V(I) \cup V(J) = V(IJ) = \operatorname{Spec}(R) = V(0) \tag{4.16}$$

We can conclude from Equation 4.15 and Lemma 4.4 that $I + J = R$. This means I and J are comaximal. Now we can apply the Chinese Remainder Theorem to see

$$R/(IJ) \cong R/I \times R/J$$

If R has no nilpotent elements, then $\sqrt{0} = (0)$, and hence $IJ = 0$ by Corollary 4.8. Substituting (0) in for IJ we are left with

$$R/(0) \cong R \cong R/I \times R/J$$

From Theorem 4.11, we know that $V(I) = \text{Spec}(R/I)$ and $V(J) = \text{Spec}(R/J)$. □

Theorem 4.17. *If* $\text{Spec}(R)$ *is disconnected, then* R *is a product of rings.*

Proof. If $\text{Spec}(R)$ is disconnected, then $\text{Spec}(R/\sqrt{0})$ is disconnected by Theorem 4.12. Since every nilpotent in $\sqrt{R}$ maps to 0 in $R/\sqrt{0}$, the ring $R/\sqrt{0}$ contains no nilpotents. Applying Theorem 4.14 to $R/\sqrt{0}$, we get $R/\sqrt{0} = S \times T$ for some rings S and T. Since $R/\sqrt{0}$ is a product of rings, it contains nontrivial idempotents by Theorem 2.28. In general, R contains nontrivial idempotents if $R/\sqrt{0}$ contains nontrivial idempotents, so $R = S' \times T'$ is also a product of rings. The proof of the fact that R contains nontrivial idempotents is intricate, and for this reason we direct the reader to [10] instead of proving it here. □

The final theorem we present in this paper will not be proven, but the reader may refer to [11] as a reference. However, we will see an application of this theorem in the following discussion.

Theorem 4.18. *The ideal* $\sqrt{0} \subset R$ *is prime if and only if* $\text{Spec}(R)$ *is irreducible.*

Corollary 4.19. *If* R *is an integral domain, then* $\text{Spec}(R)$ *is irreducible.*

We have presented several theorems that allow us to study the algebraic properties of a ring through the topology of its spectrum. Now we will provide some examples to see these theorems in action. We will not be completely rigorous in our discussion of these examples. There will be some hand-waving, and this is because we want to emphasize the geometric intuition, rather than proving everything in detail.

The ring we will be using for the following examples is the polynomial ring $\mathbb{C}[x, y]$. Since $\mathbb{C}$ is an algebraically closed field, the maximal ideals of $\mathbb{C}[x, y]$ are $(x - a, y - b)$ for $a, b \in \mathbb{C}$. The reason why this is true is because of a theorem called Hilbert's Nullstellensatz which this fact is a special case of. For more information on Hilbert's Nullstellensatz, see [6]. This means that we can identify $\text{mSpec}(\mathbb{C}[x, y])$ with $\mathbb{C} \times \mathbb{C}$ because every maximal ideal is uniquely determined by two complex numbers. Additionally, a polynomial $f(x, y)$ is contained in the ideal $(x - a, y - b)$ if and only if $f(a, b) = 0$. In other words, $V_m(f)$ contains the zeros of the polynomial f. This allows use to visualize $\text{mSpec}(\mathbb{C}[x, y]/(f))$ as the curve $f = 0$ in $\mathbb{C} \times \mathbb{C}$. As an example, if $f(x, y) = y - x^2$, then $\text{mSpec}(\mathbb{C}[x, y]/(y - x^2))$ looks like the parabola $y = x^2$. Of course, some suspension of disbelief is required here because of the extra dimensions in $\mathbb{C} \times \mathbb{C}$, but hopefully the idea is clear. What about $\text{Spec}(\mathbb{C}[x, y]/(y - x^2))$? It will contain all the individual points of $\mathbb{C} \times \mathbb{C}$ where $y - x^2 = 0$, and it will contain the irreducible component $(y - x^2)$ layered on top of these points.

On the other hand, consider the polynomial $x(x-1) \in \mathbb{C}[x, y]$. This polynomial is reducible because it has factors x and $(x - 1)$. This means that $\text{Spec}(\mathbb{C}[x, y]/(x(x-$

1))) is not irreducible, and moreover $\mathbb{C}[x,y]/(x(x-1))$ is not an integral domain. In fact, $\mathrm{Spec}(\mathbb{C}[x,y]/(x(x-1)))$ is disconnected. This makes sense when we think of the zeros of $x(x-1)$ as two separate spaces in the complex plane. On one hand we have the zeroes along the curve $x=0$, and then there is the separate curve of $x=1$. Since $\mathrm{Spec}(\mathbb{C}[x,y]/(x(x-1)))$ is disconnected, it is isomorphic to a product of rings by Theorem 4.18. In particular $\mathrm{Spec}(\mathbb{C}[x,y]/(x(x-1))) \cong \mathbb{C}[x] \times \mathbb{C}[y]$.

Acknowledgments

Firstly, I would like to thank my mentor Karl Schaefer for his time and effort helping me write and learn the material for this paper. He is a fantastic teacher, and he really made me exited to learn about math. I also thank Peter May for organizing the REU, and I am very grateful to have had the opportunity to participate in such an amazing program. Finally, I would like to thank the other graduate students who spent their time running problem sessions and mentoring other students.

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