FORCING AND THE INDEPENDENCE OF THE CONTINUUM HYPOTHESIS

F. CURTIS MASON

ABSTRACT. The purpose of this article is to develop the method of forcing and explain how it can be used to produce independence results. We first remind the reader of some basic set theory and model theory, which will then allow us to develop the logical groundwork needed in order to ensure that forcing constructions can in fact provide proper independence results. Next, we develop the basics of forcing, in particular detailing the construction of generic extensions of models of ZFC and proving that those extensions are themselves models of ZFC. Finally, we use the forcing notions $\mathbb{C}_\kappa$ and $\mathbb{K}_\alpha$ to prove that the Continuum Hypothesis is independent from ZFC.

CONTENTS

The Continuum Hypothesis (CH) is the assertion that there are no cardinalities strictly in between that of the natural numbers and that of the reals, or more formally, $2^{\aleph_0} = \aleph_1$. In 1940, Kurt Gödel showed that both the Axiom of Choice and the Continuum Hypothesis are relatively consistent with the axioms of ZF; he did this by constructing a so-called inner model L of the universe of sets V such that $(L, \in)$ is a (class-sized) model of $ZFC + CH$. And in 1963, Paul Cohen shocked the mathematical world with his discovery of the method of forcing, whereby he was able to prove that the negation of the Continuum Hypothesis is also relatively consistent with ZFC. Combined with Gödel's earlier work, this means that $\text{Con}(ZFC) \to \text{Con}(ZFC + CH)$ and $\text{Con}(ZFC) \to \text{Con}(ZFC + \neg CH)$ both hold, which in turn means that the Continuum Hypothesis is logically independent from

ZFC set theory (unless ZFC is inconsistent). This is significant because essentially all of ordinary mathematics is conducted within ZFC; in order to make the Continuum Hypothesis true or false, then, we should have to add it or its negation to our list of axioms.

Working in ZFC, one can use forcing to show that whenever $(M, \in)$ is a countable transitive model of an arbitrary finite collection of axioms of ZFC (called a *finite fragment*, denoted ZFC^*), then one can carefully choose another set G and so build a new model $M[G]$ containing M such that $(M[G], \in)$ models $ZFC^* + \neg CH$. One can also use forcing to prove that for a different G, $M[G]$ is a model of $ZFC^* + CH$. Together, these arguments are actually sufficient for proving both $\text{Con}(ZFC) \rightarrow \text{Con}(ZFC + \neg CH)$ and $\text{Con}(ZFC) \rightarrow \text{Con}(ZFC + CH)$. In this paper, we (i) remind the reader of the basic results in logic needed in order to understand why the form of argument outlined above is sufficient for proving that the Continuum Hypothesis is independent from ZFC; (ii) justify the existence of countable transitive models of arbitrarily large finite fragments of ZFC; and (iii) develop the basic theory of forcing and show how it can be used to extend any countable transitive model M of a finite fragment ZFC^* to a model of $ZFC^* + CH$ and to a model of $ZFC^* + \neg CH$. Our development closely follows those of Halbeisen [2] and Kunen [4].

1. Preliminaries

1.1. **Set Theory.** The language $\mathcal{L}$ of set theory consists of a single symbol "ϵ," which is used to denote the binary relation of set membership. In this paper, lowercase Greek letters such as "φ," "ψ," etc. will always denote formulas of set theory.

We assume that the reader is familiar with the axioms and basic results of ZFC set theory, as can be found in [3]. In particular, the reader should feel comfortable with transfinite recursion, with the basic theory of ordinals and cardinals, and with the most common uses of the Axiom of Foundation. For matters of convenience, we post an informal account of the axioms of ZFC below:

Axiom 1.1 (Extensionality). *For all sets x and y, if x and y possess precisely the same members, then $x = y$.*

Axiom 1.2 (Foundation). *Every non-empty set x has a member y such that the intersection of x and y is empty.*

Axiom 1.3 (Axiom Schema of Comprehension). *For each set x and each formula φ without y free, there exists a set y such that for all sets z, $z \in y$ if and only if $z \in x$ and $\varphi(z)$ both hold.*

Axiom 1.4 (Pairing). *For every pair of sets x and y, there exists a set z such that $x \in z$ and $y \in z$.*

Axiom 1.5 (Union). *For every set x there exists a set y such that every member of x is a subset of y.*

Axiom 1.6 (Replacement Schema). *Let φ be a formula such that x and y are free and B is not. If A is a set such that for each $x \in A$, there is a unique y satisfying $\varphi(x, y)$, then there exists a set B containing each such y.*

Axiom 1.7 (Infinity). *There exists a set x with $\emptyset \in x$ and such that $\{y\} \cup y \in x$ whenever $y \in x$.*

Axiom 1.8 (Power Set)**.** *For every set x, there exists a set y such that every subset of x is a member of y.*

Axiom 1.9 (Axiom of Choice)**.** *The cartesian product of a non-empty collection of non-empty sets is non-empty. Equivalently, if X is a non-empty collection of non-empty sets, then there exists a function f with domain X such that for each $y \in X$, $f(y) \in y$.*

1.2. **Model Theory.** We also assume that the reader is familiar with the fundamental definitions and results of first-order logic and model theory (as can be found in [1] and [4]), but remind her of the main theorems that will be needed.

Theorem 1.1 (Completeness of First-Order Logic)**.** *Let T be a collection of sentences written in a first-order (F.O.) language $\mathcal{L}$. Then T is consistent if and only if T has a set model.*

Given Theorem 1.1, we can now explain why $\text{Con}(ZFC) \to \text{Con}(ZFC + CH)$ and $\text{Con}(ZFC) \to \text{Con}(ZFC + \neg CH)$ together show that CH is independent from ZFC. If ZFC is inconsistent, then ZFC proves every F.O. sentence in the language of set theory, including both CH and $\neg CH$. We need not bother with this degenerate case, so assume that ZFC is consistent, and then suppose that CH is provable within ZFC. Combined with $\text{Con}(ZFC) \to \text{Con}(ZFC + \neg CH)$, Theorem 1.1 gives that there is a model M of $ZFC+\neg CH$. Since M is in particular a model of ZFC, we can (by assumption) prove CH within M. But then CH and $\neg CH$ both hold in M, and this is impossible. A completely analogous argument shows that $\neg CH$ cannot be proved within ZFC if ZFC is consistent, so that CH is independent from ZFC.

The following remarks are aimed at clarifying some meta-points and at further justiyfing the specific line of argument that will be employed. Standard model theory is developed in V, where for our purposes, "$\in$" will always be interpreted as the real set membership relation. We of course take the axioms of ZFC to hold for $(V, \in)$, and by Tarski's Definability Theorem, the satisfaction relation "$\models$" is formalizable within ZFC as a relation between *sets*. Hence, if $\mathfrak{U}$ is a set model and φ is a first-order sentence, then the new sentence "$\mathfrak{U} \models \varphi$" is realizable as a corresponding statement in the language of set theory. However, by Tarski's Undefinability Theorem, when B is a proper class, there is no general method for formalizing "B satisfies φ" in ZFC as a set relation. Because of this, there is no way to write "$(V, \in) \models ZFC$" in ZFC, so the statement "the axioms of ZFC hold in V" must be viewed as a collection of statements in the meta-theory. This point is closely related to Gödel's Second Incompleteness Theorem, which tells us that ZFC is unable to prove its own consistency in the first place:

Theorem 1.2 (Gödel's Second Incompleteness Theoem)**.** *Let T be a first-order theory that contains Peano Arithmetic and whose collection of axioms is recursively enumerable. Then T is able to prove its own consistency if and only if it is inconsistent.*

Theorem 1.2 explains why, when working in ZFC, we cannot pass to set models of ZFC. However, as we shall soon see, ZFC is capable of proving the existence of "nicely-behaving" set models of arbitrarily large finite fragments of ZFC.

We will make use of one more theorem in the course of our exposition:

Theorem 1.3 (Downward Löwenheim-Skolem-Tarski Theorem)**.** *Let $\mathfrak{M}$ be a model of some first-order language $\mathcal{L}$, and let the domain of $\mathfrak{M}$ be M. Then for any subset $S \subseteq M$, there exists an elementary substructure $\mathfrak{N} \preccurlyeq \mathfrak{M}$ with domain N such that $S \subseteq N$ and $|N| = |S| + |\mathcal{L}|$.*

In particular, if $|S| = \aleph_0$ and $\mathcal{L}$ is at most countable, then we can find a countable elementary substructure of $\mathfrak{M}$.

Finally, we make some simplifying remarks. A set model of a finite fragment of ZFC is typically denoted by the pair (M, E), where M is a set and E is an interpretation of the symbol "$\in$" on M. However, our development will soon show that we need only concern ourselves with models of form $(M, \in)$, where "$\in$" is the real set relation in V. In all future instances, then, "$M \models ZFC^*$" will just mean "$(M, \in) \models ZFC^*$." Furthermore, we note that the logical symbols "$\vee$," "$\rightarrow$," "$\iff$," and "$\forall$" can be defined in terms of the symbols "$\wedge$," "$\neg$," and "$\exists$." In order to simplify our work, then, we will refer only to the latter three symbols when writing definitions or proving results that rely on a recursion on the complexity of formulae.

2. The Logical Justification for Forcing

In this section, we develop the machinery needed in order to ensure that forcing constructions can indeed be used to provide proper independence results.

2.1. **The Reflection Principle.** As we mentioned before, ZFC cannot prove the existence of a set model of itself (unless it is inconsistent). However, ZFC is capable of proving the existence of set models of arbitrarily large finite fragments of ZFC; this is known as the Reflection Principle. As it turns out, this will be sufficient for our purposes. In this subsection, we prove the general form of the Reflection Principle; in the next subsection, we combine this result with the Mostowski Collapsing Theorem, which will allow us to pass to "nice" models of arbitrarily large finite fragments of ZFC.

Before proving the Reflection Theorem, we shall need some additional definitions, and eventually, a lemma.

Definition 2.1 (Relativization)**.** Let M be a class and let φ be a formula of set theory. We define the *relativization* of φ to M by recursion on the following scheme:

$$(x = y)^M \ := \ x = y,$$

$$(x \in y)^M \ := \ x \in y,$$

$$(\phi_1 \wedge \phi_2)^M \ := \ \phi_1^M \wedge \phi_2^M,$$

$$(\neg\varphi)^M \ := \ \neg\varphi^M,$$

$$(\exists x : \varphi)^M \ := \ \exists x \in M(\varphi^M).$$

Definition 2.2 (Absoluteness)**.** Let M be a subclass of N, and let $\varphi(x_1, \ldots, x_n)$ be a formula with free variables among those listed. Then we say that φ is *absolute* in M, N, written $M \preccurlyeq_\varphi N$, if and only if $\varphi^M(x_1, \ldots, x_n)$ and $\varphi^N(x_1, \ldots, x_n)$ hold for precisely the same tuples $(x_1, \ldots, x_n)$ of sets.

Note that for all tuples $(x_1, \ldots, x_n)$ of sets, $\varphi^V(x_1, \ldots, x_n)$ means the same thing as $\varphi(x_1, \ldots, x_n)$. When $N = V$, then, φ is absolute in M, V just in case $\varphi^M(x_1, \ldots, x_n)$ holds for precisely the same tuples as $\varphi(x_1, \ldots, x_n)$ does. In this case, we simply say that φ is *absolute* in M.

Definition 2.3. A set Λ of formulas $\{\varphi_1, \ldots, \varphi_n\}$ is *subformula-closed* if and only if every subformula of each φ_i is in Λ, and if additionally no $\varphi_i \in \Lambda$ makes use of the universal quantifier "$\forall$".

Using these definitions, we can now state the lemma that will aid us in the proof of the Reflection Principle:

Lemma 2.4. *Let $\{\varphi_1, \ldots, \varphi_n\}$ be subformula-closed, and let A be a non-empty subclass of B. Then the following statements are equivalent:*

(1) $A \preccurlyeq_{\varphi_i} B$ holds for $i = 1, \ldots, n$.
(2) For all existential formulae $\varphi_i(x_1, \ldots, x_{r_i})$ of form "$\exists y \varphi_j(\vec{x}, y)$" and for all tuples $(a_1, \ldots, a_{r_i})$ of sets in A, $\varphi_i^B(\vec{a})$ implies that there exists a $b \in A$ satisfying $\varphi_j^B(\vec{a}, b)$.

Proof. (1) $\to$ (2): Let φ_i be an existential formula from the given collection, and suppose that $\varphi_i^B(\vec{a})$ holds for some $a_1, \ldots, a_{r_i} \in A$. By assumption, then, $\varphi_i^A(\vec{a})$ holds. With $\{\varphi_1, \ldots, \varphi_n\}$ subformula-closed, this means that for some j, there exists a $b \in A$ for which $\varphi_j^A(\vec{a}, b)$ holds. Applying (1) to $\varphi_j(\vec{a}, b)$, we obtain $\varphi_j^B(\vec{a}, b)$. Again, with $b \in A$, the desired conclusion follows.

(2) $\to$ (1): We will prove that $A \preccurlyeq_{\varphi_i} B$ holds for each φ_i by induction on the complexity of each such formula. For atomic φ_i, the result is trivial. Now, fix a formula φ_i, and suppose that we have proved $A \preccurlyeq_{\varphi_j} B$ for each φ_j of length shorter than φ_i. If φ_i is constructed from shorter φ_j's by means of "$\neg$" or "$\wedge$," then the result is again trivial. Now, suppose that $\varphi_i(\vec{x})$ is of form "$\exists y \varphi_j(\vec{x}, y)$." Fix $a_1, \ldots, a_{r_i} \in A$. From (2) and the meaning of φ_i^B, it follows that $\varphi_i^B(\vec{a})$ holds if and only if there exists a $b \in A$ satisfying $\varphi_j^B(\vec{a}, b)$. By the induction hypothesis together with $A \subseteq B$, it follows that $\varphi_j^B(\vec{a}, b)$ holds just when $\varphi_j^A(\vec{a}, b)$ does. Since $\varphi_i \equiv \exists y \varphi_j$, the desired conclusion follows.

□

It is important to note that Theorem 2.4 is a scheme of theorems in the metatheory: Given formulae ψ_A and ψ_B that "define" the classes A and B, if $\psi_A(x)$ implies $\psi_B(x)$ for all sets $x \in V$, then the above result is a theorem of ZFC. Using this theorem scheme, we can now prove the main result of this subsection:

Theorem 2.5 (The Reflection Principle)**.** *Let $\{\varphi_1, \ldots, \varphi_n\}$ be a collection of formulae, let B be a nonempty class, and assume that A_ε is a set for every $\varepsilon \in \Omega$. Furthermore, suppose that the following conditions hold:*

(1) If $\varepsilon < \eta$, then $A_\varepsilon \subseteq A\eta$.
(2) $A_\eta = \bigcup_{\varepsilon < \eta} A_\eta$ for limit ordinals η.
(3) $B = \bigcup_{\varepsilon \in \Omega} A_\epsilon$.

Then for every ordinal ε, there exists a limit ordinal $\eta > \varepsilon$ such that A_η is nonempty and $A_\eta \preccurlyeq_{\varphi_i} B$ for $i = 1, \ldots, n$.

Proof. We may produce a subformula-closed list from $\{\varphi_1, \ldots, \varphi_n\}$ in the following fashion. For each φ_i, replace every instance of a universal quantifier "$\forall$" by the equivalent "$\neg\exists\neg$"; call the (possibly) new formula ψ_i. Then add the necessary subformulae to the end of the new list in order to make it subformula-closed. Let $\{\psi_1, \ldots, \psi_m\}$ denote the subformula-closed list obtained in this way, where $m \geq n$. If we can prove that the result holds for $\{\psi_1, \ldots, \psi_m\}$, then it will also hold for $\{\varphi_1, \ldots, \varphi_n\}$.

We will define several collections of functions in quick succession, each new collection being defined in terms of the last. For each $\psi_i(\vec{x_{r_i}})$ of form $\exists y \psi_j(\vec{x_{r_i}}, y)$, define $F_i : B^{r_i} \to \Omega$ as follows: For $\vec{a} \in B^{r_i}$, if $\psi_i^B(\vec{a})$, then $F_i(\vec{a})$ is the least $\varepsilon \in \Omega$ such that $\psi_j^B(\vec{a}, b)$ holds for some $b \in A_\varepsilon$; if $\psi_i^B(\vec{a})$ fails, then set $F_i(\vec{a}) = 0$. Now, for each i, define $G_i : \Omega \to \Omega$ by $G_i(\varepsilon) = \sup\{F_i(a_1, \ldots, a_{r_i}) \mid a_1, \ldots, a_{r_i} \in A_\varepsilon\}$ if ϕ_i is existential; otherwise, set $G_i(\varepsilon) = 0$. Finally, define $K : \Omega \to \Omega$ by setting $K(\varepsilon)$ to be the larger of ε and $\max\{G_i(\varepsilon) \mid i \leq n\}$.

Now, fix some $\varepsilon \in \Omega$. Define the sequence $\{\zeta_k \mid k \in \omega\}$ by letting ζ_0 be the smallest $\zeta > \varepsilon$ such that A_ζ is nonempty and then letting $\zeta_{k+1} = K(\zeta_k)$ for $k \in \omega$. Let $\eta = \sup\{\zeta_k \mid k \in \omega\}$. Then η is a limit ordinal for which $A_\eta \neq \emptyset$. Furthermore, (2) of Lemma 2.4 holds for A_η, B and $\{\psi_1, \ldots, \psi_m\}$: Let $\psi_i(\vec{x_{r_i}})$ be an existential formula of form "$\exists y \psi_j(\vec{x_{r_i}}, y)$," and suppose that $a_1, \ldots, a_{r_i} \in A_\eta$ are such that $\psi_i^B(\vec{a})$ holds. Since r_i is finite, there must exist a $q \in \omega$ such that $a_1, \ldots, a_{r_i} \in A_{\zeta_{q+1}}$. By our previous definitions, $\zeta_{q+1} \geq G_i(\zeta_q) \geq F_i(a_1, \ldots, a_{r_i})$. By definition of F_i, then, there must exist a $b \in A(\zeta_q)$ such that $\psi_i^B(\vec{a}, b)$ holds. With existential ψ_i and $a_1, \ldots, a_{r_i} \in A_\eta$ held arbitary, Lemma 2.4 shows that $A_\eta \preccurlyeq_{\varphi_i} B$ holds for $i = 1, \ldots, n$. Now, since $A \preccurlyeq_\varphi B$ if and only if $A \preccurlyeq_\psi B$ whenever φ and ψ are logically equivalent, the proof is complete. □

We once again remark that this theorem is actually a scheme in a metatheory: Given a collection of formulae $\{\varphi_1, \ldots, \varphi_n\}$, a class B, and a class-sized function that maps $\varepsilon \to A_\varepsilon$ for $\varepsilon \in \Omega$, the above is a theorem of ZFC. Now, before we use the Reflection Principle to show that every finite fragment of ZFC has a set model, we remind the reader of the Von Neumann Hierarchy of Sets:

Definition 2.6 (Von Neumann Hierarchy of Sets)**.** By transfinite induction, define

$$V_0 := \emptyset,$$

$$V_{\alpha+1} := \mathcal{P}(V_\alpha),$$

$$V_\alpha := \bigcup_{\beta<\alpha} V_\beta \text{ for limit ordinals } \alpha.$$

The statement "$V = \bigcup_{\alpha\in\Omega} V_\alpha$" is equivalent to the Axiom of Foundation (see [3]). Using this fact, we arrive at the desired form of the Reflection Principle:

Corollary 2.7. *Let ZFC^* be a finite fragment of ZFC. Then there exists an $\eta \in \Omega$ such that $V_\eta \models ZFC^*$.*

Proof. Letting $B = V$, $ZFC^* = \{\varphi_1, \ldots, \varphi_n\}$, and $A_\varepsilon = V_\varepsilon$ for $\varepsilon \in \Omega$, Theorem 2.5 shows that there is some limit $\eta > \omega$ such that $V_\eta \preccurlyeq_{\varphi_i} V$ holds for $i = 1, \ldots, n$. Since each axiom of ZFC holds in V, this implies that $V_\eta \models ZFC^*$. □

2.2. **The Mostowski Collapsing Theorem and Countable Transitive Models.** In this subsection, we prove the Mostowski Collapsing Theorem and show how it can be combined with the Reflection Principle to prove that each finite fragment ZFC^* of ZFC has a countable transitive model (ctm). When we develop the method of forcing, we shall always assume that we are working with a ctm M – we will need M to be countable in order to make certain key forcing arguments work, and we will need M to be transitive in order to ensure that certain arguments can be made inside of M in the first place. The latter point will be explored in the next subsection.

In order to state the Mostowski Collasping Theorem, we shall need the following definitions:

Definition 2.8. Let E be a relation on a class P.

(1) E is *well-founded* if and only if the following conditions hold:
   (a) Every nonempty set $x \subseteq P$ contains an E-*minimal* element, i.e., an element y such that $z \; E \; x$ does not hold for any $z \in x$.
   (b) The *extension* of x, defined $\mathrm{ext}_E(x) := \{z \in P \mid z \; E \; x\}$, is a set for every $x \in P$.

(2) E is *extensional* on P if and only if distinct members of P have distinct extensions.

We assume that the reader is familiar with rank functions and with well-founded induction. A discussion of these topics can be found in [3] and [4].

Theorem 2.9 (Mostowski Collapsing Theorem)**.** *Let E be a relation on a class P. If E is well-founded and extensional, then there is a transitive class M such that $(P, E) \cong (M, \in)$, i.e., there exists a bijection $\pi : P \to M$ such that $x \; E \; y$ if and only if $\pi(x) \in \pi(y)$.*[1]

Proof. By well-founded induction, define $\pi : P \to V$ by setting $\pi(x) = \{\pi(z) \mid z \, E \, x\}$ for $x \in P$. Let $M = \pi(P)$; then by the definition of π, M is a transitive class. Now, suppose that π fails to be injective. Then let $z \in M$ be of least rank such that $\pi(x) = z = \pi(y)$ for distinct $x, y \in P$. With E extensional, this means that $\mathrm{ext}_E(x) \neq \mathrm{ext}_E(y)$, so that (WLOG) there exists a $u \in \mathrm{ext}_E(x)$ such that $u \notin \mathrm{ext}_E(y)$. Let $t = \pi(u)$. Since $t \in \mathrm{ext}_E(x) = \mathrm{ext}_E(y)$, there must exist a $v \in \mathrm{ext}_E(y)$ such that $t = \pi(v)$. Thus $\pi(u) = t = \pi(v)$ and $u \neq v$; but since $t \in z$, t has smaller rank than z. This, however, contradicts our assumption about z. Thus π is injective.

Now, by construction, $x \; E \; y$ implies $\pi(x) \in \pi(y)$. Conversely, suppose that $\pi(x) \in \pi(y)$. Once again, the definition of π gives us that $\pi(x) = \pi(z)$ for some $z \; E \; y$. With π injective, it follows that $x = z$, so that $x \; E \; y$. □

We can now apply the Mostowski Collapsing Theorem to produce ctms of finite fragments of ZFC:

Corollary 2.10. *Let ZFC^* be a finite fragment of ZFC. Then there exists a ctm M of ZFC^*.*

[1] In fact, one can prove by $\in$-induction that the π constructed above is the unique isomorphism between (P, E) and $(M, \in)$ (see [3]). Although beautiful, this fact will not be needed for our development.

Proof. Let ZFC^{**} denote the finite collection of axioms of ZFC obtained by adding the Axiom of Extensionality to ZFC^*. By the Reflection Principle, there exists a model V_η of ZFC^{**} for some limit $\eta > \omega$. It follows from the Downward Löwenheim-Skolem-Tarski Theorem that there is a countable subset $N \subseteq V_\eta$ such that $N \models ZFC^{**}$. Since the Axiom of Extensionality holds in N, $\in$ is extensional on N; and by the Axiom of Foundation, $\in$ is well-founded on N. We may therefore apply the Mostowski Collapsing Theorem to $(N, \in)$ to obtain a countable transitive M such that $(N, \in) \cong (M, \in)$. Since $N \models ZFC^{**}$, so too does $M \models ZFC^{**}$; with $ZFC^* \subseteq ZFC^{**}$, the proof is complete. □

2.3. **Basic Absoluteness Results.** As mentioned earlier, our forcing constructions will take place in ctms M of finite fragments of ZFC, and we should like to know (i) that certain notions are definable in M, and (ii) that those notions "mean" the same thing in M as they do in V. Luckily, if M is a transitive model of a sufficiently large finite fragment of ZFC, then a wide variety of notions are definable in M using formulae that are absolute in M; this is precisely what we want. Now, we say that an instance of a quantifier in a formula is *bounded* if and only if it is of the form "$\exists x \in y$" or "$\forall x \in y$," and we say that a formula φ is Δ_0 if and only if every quantifier in φ is bounded. As the following theorem shows, the notion of boundedness allows us to formalize the well-behaved nature of transitive classes.

Theorem 2.11. *If M is a transitive class, then all Δ_0 formulas are absolute in M.*

Proof. We prove this result by induction on the complexity of the Δ_0 formula φ. Since $(x = y)^M = (x = y)$ and $(x \in y)^M = (x \in y)$, the result holds for all atomic formulae. Furthermore, the induction steps for the connectives "$\neg$" and "$\wedge$" are obvious. Now, suppose that $\varphi(\vec{x}, z)$ is of form "$\exists y \in z \;\wedge\; \psi(\vec{x}, y, z)$," where ψ is a Δ_0 formula and $M \preccurlyeq_\psi V$. Let $\vec{a}$ be an appropriate tuple of sets in M. It follows from the definition of relativization together with the relationship between φ and ψ that $\varphi^M(\vec{a}, z)$ holds if and only if there exists a $b \in M$ such that $b \in z$ and $\psi^M(\vec{a}, b, z)$. By the transitivity of M, if $b \in V$ is such that $b \in z$, then with $z \in M$, it follows that $b \in M$. Coupled with the induction hypothesis, this means that there exists a $b \in M$ such that $b \in z$ and $\psi^M(\vec{a}, b, z)$ if and only if there exists a $b \in V$ such that $b \in z$ and $\psi^V(\vec{a}, b, z)$. Given that ψ^V means the same thing as ψ, and given the relationship between φ and ψ, the proof is complete. □

Corollary 2.12. *If M is a transitive model for a sufficiently large finite fragment of ZFC, then the following notions are definable in M using formulae that are absolute in M:*

(1) $x \subseteq y$.
(2) $x = \emptyset$.
(3) $x \cap y$.
(4) $S(x)$.
(5) x is a transitive set.
(6) x is an ordinal.
(7) $x = \omega$.

We omit the rather routine proof of this statement. For a detailed account of absoluteness, including a greatly expanded list of notions that are definable and absolute in transitive models of sufficiently large finite fragments of ZFC, see [4] and [5]. Note that by Corollary 2.12, the set of natural numbers in M is the "standard" ω of V whenever M is a transitive model of a large enough portion of ZFC.

3. The Logical Structure of the Argument

We now explain how we can use forcing to show $\mathrm{Con}(ZFC) \to \mathrm{Con}(ZFC+CH)$ and $\mathrm{Con}(ZFC) \to \mathrm{Con}(ZFC + \neg CH)$. But first, we must point out that we were slightly disengenous in the introduction: Given a ctm of a finite fragment ZFC^*, the method of forcing does not immediately guarantee the existence of new models of ZFC^*+CH and of $ZFC^*+\neg CH$; this is because in order to carry out the forcing construction in a ctm M, we need a certain finite collection of axioms of ZFC to hold in M. So, given any finite fragment ZFC^*, we adjoin a suitable collection of axioms of ZFC to ZFC^* in order to ensure that the forcing construction can take place. Calling the finite fragment obtained in this way ZFC^{**}, forcing then shows that any ctm M of ZFC^{**} can be extended to a model of $ZF^{**} + CH$ and to a model of $ZFC^{**} + \neg CH$.

We may now proceed with our explanation. If, say, we can derive an inconsistency from $ZFC + \neg CH$, then the proof of this inconsistency must employ only a finite collection $ZFC^* + \neg CH$. For some possibly larger finite fragment ZFC^{**}, ZFC proves that there is a model N of the inconsistent set $ZFC^{**} + \neg CH$. This contradicts the Completess Theorem, which is also provable within ZFC. Thus ZFC is also inconsistent.

Throughout the rest of this paper, when we say "ctm of ZFC," we will really mean "ctm of a large enough finite fragment of ZFC to carry out the argument at hand." It is not necessary to explicitly produce such a finite list of axioms of ZFC; it is merely sufficient to check that once we fix a finite fragment ZFC^*, only finitely many theorems of ZFC are needed in order to produce the desired results, so that only finitely many axioms of ZFC are referenced in the entirety of the argument for the case of the fixed fragment ZFC^*.

4. Forcing Notions

Over the course of the next few sections, we develop the basics of forcing, in particular showing how to extend a ctm M of ZFC to a larger model $M[G]$ of ZFC. The intuitive idea behind this construction is as follows (see [3] and [4]): Suppose that there are people living in M. For them, M is the entire universe of sets, so that they reject the existence of any objects lying outside of M. Now, for some set G not belonging to M, there is a way to construct an extended universe $M[G]$ via a set-theoretic naming processes that occurs in M. Because of this, the people living in M can determine whether certain statements are true or false in $M[G]$, even though they know nearly nothing about G, and in fact, reject its existence in the first place.

We now introduce the basic notions needed in order to construct $M[G]$.

Definition 4.1. Let M be a ctm of ZFC. A *forcing notion* $\mathbb{P}$ in M is an ordered triple $(P, \leq, \mathbb{0})$ in M satisfying the following conditions:

(1) $\leq$ is a transitive, reflexive relation on P.
(2) $\mathbb{0}$ is the smallest element in P.

If, furthermore, $\mathbb{P}$ has the property that for each $p \in P$, there exist $q_1, q_2 \in P$ such that $p \leq q_1, q_2$ but $q_1, q_2 \nleq q$ for all $q \in P$, then $\mathbb{P}$ is called a *separative* forcing.

Note that if M is a ctm of ZFC and $\mathbb{P} \in M$, then the statement "$\mathbb{P}$ is a forcing notion" is absolute in M (see [4]). Note, furthermore, that the transitivity of M shows that each of P, $\leq$, and $\mathbb{0}$ are in M. The elements of P are often referred to as "conditions," and for $p, q \in P$, $p \leq q$ is usually read "p is weaker than q" or "q is stronger than p." In order to ensure that a forcing notion $\mathbb{P}$ in M produces a non-trivial extension of M, we require that $\mathbb{P}$ be separative (as will be seen in Theorem 5.2). The following two separative forcing notions will be used later to prove the independence of CH from ZFC:

Notation 4.2. Let I, J be sets. Then $\mathrm{Fn}(I, J)$ denotes the collection of functions $I \to J$ with finite domain.

Example 4.3 (The Cohen Forcing)**.** Let κ be a cardinal. Then the *Cohen Forcing* $\mathbb{C}_\kappa$ is the ordered triple $(\mathrm{Fn}(\kappa \times \omega, 2), \subseteq, \emptyset)$.

Example 4.4. Let α be an ordinal. Let K_α denote the collection of functions p from a subset of $\omega_{\alpha+1}$ to $\mathcal{P}(\omega_\alpha)$ such that $|\mathrm{dom}(p)| < \omega_{\alpha+1}$, and define $\mathbb{K}_\alpha := (K_\alpha, \subseteq, \emptyset)$.

We leave it to the reader to check that these are indeed separative forcing notions (see [2]). Now, let M once again be a ctm of ZFC, and let $\mathbb{P}$ be a forcing notion in M. For a certain subset $G \subset P$, we wish to construct an extension $M[G]$ of M by building new sets from G via set-theoretic processes definable in M. In order to do this, we consider a special class $V^{\mathbb{P}}$ that encodes how elements of the extension are constructed from G:

Definition 4.5. Let $\mathbb{P}$ be a forcing notion. Define $V^{\mathbb{P}}$, the class of $\mathbb{P}$-*names*, by recursion in V:

$$V_0^{\mathbb{P}} := \emptyset,$$

$$V_{\alpha+1}^{\mathbb{P}} := \mathcal{P}(V_\alpha^{\mathbb{P}} \times P),$$

$$V_\alpha^{\mathbb{P}} := \bigcup_{\beta<\alpha} V_\beta^{\mathbb{P}} \text{ for limit ordinals } \alpha,$$

and set

$$V^{\mathbb{P}} := \bigcup_{\alpha \in ON} V_\alpha^{\mathbb{P}}.$$

"Ordinary" $\mathbb{P}$-names will be denoted by the symbols $\underset{\sim}{x}$, $\underset{\sim}{y}$, $\underset{\sim}{f}$, and so on. We will reserve the symbols $\underset{\cdot}{x}$, $\underset{\cdot}{y}$, $\underset{\cdot}{f}$, etc. for a special subclass of $V^{\mathbb{P}}$ (which we will soon introduce). Now, the defining structure of $V^{\mathbb{P}}$ allows us to define a rank function on this class, which will in turn allow us to define the elements of $M[G]$.

Definition 4.6. Let $\mathbb{P}$ be a forcing notion. Define the rank function rk on $V^{\mathbb{P}}$ by setting

$$\mathrm{rk}(\underset{\sim}{x}) := \max\Big\{\mathrm{rk}(\underset{\sim}{y}) + 1 \mid \exists p \in P\big((\underset{\sim}{y}, p) \in \underset{\sim}{x}\big)\Big\}.$$

In particular, if $(\underset{\sim}{y}, p) \in \underset{\sim}{x}$ for some $p \in P$, then $\mathrm{rk}(\underset{\sim}{y}) < \mathrm{rk}(\underset{\sim}{x})$ (see [2]). Using this fact, we are able to make the following definition.

Definition 4.7. Let $G \subseteq P$. By recursion on the ranks of $\mathbb{P}$-names, define

$$\underset{\sim}{x}[G] := \Big\{\underset{\sim}{y}[G] \mid \exists q \in G\big((\underset{\sim}{y}, q) \in \underset{\sim}{x}\big)\Big\}.$$

Then set

$$V[G] := \{\underset{\sim}{x}[G] \mid \underset{\sim}{x} \in V^{\mathbb{P}}\}.$$

Because the notion of being a $\mathbb{P}$-name is also absolute in M (see [4]), the following definition means the same thing in M as it does in V:

Definition 4.8. Let $\mathbb{P}$ be a forcing notion in a ctm M of ZFC. Then we define $M^{\mathbb{P}} := M \cap V^{\mathbb{P}} = \{x \in M \mid (x \text{ is a } \mathbb{P}\text{-name})^M\}$.

Using this definition, we can then set $M[G] := \{\underset{\sim}{x}[G] \mid \underset{\sim}{x} \in M^{\mathbb{P}}\}$. Now, in order to show that $M \subseteq M[G]$, we will first have to produce a special class of $\mathbb{P}$-names that identify the elements of V. If $\mathbb{0} \in G$, then there is a particularly slick way to do this.

Definition 4.9. Let $G \subseteq P$ and suppose that $\mathbb{0} \in G$. By rank-recursion, define

$$\underset{\cdot}{x} := \big\{(\underset{\cdot}{y}, \mathbb{0}) \mid y \in x\big\}.$$

So, for example, $\underset{\cdot}{\emptyset} = \emptyset$, $\underset{\cdot}{1} = 1$, $\underset{\cdot}{2} = 2$, and so on. In what follows, we limit ourselves to working with G that contain $\mathbb{0}$.

Theorem 4.10. *Suppose that $G \subseteq P$ and that $\mathbb{0} \in G$. Then $\underset{\cdot}{x} = x$ for all $x \in V$.*

Proof. Work by induction on the rank of $\underset{\cdot}{x} \in V^{\mathbb{P}}$. First, note that since $\mathbb{0} \in G$, we have that $\underset{\cdot}{x} = \big\{(\underset{\cdot}{y}, \mathbb{0}) \mid y \in x\big\}$. Now, if $\mathrm{rk}(\underset{\cdot}{x}) = 0$, then $\underset{\cdot}{x} = \underset{\cdot}{0}$, so that

$$\underset{\cdot}{x}[G] = \big\{(\underset{\cdot}{y}, \mathbb{0}) \mid y \in \emptyset\big\} = \emptyset.$$

Now, suppose that $\mathrm{rk}(\underset{\cdot}{x}) = \alpha$ and that $\underset{\cdot}{y}[G] = y$ for all $\mathbb{P}$-names $\underset{\cdot}{y}$ of rank smaller than α. Then

$$\underset{\cdot}{x} = \big\{(\underset{\cdot}{y}, \mathbb{0}) \mid y \in x\big\} = \{y \mid y \in x\} = x.$$

By induction, the proof is complete. □

Using an absoluteness argument, one can check that if x is a member of M, then so is $\underset{\cdot}{x}$ (see [4]). Together with Theorem 4.10, this immediately gives the following corollary:

Corollary 4.11. *Let $\mathbb{P}$ be a forcing notion in a ctm M of ZFC, and suppose that G is a subset of P containing $\mathbb{0}$. Then $M \subseteq M[G]$.*

We should also like to know that $G \in M[G]$. In order for this to be the case, we need to find a $\mathbb{P}$-name $\underset{\sim}{G}$ in $M^{\mathbb{P}}$ such that $\underset{\sim}{G}[G] = G$. Now, if we make the assignment $\underset{\sim}{G} := \big\{(\underset{\cdot}{p}, p) \mid p \in P\big\}$, then Theorem 4.10 shows that $\underset{\sim}{G}[G] = G$.

Before we move on to consider the particular characteristics that G needs to have in order to make forcing arguments work in $M[G]$, we mention two special kinds of $\mathbb{P}$-names that identify pairs in $M[G]$. For $\mathbb{P}$-names $\underset{\sim}{x}, \underset{\sim}{y}$, define

$$\mathrm{up}(\underset{\sim}{x}, \underset{\sim}{y}) := \big\{(\underset{\sim}{x}, \mathbb{0}), (\underset{\sim}{y}, \mathbb{0})\big\}$$

and

$$\mathrm{op}(\underset{\sim}{x}, \underset{\sim}{y}) := \left\{ \Big(\big\{ (\underset{\sim}{x}, \mathbb{0}) \big\}, \mathbb{0} \Big), \Big(\big\{ (\underset{\sim}{x}, \mathbb{0}), (\underset{\sim}{y}, \mathbb{0}) \big\}, \mathbb{0} \Big) \right\}.$$

Through a tedious check, one can verify that if $G \subseteq P$ and $\mathbb{0} \in G$, then $\mathrm{up}(\underset{\sim}{x}, \underset{\sim}{y}) = \{\underset{\sim}{x}[G], \underset{\sim}{y}[G]\}$ and $\mathrm{op}(\underset{\sim}{x}, \underset{\sim}{y}) = (\underset{\sim}{x}[G], \underset{\sim}{y}[G])$ (see [2]).

5. Generic Extensions

In this section, we consider the basic theory of generic filters, these being the special sets that are needed in order to make forcing arguments work. Now, if M is a ctm of ZFC and G is a generic filter, then $M[G]$ is called a *generic extension* of M. Combined with the fact that $M[G]$ is constructed using a naming process that occurs in M, the special characteristics of G allow us to develop a technique within M for determining many important characteristics of the generic extension $M[G]$. For the people living in M, this technique − called the *forcing relation* − is mere formalism; they do not believe that it really says anything about objects outside of M. We will develop the forcing relation in the next section.

Although the verification of some of the following results is left to the reader, a complete account of the topics presented in this section can be found in [2].

Definition 5.1. Let $\mathbb{P}$ be a forcing notion.

(1) Two conditions $p_1, p_2 \in P$ are *compatible* if and only if there exists a $q \in P$ such that $p_1 \leq q \geq p_2$. In this case, we write $p_1 \mid p_2$. Otherwise, we say that p_1 and p_2 are *incompatible*, written $p_1 \perp p_2$.

(2) A set $A \subseteq P$ is an *anti-chain* if and only if every pair of conditions in A are incompatible. If, additionally, A is not a proper subset of any antichain A' in P, then A is called a *maximal antichain*.

(3) A set $D \subseteq P$ is *open dense* if and only if the following conditions hold:
   (a) If $p \in D$ and $q \geq p$, then $q \in D$ (*open*).
   (b) For every $p \in P$, there exists a $q \in D$ such that $q \geq p$ (*dense*).

(4) Let $p \in P$. A subset $D \subseteq P$ is *dense above* p if and only if for every $q \geq p$, there is a $q' \in D$ such that $q' \geq q$.

(5) A nonempty subset $G \subseteq P$ is a *filter on* P if and only if the following conditions hold:
   (a) If $p \in G$ and $q \leq p$, then $q \in G$ (*downwards closed*).
   (b) For every pair of conditions $p_1, p_2 \in G$, $p_1 \mid p_2$ (*directed*).

(6) A filter $G \subseteq P$ is $\mathbb{P}$-*generic over* M if and only if $G \cap D \neq \emptyset$ for every open dense $D \subseteq P$ in M.

In the language of compatibility, a forcing notion $\mathbb{P}$ is separative if and only if for each condition $p \in P$, there exist incompatible conditions q_1, q_2 that are stronger than p. Now that we know what generic filters are, we make good on our earlier claim that the people living in M reject the existence of G:

Theorem 5.2. *Let $\mathbb{P}$ be a separative forcing notion in a ctm M of ZFC. If G is a $\mathbb{P}$-generic filter over M, then $G \notin M$.*

Proof. Set $D_G := P - G$, and let $p \in P$. With $\mathbb{P}$ a separative forcing notion, there are incompatible elements above p, i.e., there exist $q_1, q_2 \in P$ such that $p \leq q_1, q_2$ and $q_1 \perp q_2$. With G directed, at most one of q_1, q_2 lives in G, so that at least one of these two elements is in D_G. This shows that D_G is dense in P. And with G downwards-closed, D_G must be open. Now, if $G \in M$, then $D_G \in G$ too. By the definition of $\mathbb{P}$-generic filters, it would then follow that G intersects D_G; this, however, contradicts the definition of D_G. Hence $G \notin M$. $\square$

The people in M therefore have good reason for rejecting the existence of $\mathbb{P}$-generic filters; but what about us, the people living in V? If $\mathbb{P}$ is a forcing notion in an arbitrary model N of ZFC, then we are not necessarily guaranteed the existence of a $\mathbb{P}$-generic filter over N. Luckily for us, the "small" size of M allows us to explicitly construct such filters:

Theorem 5.3. *Let $\mathbb{P}$ be a forcing notion in a ctm M of ZFC and let p_0 be a condition in P. Then there exists a $\mathbb{P}$-generic filter G over M containing p_0.*

Proof. With M countable, we may write down all of the open dense subsets $D \subseteq P$ in M in a sequence $\Big\{D_n \mid n \in \big(\omega - \{\emptyset\}\big)\Big\}$. With D_1 dense in P, there must be some condition $p_1 \in D_1$ such that $p_1 \geq p_0$. Additionally, whenever n is nonzero and $p_n \in D_n$, the density of D_{n+1} ensures that there exists a condition $p_{n+1} \in D_{n+1}$ for which $p_{n+1} \geq p_n$. Setting $G := \big\{q \in P \mid \exists n \in \omega(q \leq p_n)\big\}$, G is easily seen to be a $\mathbb{P}$-generic filter over M containing p_0. $\square$

Note that the enumeration of the open dense subsets of P occurs in V; this explains how it is possible for us to know that $\mathbb{P}$-generic filters exist even though the people living in M reject any such possibility.

In the course of proving some of the basic forcing results, it will be of great use to have a few alternative characterizations of $\mathbb{P}$-generic filters.

Theorem 5.4. *Let $\mathbb{P}$ be a forcing notion in a ctm M, and let G be a filter on P. Then the following statements are equivalent:*

(1) G is $\mathbb{P}$-generic over M.
(2) G intersects every maximal anti-chain $A \subseteq P$ that is in M.
(3) G intersects every dense $D \subseteq P$ that belongs to M.

Proof. (1) $\to$ (2): Let A be a maximal anti-chain in P that belongs to M. Then consider the set $D_A := \{p \in P \mid \exists q \in A(q \leq p)\}$; D_A is clearly open in P. We want to show that D_A is dense in P, i.e., that for every condition in P, there is a stronger condition in D_A. So, let $p_0 \in P$. If $p_0 \in A$, then $p_0 \in D_A$, and our search is complete. If $p_0 \notin D_A$, then with A a maximal anti-chain in P, there is some condition q_0 in A such that $q_0 \mid p_0$. With G a filter on P, there must in turn be some $p \in G$ such that $p_0 \leq p \geq q_0$; by the definition of D_A, $p \in D_A$. Thus D_A is dense in P, so that G intersects D_A. With G downwards-closed, this means that G must also intersect A.

(2) $\to$ (3): Let D be a dense subset of P that belongs to M. Let A be a maximal anti-chain in D. If A is not a maximal anti-chain in P, then there is some $p_0 \in P - A$ such that $p_0 \perp q$ for every $q \in A$. With D dense in P, there is then some $q_0 \in D$ such that $p_0 \leq q_0$. It follows that q_0 must be incompatible with every condition in A, for whenever $q_0 \mid p$ for some condition p, $p_0 \mid p$ too. But then $q_0 \notin A$ and $A \cup \{q_0\}$ is an anti-chain; this contradicts the maximality of A. Hence A is a maximal anti-chain in P too, so that G intersects A. With $A \subseteq D$, we are done.

(3) $\to$ (1): This portion of the proof is obvious, for open dense sets are in particular dense. $\square$

Using Theorem 5.4, we can verify one more especially helpful characterization of $\mathbb{P}$-generic filters:

Corollary 5.5. *Let $\mathbb{P}$ be a forcing notion in a ctm M of ZFC, and suppose that $p \in P$. Then a filter G in P is $\mathbb{P}$-generic over M if and only if G intersects every set $D \subseteq P$ in M that is dense above p.*

Proof. Suppose that G intersects every set $D \subseteq P$ in M that is dense above p. Since dense sets in P are in particular dense above p, this means that G intersects every dense subset of P that is in M. By Theorem 5.4.3, then, G is $\mathbb{P}$-generic over M. Conversely, suppose that G is $\mathbb{P}$-generic over M, and let $D \in M$ be dense above p. Define $E := D \cup \{x \in P \mid x \perp p\}$. Then $E \in M$, and one easily checks that E is dense in P, so that by Theorem 5.4.3, $G \cap E \neq \emptyset$. Let $r \in G \cap E$. With $r, p \in G$ and G directed, this means that $r \in D$. □

6. The Forcing Relation

We are now ready to develop the relation that enables the people living in M to determine many of the characteristics of $M[G]$. But before we actually define the forcing relation, we must first formalize what it means for the people living in M to make assertions about $M[G]$:

Definition 6.1 (The Forcing Language)**.** Let $\mathbb{P}$ be a forcing notion. The *forcing language* in $\mathbb{P}$, denoted $\mathcal{FL}_{\mathbb{P}}$, is the collection of F.O. set-theoretic formulae whose constant symbols are $\mathbb{P}$-names. The *forcing language* in $\mathbb{P}, M$, denoted $\mathcal{FL}_{\mathbb{P},M}$, is the collection of F.O. set-theoretic formulae whose constant symbols are $\mathbb{P}$-names in M.

Hence, the people living in M make statements about $M[G]$ using $\mathcal{FL}_{\mathbb{P},M}$. Now, the forcing relation:

Definition 6.2 (The Forcing Relation)**.** Let $\mathbb{P}$ be a forcing notion in a ctm M of ZFC, let $p_0 \in P$, let $\varphi(x_1, \ldots, x_n)$ be a formula of $\mathcal{FL}_{\mathbb{P},M}$, and let $\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_n \in M^{\mathbb{P}}$. Define $p_0 \Vdash_{\mathbb{P}} \varphi(\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_n)$, read "$p_0$ *forces* $\varphi(\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_n)$," by a double recursion – first on the ranks of $\mathbb{P}$-names, then on the complexity of the formula φ:

(1) $\boldsymbol{\varphi(\underset{\sim}{x}_1, \underset{\sim}{x}_2) \equiv (\underset{\sim}{x}_1 = \underset{\sim}{x}_2)}$**:** $p_0 \Vdash_{\mathbb{P}} \underset{\sim}{x}_1 = \underset{\sim}{x}_2$ if and only if the following two conditions hold:
  (α) For each $(\underset{\sim}{y}_1, s_1) \in \underset{\sim}{x}_1$, the set
  $$\{q \geq p_0 \mid q \geq s_1 \to \exists(\underset{\sim}{y}_2, s_2) \in \underset{\sim}{x}_2(q \geq s_2 \wedge q \Vdash_{\mathbb{P}} \underset{\sim}{y}_1 = \underset{\sim}{y}_2)\}$$
  is dense above p_0.
  (β) An analogous statement holds for each $(\underset{\sim}{y}_2, s_2) \in \underset{\sim}{x}_2$.

(2) $\boldsymbol{\varphi(\underset{\sim}{x}_1, \underset{\sim}{x}_2) \equiv (\underset{\sim}{x}_1 \in \underset{\sim}{x}_2)}$**:** $p_0 \Vdash_{\mathbb{P}} \underset{\sim}{x}_1 \in \underset{\sim}{x}_2$ if and only if the set
$$\{q \geq p_0 \mid \exists(\underset{\sim}{y}, s) \in \underset{\sim}{x}_2(q \geq s \wedge q \Vdash_{\mathbb{P}} \underset{\sim}{y} = \underset{\sim}{x}_1)\}$$
is dense above p_0.

(3) $\boldsymbol{\varphi(\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_n) \equiv \neg\psi}$**:** $p_0 \Vdash_{\mathbb{P}} \neg\psi$ if and only if $q \nVdash_{\mathbb{P}} \psi$ for all $q \geq p_0$, i.e., $q \Vdash_{\mathbb{P}} \psi$ does not hold for any $q \geq p_0$.

(4) $\boldsymbol{\varphi(\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_n) \equiv \psi_1 \wedge \psi_2}$**:** $p_0 \Vdash_{\mathbb{P}} \psi_1 \wedge \psi_2$ if and only if $p_0 \Vdash_{\mathbb{P}} \psi_1$ and $p_0 \Vdash_{\mathbb{P}} \psi_2$.

(5) $\boldsymbol{\varphi(\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_n) \equiv \exists z \psi(z)}$**:** $p_0 \Vdash_{\mathbb{P}} \exists z \psi(z)$ if and only if the set

$$\left\{ q \geq p_0 \mid \exists \underset{\sim}{z} \in M^{\mathbb{P}} \left(q \Vdash_{\mathbb{P}} \psi(\underset{\sim}{z}) \right) \right\}$$

is dense above p_0.

Using Definition 6.2, we can quickly prove:

Theorem 6.3. *Let $\mathbb{P}$ be a forcing notion in a ctm M of ZFC and let φ be a sentence of $\mathcal{FL}_{\mathbb{P},M}$. Then the following statements hold:*

(1) If $p \Vdash_{\mathbb{P}} \varphi$ and $q \geq p$, then $q \Vdash_{\mathbb{P}} \varphi$.
(2) $p \Vdash_{\mathbb{P}} \varphi$ if and only if the set of conditions that force φ is dense above p.
(3) The set $\Delta_{\varphi} := \{p \in P \mid (p \Vdash_{\mathbb{P}} \varphi) \vee (p \nVdash_{\mathbb{P}} \varphi)\}$ is open dense in P.

Proof. Exercise (see [2] and [4]). □

Using Theorem 6.3, we are now able to prove the first key result about the forcing relation:

Theorem 6.4 (The Forcing Theorem)**.** *Let $\mathbb{P}$ be a forcing notion in a ctm M of ZFC, let $\varphi(x_1, \ldots, x_n)$ be a formula of $\mathcal{FL}_{\mathbb{P},M}$, let $\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_n$ be $\mathbb{P}$-names in M, and let G be a $\mathbb{P}$-generic filter over M. Then the following statements hold:*

(1) If $p \in G$ and $p \Vdash_{\mathbb{P}} \varphi(\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_n)$, then $M[G] \models \varphi(\underset{\sim}{x}_1[G], \ldots, \underset{\sim}{x}_n[G])$.
(2) If $M[G] \models \varphi(\underset{\sim}{x}_1[G], \ldots, \underset{\sim}{x}_n[G])$, then there exists a $p \in G$ such that p forces $\varphi(\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_n)$.

Proof. The overall structure of the proof is induction on the complexity of the formula φ, but for the case in which φ is an atomic formula of form "$x_1 = x_2$," we will have to carry out a separate induction on the ranks of the $\mathbb{P}$-names in M that are substituted for the free variables in φ. Since the other induction steps are much easier, we leave their verification to the reader. Once again, a complete proof of this theorem can be found in [2].

Set $\mathrm{rk}'(\underset{\sim}{x}_1, \underset{\sim}{x}_2) := \max\left\{\mathrm{rk}(\underset{\sim}{x}_1), \mathrm{rk}(\underset{\sim}{x}_2)\right\}$. If $\mathrm{rk}'(\underset{\sim}{x}_1, \underset{\sim}{x}_2) = 0$, then $\underset{\sim}{x}_1 = \underset{\sim}{x}_2 = \emptyset$. Since $\emptyset[G] = \emptyset$, and $\emptyset = \emptyset$ is clearly true in $M[G]$, so that (1) holds. By Definition 6.2.1, it is clear that $p \Vdash_{\mathbb{P}} \emptyset = \emptyset$ for all $p \in P$, so that (2) holds. For rk'$(\underset{\sim}{x}_1, \underset{\sim}{x}_2) > 0$, we will verify (1) and (2) separately.
(1): Suppose that $p \in G$, that $p \Vdash_{\mathbb{P}} \underset{\sim}{x}_1 = \underset{\sim}{x}_2$, and that (1) holds for all $\mathbb{P}$-names $\underset{\sim}{y}_1, \underset{\sim}{y}_2$ in M with $\mathrm{rk}'(\underset{\sim}{y}_1, \underset{\sim}{y}_2) < \mathrm{rk}'(\underset{\sim}{x}_1, \underset{\sim}{x}_2)$. We will prove that $M[G] \models \underset{\sim}{x}_1[G] = \underset{\sim}{x}_2[G]$ by showing that $M[G] \models \underset{\sim}{x}_1[G] \subseteq \underset{\sim}{x}_2[G]$ and $M[G] \models \underset{\sim}{x}_2[G] \subseteq \underset{\sim}{x}_1[G]$. Now, suppose that $x \in \underset{\sim}{x}_1[G]$; then x is of form $\underset{\sim}{y}_1[G]$, where $(\underset{\sim}{y}_1, s_1) \in \underset{\sim}{x}_1$ for some $s_1 \in G$. With G directed, there is some $r \in G$ such that $s_1 \leq r \geq p$. By Theorem 6.3.1, $r \Vdash_{\mathbb{P}} \underset{\sim}{x}_1 = \underset{\sim}{x}_2$. Using Definition 6.2.1 and Corollary 5.5, it follows that there is a condition $q \in G$ and an ordered pair $(\underset{\sim}{y}_2, s_2) \in \underset{\sim}{x}_2$ such that $s_1 \leq r \leq q \geq s_2$ and q forces $\underset{\sim}{y}_1 = \underset{\sim}{y}_2$. Fix such a $(\underset{\sim}{y}_2, s_2)$. Then $\mathrm{rk}'(\underset{\sim}{y}_1, \underset{\sim}{y}_2) < \mathrm{rk}'(\underset{\sim}{x}_1, \underset{\sim}{x}_2)$, so that by our induction hypothesis, $M[G] \models \underset{\sim}{y}_1[G] = \underset{\sim}{y}_2[G]$. Furthermore, with $q \geq s_2$ and G downwards-closed, $s_2 \in G$, which gives us that $\underset{\sim}{y}_2[G] \in \underset{\sim}{x}_2[G]$. Thus $M[G] \models \underset{\sim}{y}_1[G] \in \underset{\sim}{x}_2[G]$, so that $M[G] \models \underset{\sim}{x}_1[G] \subseteq \underset{\sim}{x}_2[G]$. Since our argument is symmetric about $\underset{\sim}{x}_1, \underset{\sim}{x}_2$, we have in fact shown that $M[G] \models \underset{\sim}{x}_1[G] = \underset{\sim}{x}_2[G]$; by induction, then, (1) holds for all φ of form $\underset{\sim}{x}_1 = \underset{\sim}{x}_2$, where $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$ are in M.
(2): Assume that $M[G] \models \underset{\sim}{x}_1[G] = \underset{\sim}{x}_2[G]$ and that (2) holds for all names $\underset{\sim}{y}_1, \underset{\sim}{y}_2$

in M with $\mathrm{rk}'(\underset{\sim}{y}_1, \underset{\sim}{y}_2) < \mathrm{rk}'(\underset{\sim}{x}_1, \underset{\sim}{x}_2)$. Let $D_{\underset{\sim}{x}_1, \underset{\sim}{x}_2}$ denote the collection of conditions $r \in P$ such that at least one of the following conditions holds:

(i) $r \Vdash_{\mathbb{P}} \underset{\sim}{x}_1 = \underset{\sim}{x}_2$.

(ii) There exists a name $(\underset{\sim}{y}_1, s_1) \in \underset{\sim}{x}_1$ such that $r \geq s_1$ and such that for all $(\underset{\sim}{y}_2, s_2) \in \underset{\sim}{x}_2$ and $q \in P$, if $q \geq s_1$ and $q \Vdash_{\mathbb{P}} \underset{\sim}{y}_1 = \underset{\sim}{y}_2$, then $q \perp r$.

(iii) There exists a name $(\underset{\sim}{y}_2, s_2) \in \underset{\sim}{x}_2$ such that $r \geq s_2$ and such that for all $(\underset{\sim}{y}_1, s_1) \in \underset{\sim}{x}_1$ and $q \in P$, if $q \geq s_2$ and $q \Vdash_{\mathbb{P}} \underset{\sim}{y}_1 = \underset{\sim}{y}_2$, then $q \perp r$.

We seek to show, first, that no condition in G can satisfy either (ii) or (iii), and second, that $D_{\underset{\sim}{x}_1, \underset{\sim}{x}_2}$ is dense in P. Towards the former: Suppose that there is some condition $r \in G$ and some $(\underset{\sim}{y}_1, s_1) \in \underset{\sim}{x}_1$ that satisfy (i). Then $s_1 \in G$, so that $\underset{\sim}{y}_1[G] \in \underset{\sim}{x}_1[G]$. By our assumption, then, $M[G] \models \underset{\sim}{y}_1[G] \in \underset{\sim}{x}_2[G]$. By the definition of $\underset{\sim}{x}_2[G]$, there is some $(\underset{\sim}{y}_2, s_2) \in \underset{\sim}{x}_2$ such that $s_2 \in G$ and $M[G] \models \underset{\sim}{y}_1[G] = \underset{\sim}{y}_2[G]$. With $\mathrm{rk'}(\underset{\sim}{y}_1, \underset{\sim}{y}_2) < \mathrm{rk'}(\underset{\sim}{x}_1, \underset{\sim}{x}_2)$, the induction hypothesis admits the existence of a condition $q_0 \in G$ for which $q_0 \Vdash_{\mathbb{P}} \underset{\sim}{y}_1 = \underset{\sim}{y}_2$. With G directed, there is another condition $q \in G$ such that $q_0 \leq q \geq s_2$. By Theorem 6.3.1, it follows that q forces $\underset{\sim}{y}_1 = \underset{\sim}{y}_2$. By (i), we must then have $q \perp r$. This, however, contradicts the fact that G is directed. Hence no $r \in G$ can satisfy (i), and an analogous argument shows the same for (ii).

Now, towards showing that $D_{\underset{\sim}{x}_1, \underset{\sim}{x}_2}$ is dense in P: Fix a condition $p \in P$; we wish to find a condition $r \geq p$ in $D_{\underset{\sim}{x}_1, \underset{\sim}{x}_2}$. If $p \Vdash_{\mathbb{P}} \underset{\sim}{x}_1 = \underset{\sim}{x}_2$, then we are done. If this is not the case, then either (α) or (β) of Definition 6.2.1 fails. Suppose that (α) fails. Then there are $(\underset{\sim}{y}_1, s_1) \in \underset{\sim}{x}_1$ and $r \geq p$ such that $r \geq s_1$ and such that for all $q \geq r$ and $(\underset{\sim}{y}_2, s_2) \in \underset{\sim}{x}_2$, either $q \geq s_2$ fails or $q \Vdash_{\mathbb{P}} \underset{\sim}{y}_1 = \underset{\sim}{y}_2)$ fails. Now, let $(\underset{\sim}{y}_2, s_2) \in \underset{\sim}{x}_2$. If there is some condition $q' \in P$ for which $q' \geq s_2$ and $q' \Vdash_{\mathbb{P}}$, then we must have that $q' \perp r$, for a common extension of q and r would contradict the above equation. Thus $r \geq p$ and r satisfies (i), so that $r \in D_{\underset{\sim}{x}_1, \underset{\sim}{x}_2}$. A similar argument holds in the case that (β) fails, so that $D_{\underset{\sim}{x}_1, \underset{\sim}{x}_2}$ is dense in P. By Theorem 5.4.3, G must then intersect $D_{\underset{\sim}{x}_1, \underset{\sim}{x}_2}$; by our earlier work, this means that there is some $r \in G$ for which $r \Vdash_{\mathbb{P}}$. By induction, (2) holds for all φ of form $\underset{\sim}{x}_1 = \underset{\sim}{x}_2$, where $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$ are in M. □

Using Forcing Theorem, one can quickly prove the following lemma, which is a standard result in the forcing literature (see [2]).

Corollary 6.5. *Let $\mathbb{P}$ be a forcing notion in a ctm M of ZFC, let G be $\mathbb{P}$-generic over M, and let p be a condition in G. Then we have the following:*

(1) If $p \Vdash_{\mathbb{P}} \underset{\sim}{z} \in \underset{\sim}{y}$, then there exist a $\mathbb{P}$-name $\underset{\sim}{x}$ and a condition $q \in G$ such that $q \geq p$, $rk(\underset{\sim}{x}) < rk(\underset{\sim}{y})$, and $q \Vdash_{\mathbb{P}} \underset{\sim}{x} = \underset{\sim}{z}$.

(2) If $p \Vdash_{\mathbb{P}} (\underset{\sim}{f} \in \underset{\sim}{B}^{\underset{\sim}{A}} \wedge \underset{\sim}{x}_0 \in \underset{\sim}{A})$, then there exist a $\mathbb{P}$-name $\underset{\sim}{y} \in \underset{\sim}{B}$ and conditions $p, r \in G$ such that $q \geq p$, $(\underset{\sim}{y}, r) \in \underset{\sim}{B}$, and $q \Vdash_{\mathbb{P}} \underset{\sim}{f}(\underset{\sim}{x}_0) = \underset{\sim}{y}$.

Together with Corollary 6.5, the Forcing Theorem now allows us to prove that generic extensions of ctms of ZFC are also models of ZFC:

Theorem 6.6 (The Generic Model Theorem)**.** *Let $\mathbb{P}$ be a forcing notion in a ctm M of ZFC and let G be a $\mathbb{P}$-generic filter over M. Then $M[G] \models ZFC$.*

Proof. We will prove that the Axioms of Extensionality, Comprehension, Pairing, and Choice all hold in $M[G]$, and leave the verification of the rest to the reader. A complete proof of this theorem can be found in [2]; there, one sees that the Forcing Theorem is really only needed for proving that the Axioms of Comprehension, Union, Replacement, Powerset, and Choice hold in $M[G]$.

Axiom of Extensionality: In order to prove that the Axiom of Extensionality holds in $M[G]$, we first show that $M[G]$ is transitive. So, let $x \in M[G]$, and suppose that $y \in x$. By the definition of $M[G]$, there is a $\mathbb{P}$-name $\underset{\sim}{x}$ such that $x = \underset{\sim}{x}[G]$. And by the definition of $\underset{\sim}{x}[G]$, there exists a $\mathbb{P}$-name $\underset{\sim}{y}$ and a condition $p \in G$ such that $(\underset{\sim}{y}, p) \in \underset{\sim}{x}$ and $y = \underset{\sim}{y}[G]$. But then $y \in M[G]$ too, so that $M[G]$ is transitive.

Now, let $x, y \in M[G]$, and suppose that for each $z \in M[G]$, $z \in x \iff z \in y$. If $z \in x$, then by the transitivity of $M[G]$, $z \in M[G]$. By assumption, then, $z \in y$. An analogous argument shows that if $z \in y$, then $z \in x$ too. By the Axiom of Extensionality in V, then, $x = y$, so that the Axiom of Extensionality holds in $M[G]$.

Axiom Schema of Comprehesion: Let $\varphi(z, p_1, \ldots p_n)$ be an arbitrary formula in the language of set theory. We need to show that for all tuples $(x, p_1, \ldots, p_n)$ of sets in $M[G]$, the set $\big\{z \in M[G] \mid z \in x \wedge \varphi(z, p_1, \ldots, p_n)\big\}$ lives in $M[G]$. For the sake of brevity, we shall omit the parameters $p_1, \ldots, p_n$ in the rest of this proof. Now, let $x \in M[G]$, and let $\underset{\sim}{x}$ be a $\mathbb{P}$-name for x, i.e., $x = \underset{\sim}{x}[G]$. Let $\operatorname{dom}(\underset{\sim}{x})$ denote the collection of $\mathbb{P}$-names $\underset{\sim}{z} \in M^{\mathbb{P}}$ such that $(\underset{\sim}{z}, q) \in \underset{\sim}{x}$ for some $q \in P$, and consider the set of ordered pairs $(\underset{\sim}{z}, p) \in \operatorname{dom}(\underset{\sim}{x}) \times P$ such that p forces both $\underset{\sim}{z} \in \underset{\sim}{x}$ and $\varphi(\underset{\sim}{z})$. This set is a $\mathbb{P}$-name; call it $\underset{\sim}{y}$. We will be done if we can show that $\big\{z \in M[G] \mid z \in x \wedge \varphi(z)\big\} = \underset{\sim}{y}[G]$. So, let $z \in \underset{\sim}{y}[G]$; then by the definition of $\underset{\sim}{y}[G]$, there exists a $\mathbb{P}$-name $\underset{\sim}{z}$ and a condition $p \in G$ such that $(\underset{\sim}{z}, p) \in \underset{\sim}{y}$ and $z = \underset{\sim}{z}[G]$. Given the definition of $\underset{\sim}{y}$, this in turn means that p forces both $\underset{\sim}{z} \in \underset{\sim}{x}$ and $\varphi(\underset{\sim}{z})$. By (1) of the Forcing Theorem, we then have $M[G] \models \underset{\sim}{z}[G] \in \underset{\sim}{x}[G] \wedge \varphi\big(\underset{\sim}{z}[G]\big)$, so that $\underset{\sim}{y}[G] \subseteq \big\{z \in M[G] \mid z \in x \wedge \varphi(z)\big\}$. Conversely, suppose that $z \in M[G]$, $z \in x$, and that $\varphi(z)$ holds. In particular, these statements are true in $M[G]$, so that by (2) of the Forcing Theorem, there exists a $\mathbb{P}$-name $\underset{\sim}{z}$ and a condition $p \in G$ such that p forces both $\underset{\sim}{z} \in \underset{\sim}{x}$ and $\varphi(\underset{\sim}{z})$. By the definition of $\underset{\sim}{y}$, it follows that $(\underset{\sim}{z}, p) \in \underset{\sim}{y})$. With $p \in G$, we then have that $M[G] \models \underset{\sim}{z}[G] \in \underset{\sim}{y}[G]$, which is to say that $\big\{z \in M[G] \mid z \in x \wedge \varphi(z)\big\} \subseteq \underset{\sim}{y}[G]$. Thus $\big\{z \in M[G] \mid z \in x \wedge \varphi(z)\big\} = \underset{\sim}{y}[G]$, so that the proof is complete.

Axiom of Pairing: Suppose that $x, y \in M[G]$. Then there are $\mathbb{P}$-names $\underset{\sim}{x}, \underset{\sim}{y}$ such that $x = \underset{\sim}{x}[G]$ and $y = \underset{\sim}{y}[G]$. With G downwards-closed, $\mathbb{0} \in G$, so that

$$\operatorname{up}(\underset{\sim}{x}, \underset{\sim}{y})[G] = \big\{\underset{\sim}{x}[G], \underset{\sim}{y}[G]\big\} = \{x, y\}.$$

Since $\operatorname{up}(\underset{\sim}{x}, \underset{\sim}{y}) \in M[G]$, $M[G]$ is a model for the Axiom of Pairing.

Axiom of Choice: Since ZFC proves that the Well-Ordering Theorem is equivalent to the Axiom of Choice (see [3]), it will be sufficient to prove that the Well-Ordering Theorem holds in $M[G]$. Towards this end, we first show that for each $x \in M[G]$, there exists an injection from x into Ω^M. Fix an $x \in M[G]$, and let $\underset{\sim}{x}$ be a $\mathbb{P}$-name for x. Because $\operatorname{dom}(\underset{\sim}{x})$ resides in M, and since the Axiom of Choice holds in M, it follows that there is a bijection between $\operatorname{dom}(\underset{\sim}{x})$ and κ, where $\kappa \in \Omega^M$ and

$|\text{dom}(\underset{\sim}{x})| = \kappa$. This bijection in turn allows us to write down $\text{dom}(\underset{\sim}{x})$ as a sequence of $\mathbb{P}$-names $\{\underset{\sim}{y}_\alpha \mid \alpha \in \kappa\}$. Now, set

$$\underset{\sim}{R} := \big\{\text{op}(\check{\alpha}, \underset{\sim}{y}_\alpha) \mid \alpha \in \kappa\big\} \times \{\mathbb{0}\}.$$

Then with $\mathbb{0} \in G$, $\underset{\sim}{R}[G] = \big\{(\alpha, \underset{\sim}{y}_\alpha) \mid \alpha \in \kappa\big\}$. By the Axiom of Comprehension in $M[G]$, the set

$$\Big\{(\alpha, \underset{\sim}{y}_\alpha[G]) \in \underset{\sim}{R}[G] \mid \exists p \in G\big((\underset{\sim}{y}_\alpha, p) \in \underset{\sim}{x}\big)\Big\}$$

is in $M[G]$; furthermore, this set is a bijection between a subset of κ and $\underset{\sim}{x}[G] = x$. Hence, for each $x \in M[G]$, there is a bijection between x and some ordinal in Ω^M.

Finally, since Ω^M is well-ordered by inclusion, any bijection from a set x onto a subset of Ω^M naturally induces a well-order of x: Letting f denote this bijection, then simply require that for $x_1, x_2 \in x$, $x_1 \leq x_2$ if and only if $f(x_1) \leq f(x_2)$. The proof is thus complete. □

Note that in order to prove that the Well-Ordering Theorem holds in $M[G]$, we only needed to make use of the ordinals in M. This suggests the following result:

Theorem 6.7. *Let $\mathbb{P}$ be a forcing notion in a ctm M of ZFC, and let G be a $\mathbb{P}$-generic filter over M. Then $\Omega^M = \Omega^{M[G]}$.*

Proof. By Theorem 6.6, we know that $M[G]$ is a transitive model of ZFC. Since $M \subseteq M[G]$, and since the notion of being an ordinal is absolute in transitive models of ZFC, it then follows that $\Omega^M \subseteq \Omega^{M[G]}$. We will prove the reverse set inclusion by transfinite induction. Let X denote the collection of ordinals α such that $\alpha \in \Omega^{M[G]}$ implies $\alpha \in \Omega^M$. Clearly, $0 \in X$. Now, let γ be a nonzero ordinal, and suppose that $\alpha \in X$ for every ordinal $\alpha < \gamma$. If $\gamma \notin M[G]$, then $\gamma \in X$. Suppose, on the other hand, that $\gamma \in M[G]$. Let $\underset{\sim}{\gamma}$ be a $\mathbb{P}$-name for γ, so that $\underset{\sim}{\gamma}[G] = \underset{\sim}{\gamma}$. By the Axiom Schema of Comprehension in M, it follows that

$$T := \Big\{\underset{\sim}{x} \in M^{\mathbb{P}} \mid \exists p \in P\big((\underset{\sim}{x}, p) \in \underset{\sim}{\gamma}_0\big)\Big\}$$

is a set in M. As a result, we may in turn form the set

$$T' := \Big\{\alpha \in \Omega^M \mid \exists \underset{\sim}{x} \in T \exists p \in P\big(p \Vdash_{\mathbb{P}} \check{\alpha} = \underset{\sim}{x}\big)\Big\}$$

in M. By the induction hypothesis, T' contains all of the ordinals in γ, so that $\gamma \subseteq \cup T'$ (in V). Since the union of a set of ordinals is once again an ordinal, T' must be an ordinal. And since γ and T' are ordinals with $\gamma \subset T'$, it follows that either $\gamma \in T'$ or $\gamma = T'$ (see [3]). Finally, the transitivity of M guarantees that $\gamma \in M$. By induction, $\Omega^{M[G]} \subseteq \Omega^M$, and the proof is complete. □

7. The Independence of the Continuum Hypothesis

Once again, let M be a ctm of ZFC, and let κ be a cardinal in M. We wish to show that $\mathbb{C}_\kappa$ adds κ reals to M, so that whenever G is $\mathbb{C}_\kappa$-generic over M, $M[G] \models 2^{\aleph_0} \geq \kappa$ (where $2^{\aleph_0}$ is the cardinality of the continuum). In particular, for $\kappa > \aleph_1$, this shows that $M[G] \models \neg CH$. In order to do this, however, we first have to show that κ remains a cardinal in $M[G]$. The problem is that cardinalities are defined in terms of the existence of certain bijective functions between ordinals, and *a priori*, there is no reason to believe that $M[G]$ contains the same such bijections as M does. If $M[G]$ has bijections of this form that are not contained in M, then we might then have $M \models |\kappa| = \kappa$ but $M[G] \models |\kappa| < \kappa$. As a reminder, if N is any model of ZFC containing κ, then $|\kappa|^N$ denotes the cardinality of κ as it is defined in N.

Definition 7.1. Let $\mathbb{P}$ be a forcing notion in a ctm M of ZFC, and let κ be an ordinal in M that M thinks is a cardinal, i.e., for which $|\kappa|^M = \kappa$. We say that $\mathbb{P}$ *preserves* κ if and only if $|\kappa|^{M[G]} = \kappa$ for all G that are $\mathbb{P}$-generic over M. Otherwise, we say that $\mathbb{P}$ *collapses* κ. If $\mathbb{P}$ preserves each ordinal that M thinks is a cardinal, then we say that $\mathbb{P}$ *preserves cardinalities* in M.

Our next goal, then, is to show that $\mathbb{C}_\kappa$ preserves cardinalities. In order to do this, we first remind the reader of the notion of cofinality, in terms of which we can then state (and prove) a sufficient condition for $\mathbb{P}$ to preserve cardinalities.

Definition 7.2 (Cofinality)**.** Let λ be a nonzero limit ordinal.

(1) A subset $\mathcal{C}$ is *cofinal* in λ if and only if $\cup\mathcal{C} = \lambda$.
(2) The *cofinality* of λ denoted $\mathrm{cf}(\lambda)$, is the cardinality of the smallest cofinal set $\mathcal{C} \subseteq \lambda$.
(3) An infinite cardinal κ is *regular* if and only if $\mathrm{cf}(\kappa) = \kappa$.

We will say that a forcing notion $\mathbb{P}$ in a ctm M of ZFC *preserves cofinalities* if and only if $\mathrm{cf}(\lambda)^{M[G]} = \mathrm{cf}(\lambda)^M$ for all nonzero limit ordinals $\lambda \in \Omega^M$. Additionally, we shall make use of the following result in the basic theory of cofinalities (see [2] and [4]):

Theorem 7.3. $ZFC \vdash$ (*If κ is an infinite cardinal, then $\kappa^+$is regular*).

Theorem 7.4. *Let $\mathbb{P}$ be a forcing notion in a ctm M of ZFC. If $\mathbb{P}$ preserves cofinalities in M, then $\mathbb{P}$ preserves cardinalities in M.*

Proof. Supppose that $\mathbb{P}$ preserves cofinalities in M, let G be $\mathbb{P}$-generic over M, and let κ be a regular cardinal in M, i.e., $\kappa = |\kappa|^M = \mathrm{cf}(\kappa)^M$. By assumption, $\mathrm{cf}(\kappa)^M = \mathrm{cf}(\kappa)^{M[G]}$. Furthermore, it is clear that $\mathrm{cf}(\kappa)^{M[G]} \leq |\kappa|^{M[G]} \leq |\kappa|^M$. All together, we then have that

$$|\kappa|^{M[G]} \leq |\kappa|^M = \mathrm{cf}(\kappa)^M = \mathrm{cf}(\kappa)^{M[G]} \leq |\kappa|^{M[G]}.$$

Thus all the inequalities must in fact be equalities, so that $|\kappa|^M = |\kappa|^{M[G]}$ and κ remains regular in $M[G]$. Now, if κ is a successor cardinal in M, then by Theorem 7.3, $|\kappa|^M = |\kappa|^{M[G]}$. And with M and $M[G]$ both transitive models of ZFC, it follows that $|\aleph_0|^{M[G]} = \aleph_0 = |\aleph_0|^M$. Finally, suppose that κ is a limit cardinal in M such that $\kappa > \omega$. Since successor cardinals are regular, it follows that the set $\mathcal{C} := \{\eta < \kappa \mid \eta \text{ is regular}\}$ is cofinal in κ. Because $\mathbb{P}$ preserves regular cardinals and $\Omega^M = \Omega^{M[G]}$, it follows that $\mathcal{C}^M = \mathcal{C}^{M[G]}$. Hence $M[G]$ thinks that κ is the supremum of a collection of cardinals, so that $M[G]$ must in turn think that κ is itself a cardinal. □

So, if we can prove that $\mathbb{C}_\kappa$ preserves cofinalities, then we will know that $\mathbb{C}_\kappa$ also preserves cardinalities. Once we do this, we will be on track to prove that $\mathbb{C}_\kappa$ adds κ reals to M. Towards the end of showing that $\mathbb{C}_\kappa$ preserves cofinalities, we will need one last bit of theory.

Definition 7.5. A collection of sets $\mathcal{A}$ is said to form a *delta system* with *root* R if and only if $X \cap Y = R$ for every pair of sets $X, Y \in \mathcal{A}$.

Lemma 7.6 (Delta System Lemma)**.** *Let κ be an uncountable regular cardinal, and let $\mathcal{A}$ be a collection of finite sets such that $|\mathcal{A}| = \kappa$. Then there exists a set $\mathcal{B} \subseteq \mathcal{A}$ such that $|\mathcal{B}| = \kappa$ and $\mathcal{B}$ forms a delta system.*

Proof. Let f be a bijection between $\mathcal{A}$ and κ. For each $n \in \omega$, let D_n denote the set $\{X \in \mathcal{A} \mid |X| = n\}$; then let A_n denote $f(D_n)$. Then for each $n \in \omega$, $|D_n| = |A_n|$. If $|D_n| < \kappa$ for every $n \in \omega$, then the regularity of κ implies that $\bigcup_{n\in\omega} A_n \subsetneq \kappa$; this, however, contradicts the fact that f is a bijection from the set $\mathcal{A} = \bigcup_{n\in\omega} D_n$ onto κ. Hence, there must be an $n \in \omega$ for which $|D_n| = \kappa$.

We now use ω-induction to prove that D_n is a delta system. If $n = 1$, then D_n is a delta system with empty root: Each element of D_n is a singleton, so that the intersection of distinct members of D_n is empty. Now, suppose that $n > 1$ and that the result holds for $n - 1$. For $p \in V$, define $S_p := \{X \in D_n \mid p \in X\}$. We consider two cases:

Case 1: Suppose that $|S_p| = \kappa$ for some $p \in V$. Then $\mathcal{E} := \{X - \{p\} \mid X \in S_p\}$ is a collection of κ sets, each of size $n - 1$. By the induction hypothesis, $\mathcal{E}$ possesses a delta system $\mathcal{C}$ with some root $R_{\mathcal{C}}$. It follows that the collection $\{Z \cup \{p\} \mid Z \in \mathcal{C}\}$ is a subset of $\mathcal{A}$ of size κ and is a delta system with root $R_{\mathcal{C}} \cup \{p\}$.

Case 2: Suppose that $|S_p| < \kappa$ for every $p \in V$. First, note that for any set T, $\{X \in D_n \mid X \cap T \neq \emptyset\} = \bigcup_{p\in T} S_p$. With κ regular, this set has size $< \kappa$ whenever T has size $< \kappa$. Fix any $X_0 \in D_n$; then $|\{X \in D_n \mid X \cap T \neq \emptyset\}| < \kappa$, so that there must be an $X_1 \in D_n$ such that $X_1 \cap X_0 = \emptyset$. Indeed, the same argument can be carried out for any $\alpha < \kappa$, so that there exists a κ-sequence of pairwise disjoint sets in D_n. This sequence forms a delta system in $\mathcal{A}$ with empty root, and is clearly of size κ. □

Definition 7.7. Let $\mathbb{P}$ be a forcing notion. We say that $\mathbb{P}$ satisfies the *countable chain condition (ccc)* if and only if every antichain in P is at most countable.

Using the Delta System Lemma, we can now prove that the Cohen Forcing $\mathbb{C}_\kappa$ satisfies ccc:

Theorem 7.8. *$\mathbb{C}_\kappa$ satisfies ccc.*

Proof. Suppose, for a contradiction, that there exists an uncountable collection of pairwise incompatible conditions of $\mathrm{Fn}(\kappa \times \omega, 2)$. Then in particular, there exists such a collection of size $\aleph_1$. Write this collection down as the sequence $\{p_\varepsilon \mid \varepsilon < \omega_1\}$, and for each $\varepsilon < \omega_1$, define $S_\alpha := \mathrm{dom}(p_\varepsilon)$. Then with $\aleph_1$ a regular cardinal, the Delta System Lemma applied to $\{S_\varepsilon \mid \varepsilon < \omega_1\}$ gives that there is an uncountable $B \subseteq \{S_\varepsilon \mid \varepsilon < \omega_1\}$ and some (finite) R such that $S_\alpha \cap S_\beta = R$ whenever α, β are distinct ordinals with $S_\alpha, S_\beta \in B$. With R^2 finite and $\{p_\varepsilon\}$ uncountable, there must be $\alpha < \beta < \omega_1$ such that $p_\alpha|_R = p_\beta|_R$, which means that $p_\alpha \mid p_\beta$. This is the desired contradiction. □

Theorem 7.9. *Let $\mathbb{P}$ be a forcing notion in a ctm M of ZFC. If $\mathbb{P}$ satisfies ccc, then $\mathbb{P}$ preserves cofinalities in M.*

Proof. Let G be a $\mathbb{P}$-generic filter over M, let κ be an infinite cardinal in M, let $\mathrm{cf}(\kappa)^{M[G]} = \lambda$, and let $\underset{\sim}{S}$ be a $\mathbb{P}$-name for a strictly increasing sequence of length λ that is cofinal in κ. Then $\underset{\sim}{S}[G]$ is a function from λ to κ such that $\bigcup\left\{\underset{\sim}{S}[G](\alpha) \mid \alpha \in \lambda\right\} = \kappa$. By (2) of the Forcing Theorem, there must exist a condition $p \in G$ that forces both $\underset{\sim}{S} \in \underset{\sim}{\kappa}^{\lambda}$ and $\bigcup\{\underset{\sim}{S}(\underset{\cdot}{\alpha}) \mid \underset{\cdot}{\alpha} \in \underset{\cdot}{\lambda}\}$. Now, for each $\alpha \in \lambda$, consider the set

$$D_\alpha := \Big\{q \geq p \mid \exists\gamma \in \kappa\big(q \Vdash_{\mathbb{P}} \underset{\sim}{S}(\underset{\cdot}{\alpha}) = \underset{\cdot}{\gamma}\big)\Big\}.$$

Then use D_α to define

$$Y_\alpha := \Big\{\gamma \in \lambda \mid \exists q \in D_\alpha\big(q \Vdash_{\mathbb{P}} \underset{\sim}{S}(\underset{\cdot}{\alpha}) = \underset{\cdot}{\gamma}\big)\Big\}.$$

By Theorem 6.3.2, D_α is dense above p for each $\alpha \in \lambda$. And since each D_α lives in M, so too is each Y_α in M. Now, we would like to show that each Y_α is at most countable. In this vein, fix a Y_α, and let $q_1, q_2 \in Y_\alpha$ be such that $q_1 \Vdash_{\mathbb{P}} \underset{\sim}{S}(\underset{\cdot}{\alpha}) = \underset{\cdot}{\gamma}_1$ and $q_2 \Vdash_{\mathbb{P}} \underset{\sim}{S}(\underset{\cdot}{\alpha}) = \underset{\cdot}{\gamma}_2$ for distinct $\gamma_1, \gamma_2 \in \kappa$. If $q_1 \leq p_0 \geq q_2$ for some condition $p_0 \in P$, then by Theorem 5.3, we would be able to construct a $\mathbb{P}$-generic filter G_0 with base condition p_0. By (2) of the Forcing Theorem, we would then have $M[G_0] \models \gamma_1 = \gamma_2$, which is impossible. Hence $q_1 \perp q_2$. Since $\mathbb{P}$ satisfies ccc and $Y_\alpha \in M$, it follows that Y_α is at most countable.

For each $\alpha \in \lambda$, let A_α be a maximal antichain in D_α. By the proof of Theorem 5.4.2, G must intersect every such A_α; this implies that $M[G] \models \underset{\sim}{S}[G](\alpha) \in Y_\alpha$ for each $\alpha \in \lambda$. Setting $Y := \bigcup\{Y_\alpha \mid \alpha \in \lambda\}$, we then have that Y is cofinal in κ. By the construction of Y, $|Y| \leq \lambda \cdot \omega$; and by cardinal arithmetic, $\lambda \cdot \omega = \lambda$ (see [3]). Since Y is constructed in M, this implies that $\mathrm{cf}(\kappa)^M \leq \lambda$. And since $\lambda = \mathrm{cf}(\kappa)^{M[G]} \leq \mathrm{cf}(\kappa)^M$, we must have $\mathrm{cf}(\lambda)^M = \mathrm{cf}(\lambda)^{M[G]}$. □

Combining Theorem 7.9 with Theorem 7.4 shows that $\mathbb{C}_\kappa$ preserves cardinalities in M. Using this fact, we can now prove that $\mathbb{C}_\kappa$ adds κ reals to M:

Theorem 7.10. *Let M be a ctm of ZFC, let G be $\mathbb{P}$-generic over M, and let κ be a cardinal in M. Then $M[G] \models 2^{\aleph_0} \geq \kappa$. In particular, if $\kappa > \aleph_1$, then $M[G] \models \neg CH$.*

Proof. To shorten notation, set $C_\kappa := \mathrm{Fn}(\kappa \times \omega, 2)$. First, we will show that $\cup G$ is a function from $\kappa \times \omega$ to 2 For each $\alpha \in \kappa$ and $n \in \omega$, set

$$D_{\alpha,n} := \{p \in \mathbb{C}_\kappa \mid (\alpha, n) \in \mathrm{dom}(p)\}.$$

One can quickly check that for each $\alpha \in \kappa$ and $n \in \omega$, $D_{\alpha,n}$ is an open dense subset of P in M. With G $\mathbb{P}$-generic, it follows that $G \cap D_{\alpha,n} \neq \emptyset$. Hence, for every pair $(\alpha, n) \in \kappa \times \omega$, there is a function $p \in G$ that is defined on (α, n). Since G is directed, it follows that $\cup G$ is a function with domain $\kappa \times \omega$.

We can now use the fact that $\cup G$ is a function to explicitly produce κ distinct real numbers in $M[G]$. For $\alpha \in \kappa$, define $\mathrm{r}_\alpha \in {}^\omega 2$ by setting

$$\mathrm{r}_\alpha(n) := \cup G\big((\alpha, n)\big) \text{ for } n \in \omega.$$

Furthermore, for $\alpha, \beta \in \kappa$, let $D_{\alpha,\beta}$ denote the collection of conditions $p \in C_\kappa$ that are defined on both (α, n) and (β, n) for some $n \in \omega$, and are such that $p\big((\alpha, n)\big) \neq p\big((\beta, n)\big)$. Once again, one easily checks that $D_{\alpha,\beta}$ is an open dense subset of C_κ in M, so that $G \cap D_{\alpha,\beta}$ must be nonempty. Hence, for any distinct $\alpha, \beta \in \kappa$, there is a natural number n and a condition $p \in C_\kappa$ such that $p\big((\alpha, n)\big)$ and $p\big((\beta, n)\big)$ are distinct. By the definition of r_γ for $\gamma \in \kappa$, it follows that $\mathrm{r}_\alpha(n) \neq \mathrm{r}_\beta(n)$, so that there are at least κ distinct reals in $M[G]$. With $|\omega| = \aleph_0$, this means that $M[G] \models 2^{\aleph_0} \geq \kappa$. As mentioned in the statement of the proof, fixing $\kappa > \aleph_1$ then gives $M[G] \models \neg CH$. □

By the work of Section 3, Theorem shows $\mathrm{Con}(ZFC) \to \mathrm{Con}(ZFC + \neg CH)$. Now, our next goal is to show that the forcing notion $\mathbb{K}_\alpha$ can be used to prove $\mathrm{Con}(ZFC) \to \mathrm{Con}(ZFC + CH)$. In fact, we will prove that if M is a ctm of ZFC, α is an ordinal in M, and G_α is a $\mathbb{K}_\alpha$-generic filter over M, then a generalized form of the Continuum Hypothesis holds in $M[G_\alpha]$. But in order for our arguments to work, we once again need to know that $\mathbb{K}_\alpha$ preserves certain cardinals.

Definition 7.11. Let $\mathbb{P}$ be a forcing notion, $\gamma \in \Omega$, and let $\{p_\varepsilon \mid \varepsilon < \mu\}$ be a sequence of conditions in P.

1. (1) $\{p_\varepsilon \mid \varepsilon < \gamma\}$ is *increasing* if and only if $p_{\varepsilon_1} \leq p_{\varepsilon_2}$ whenever $\varepsilon_1 < \varepsilon_2 < \gamma$.
2. (2) $\{p_\varepsilon \mid \varepsilon < \gamma\}$ is *bounded above* in P if and only if there exists a condition $q \in P$ such that $q \geq p_\varepsilon$ for every $\varepsilon < \gamma$.
3. (3) Let κ be an infinite cardinal. $\mathbb{P}$ is *κ-closed* if and only if every increasing sequence of conditions in P of length shorter than κ is bounded above in P.

The following theorem will allow us to prove that $\mathbb{K}_\alpha$ preserves all cardinals $\leq \aleph_{\alpha+1}$ by showing that $\mathbb{K}_\alpha$ is $\aleph_{\alpha+1}$-closed.

Theorem 7.12. *Let $\mathbb{P}$ be a κ-closed forcing notion in a ctm M for ZFC, let G be a $\mathbb{P}$-generic filter over M, let $\mu < \kappa$ be a cardinal, let $X \in M$, and let $f : \mu \to X$ be a function in $M[G]$. Then f belongs to M.*

Proof. Let $f \in X^\mu$ be a function in $M[G]$. Then there are $\mathbb{P}$-names $\underset{\sim}{f}, \underset{\sim}{X^\mu}$ for f and X^μ, respectively. By (2) of the Forcing Theorem, there is a condition $p \in G$ such that $p \Vdash_{\mathbb{P}} \underset{\sim}{f} \in \underset{\sim}{X^\mu}$. We will show that for every $p' \geq p$, there exist a condition $q \geq p'$ and a function $g : \mu \to X$ in M such that $q \Vdash_{\mathbb{P}} \underset{\sim}{f} = g$. So, fix a condition $p' \geq p$. Since $M[G] \models \underset{\cdot}{0}[G] \in \underset{\cdot}{\mu}[G]$, (2) of the Forcing Theorem shows that there is some condition $r \in G$ such that $r \Vdash_{\mathbb{P}} \underset{\cdot}{0} \in \underset{\cdot}{\mu}$. With G directed, there exists a condition $r' \in G$ such that $r' \Vdash_{\mathbb{P}} \underset{\sim}{f} \in \underset{\sim}{X^\mu} \wedge \underset{\cdot}{0} \in \underset{\cdot}{\mu}$. By Corollary 6.5.2, it follows that there is a condition $p_0 \in G$ such that $p_0 \Vdash_{\mathbb{P}} \underset{\sim}{f}(\underset{\cdot}{0}) = \underset{\cdot}{x}_0$ for some $x_0 \in X$. Similarly, for each $\alpha \in \mu$ and condtion $p_\alpha \in G$, the same argument produces a $p_{\alpha+1} \geq p_\alpha$ such that $p_{\alpha+1} \Vdash \underset{\sim}{f}(\underset{\cdot}{p}_{\alpha+1}) = \underset{\cdot}{x}_{\alpha+1}$. By recursion in M, $\{p_\alpha \mid \alpha \in \mu\}$ is a well-defined, increasing sequence that lives in M. Since $\mathbb{P}$ is a κ-closed forcing notion, it follows that there exists a condition $q \in P$ such that $q \geq p_\alpha$ for all $\alpha \leq \mu$. By Theorem 6.3.1, it follows that $q \Vdash_{\mathbb{P}} \underset{\sim}{f} \in \underset{\cdot}{X}^\mu$.

We have thus shown that the collection of conditions in P that force $\underset{\sim}{f} = g$ for some function $g \in X^\mu$ living in M is dense above p. By Corollary 5.5, there is a condition $s \in G$ such that $s \Vdash_{\mathbb{P}} \underset{\sim}{f} \in \underset{\cdot}{X}^\mu$. Since $\underset{\cdot}{X}^\mu$ is the canonical $\mathbb{P}$-name for X^μ as it is defined in M, it follows that $M[G] \models f \in M$. $\square$

Corollary 7.13. *Let M be a ctm of ZFC, and suppose that $\alpha \in M$. Then $\mathbb{K}_\alpha$ preserves all cardinals $\leq \omega_{\alpha+1}$.*

Proof. Let $\gamma < \omega_{\alpha+1}$, and let $\{p_\varepsilon \mid \varepsilon < \gamma\}$ be an increasing sequence in K_α. First, note that $q := \bigcup_{\varepsilon<\gamma} p_\varepsilon$ is a function from a subset of $\omega_{\alpha+1}$ to $\mathcal{P}(\omega_\alpha)$. By Theorem 7.3, $\omega_{\alpha+1}$ is a regular cardinal, so that $\bigcup_{\varepsilon<\gamma} \mathrm{dom}(p_\varepsilon) \subsetneq \omega_{\alpha+1}$. This ensures that q is indeed a condition in $\mathbb{K}_\alpha$, and by construction, $q \geq p_\varepsilon$ for every $\varepsilon < \gamma$. With $\gamma < \omega_{\alpha+1}$ arbitrary, this means that $\mathbb{K}_\alpha$ is $\aleph_{\alpha+1}$-closed.

Now, let G be a $\mathbb{K}_\alpha$-generic filter over M, let $\mu < \omega_{\alpha+1}$ be a cardinal, and let $\gamma \in \Omega^M$. If $f : \mu \to \gamma$ is a function in $M[G]$, then Theorem 7.12 shows that f actually lives in M. Fixing $\mu < \omega_{\alpha+1}$ and varying $\gamma \in \Omega^M$ shows that μ remains a cardinal in $M[G]$, while setting $\gamma = \omega_{\alpha+1}$ and varying $\mu < \omega_{\alpha+1}$ shows that $\omega_{\alpha+1}$ remains a cardinal in $M[G]$.[2] □

Now that we know that $\mathbb{K}_\alpha$ preserves all cardinals $\leq \omega_{\alpha+1}$, we are ready to prove that the continuum hypothesis holds in $\mathbb{K}_\alpha$-generic extensions of ctms of ZFC:

Theorem 7.14. *Let M be a ctm of ZFC, and suppose that G_α is a $\mathbb{K}_\alpha$-generic filter over M. Then $M[G_\alpha] \models 2^{\aleph_\alpha} = \aleph_{\alpha+1}$. In particular, $M[G_0] \models CH$.*

Proof. We will first show that $\cup G_\alpha$ is a surjective function from $\omega_{\alpha+1}$ onto $\mathcal{P}(\omega_\alpha)$. Let $\varepsilon \in \omega_{\alpha+1}$, $x \in \mathcal{P}(\omega_\alpha)$, and set

$$D_{\varepsilon,x} := \big\{p \in \mathbb{K}_\alpha \mid \varepsilon \in \mathrm{dom}(p) \wedge x \in \mathrm{ran}(p)\big\}.$$

One easily checks that $D_{\varepsilon,x}$ is an open dense subset of $\mathbb{K}_\alpha$, so that $G \cap D_{\varepsilon,x} \neq \emptyset$. Hence, for all $\varepsilon \in \omega_{\alpha+1}$ and $x \in \mathcal{P}(\omega_\alpha)$, there is a condition $p \in G_\alpha$ such that $\varepsilon \in \mathrm{dom}(p)$ and $x \in \mathrm{ran}(p)$. Now working in $M[G]$, the fact that G_α directed implies that $\cup G_\alpha$ is a surjective function from $\omega_{\alpha+1}$ onto $\mathcal{P}(\omega_\alpha)$. Thus $|\mathcal{P}(\omega_\alpha)| \leq \omega_{\alpha+1}$ holds in $M[G]$, or in other words, $M[G] \models 2^{\aleph_\alpha} \leq \aleph_{\alpha+1}$. And since $2^{\aleph_\alpha} \geq \aleph_{\alpha+1}$ holds by the definition of $\aleph_{\alpha+1}$ together with $2^{\aleph_\alpha} > \aleph_\alpha$ (see [2]), we have that $M[G] \models 2^{\aleph_\alpha} = \aleph_{\alpha+1}$.

Letting $\alpha = 0$ then gives us that CH is true in $M[G]$. □

By the work of Section 3, Theorem 7.14 shows $\mathrm{Con}(ZFC) \to \mathrm{Con}(ZFC + CH)$, so that overall, we have proved that CH is logically independent from ZFC set theory.

Acknowledgements

A special thanks is in order to my mentor Isabella Scott – She not only provided me with crucial insights throughout the course of my research, but also gave me helpful suggestions in writing this paper. I would also like to thank fellow REU participants Alexander Burka, Gabriel Day, John Hopper, and Aleksander Skenderi for helping me refine the specific pedagogical approach employed in this paper. Finally, I would like to thank Peter May for his unflinching commitment to both the REU program and its participants.

References

[1] C.C. Chang and H. Jerome Keisler. Model Theory. Dover Publications. 2012.
[2] Lorenz J. Halbeisen. Combinatorial Set Theory with a Gentle Introduction to Forcing. Springer International Publishing. 2017.
[3] Thomas Jech. Set Theory. Springer-Verlag. 1997.
[4] Kenneth Kunen. Set Theory. College Publications. 2011.
[5] Kenneth Kunen. Set Theory: An Introduction to Independence Proofs. Elsevier. 1980.
[6] Justin Tatch Moore. The Method of Forcing. https://arxiv.org/pdf/1902.03235.pdf

[2]In fact, this argument also shows that $\mathbb{K}_\alpha$ does not add new subsets of ω_α to M.