FIXED POINT THEOREMS AND THE EXISTENCE OF A GENERAL EQUILIBRIUM

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ABSTRACT. An economy with finitely many commodity markets reaches a general equilibrium only when, at a given set of prices, the aggregate supply equals the aggregate demand within all of these individual markets. In this expository paper, we will take an algebraic topological approach in understanding and proving a useful tool in Brouwer's Fixed Point Theorem. We then generalize this result into Kakutani's Fixed Point Theorem, which we will ultimately use to prove the existence of a general equilibrium in an economy.

CONTENTS

1. BROUWER'S FIXED POINT THEOREM

We will start by developing the algebraic topology preliminaries required to prove Brouwer's Fixed Point Theorem.

Definition 1.1. A **path** in a space X is a continuous map $f : I = [0, 1] \to X$. We call a path that starts and ends at the same point (i.e. $f(0) = f(1)$) a **loop**.

Definition 1.2. Let $f : X \to Y$ and $g : Z \to Y$ be continuous maps. A **lift** of f to Z is a continuous map $h : X \to Z$ such that $g \circ h(x) = f(x)$ for all $x \in X$.

Definition 1.3. Given spaces X, Y where $Y \subset X$, a continuous map $r : X \to Y$ is a **retraction** if $r(y) = y$ for all $y \in Y$.

Definition 1.4. A **homotopy** between two paths in a space X is a collection of paths, $\{f_t\}$ for $t \in [0, 1]$ that satisfies the following properties:

(1) The start and end points of all the paths of $\{f_t\}$ are independent of t. In other words, for some $x_0, x_1 \in X$, we require that $f_t(0) = x_0$ and $f_t(1) = x_1$ for all $t \in [0, 1]$.
(2) The map $F : [0, 1] \times [0, 1] \to X$ defined such that $F(s, t) = f_t(s)$, is continuous.

In other words, a homotopy can be thought of as a collection of paths resulting from continuously deforming a path f_0 to a path f_1 while keeping their starting/ending points fixed.

Two paths f_0 and f_1 are **homotopic** ($f_0 \simeq f_1$) if there exists a homotopy $\{f_t\}_{t\in[0,1]}$ between them.

Definition 1.5. The **fundamental group** of a space X with base point x_0 is defined as the set

$$\pi_1(X, x_0) = \{[f] \mid f \text{ is a loop with base point } x_0\},$$

where $[f]$ is defined as the equivalence class with the equivalence relation being homotopic equivalence, "$\simeq$".

Remark 1.6. The fundamental group can be shown to be a group equipped with group multiplication ($*$) defined for homotopy classes $[f], [g] \in \pi_1(X, x_0)$ by $[f] * [g] = [f \Diamond g]$, where

$$(f \Diamond g)(t) = \begin{cases} f(2t) & 0 \le t \le \frac{1}{2} \\ g(2t-1) & \frac{1}{2} < t \le 1 \end{cases}.$$

Under this definition, it follows that the group is closed under multiplication, has an identity element (the homotopy class of the constant loop at x_0), and for every $[f] \in \pi_1(X, x_0)$, there is an inverse $[f^{-1}]$ under multiplication, where $f^{-1}(t) = f(1-t)$. Thus, the fundamental group is a group.

For this section, we will let the continuous map $p : \mathbb{R} \to S^1$ be defined by $p(r) = (cos(2\pi r), sin(2\pi r))$.

We can then prove the following lemma in order to help us compute that the fundamental group of a circle is non-trivial in theorem 1.9.

Lemma 1.7. *For the unit interval $I = [0,1]$, a function $F : I \times I \to S^1$, and a map $F' : I \times \{0\} \to \mathbb{R}$ that lifts $F \mid_{I\times\{0\}}$ through the map p, there exists a unique map $H : I \times I \to \mathbb{R}$ that lifts F and is an extension of F'.*

The proof of this lemma lies beyond the focus of this paper, but an inductive proof can be found in the first chapter of Hatcher's textbook [2].

Remark 1.8. Note that the result from Lemma 1.7 can be applied to path homotopies, and we can write the following:

Let $\{f_t : [0,1] \to S^1\}$ be a homotopy of paths where $f_t(0) = x_0$ and the map $p : \mathbb{R} \to S^1$ be defined by $p(r) = (\cos(2\pi r), \sin(2\pi r))$. Then for each $x_0' \in p^{-1}(x_0)$, there exists a unique homotopy of lifts ${f'}_t$ of f_t onto $\mathbb{R}$ where ${f'}_t(0) = x_0'$ for all $t \in [0,1]$.

Theorem 1.9. *The fundamental group of S^1 is non-trivial. (It is in fact isomorphic with the integers, $\mathbb{Z}$, but we only need to prove non-triviality here.)*

Proof. Let the map $\Phi : \mathbb{Z} \to \pi(S^1)$ be defined such that, for all $n \in \mathbb{Z}$, $\Phi(n) = [w_n] \in \pi(S^1)$, where $[w_n]$ is the homotopy class of loops on the circle with the basepoint $(1,0)$ and the loop $w_n(s) = (\cos(2\pi ns), \sin(2\pi ns))$ with $s \in [0,1]$.

Let $p : \mathbb{R} \to S^1$ where $p(r) = (\cos(2\pi r), \sin(2\pi r))$. Let $\widetilde{w}_n : I \to \mathbb{R}$ where $\widetilde{w}_n(s) = ns$.

We will then show that Φ is injective. Suppose that there existed $m, n \in \mathbb{Z}$ such that $\Phi(m) = \Phi(n)$ or in other words that $w_m \simeq w_n$.

It follows from lemma 1.7, because $f_t(0) = w_m(0) = w_n(0) = (1,0)$ and $0 \in p^{-1}((1,0)) = \mathbb{Z}$, that must exist a unique homotopy of lifts $\{f_t'\}$ such that $f_t'(0) = 0$ for all $t \in [0,1]$.

Because this lift is unique, it follows that because $w'_m(s) = ms$ is a lift of w_m, it follows that $f'_0(s) = w'_m(s) = ms$ and $f'_1(s) = w'_n(s) = ns$.

We could then conclude that because the endpoints of the path homotopy $\{f'_t\}$ are fixed, that $f'_t(1) = f'_0(1) = f'_1(1) = m(1) = n(1)$ and that $m = n$. Thus, Φ is injective.

Because Φ is injective, it follows that $\pi_1(S^1)$ must be non-trivial. For distinct integers, such as 0 and 1, it follows that their corresponding loops, $w_0, w_1 \in \pi_1(S^1)$, but these loops are not equivalent because Φ is injective. Thus, $\pi_1(S^1)$ must consist of more than one element and be non-trivial. □

We can then proceed to use our computation of the fundamental groups to prove the following:

Theorem 1.10. *Brouwer's Fixed Point Theorem*

If $K \subset \mathbb{R}^n$ is some compact, convex set, then any continuous function $f : K \to K$ has a fixed point.

Because any compact, convex set K is homeomorphic to the closed n-dimensional ball, D^n, we can consider K to be the closed ball D^n.

Seeing this result in the 1-dimensional $(n = 1)$ case is simple. It is equivalent to saying that some continuous function $f[0, 1] \to [0, 1]$, when plotted on a graph, must pass through the line $y = x$. When we move to higher dimensions however, the proof is not so simple. Let us consider, for instance, the two dimensional disk, D^2, where we use the fundamental groups to find a contradiction.

Proof. (n=2 case) Suppose by contradiction that, for $K = D^2$, f does not have a fixed point. In other words, for all $x \in K$, $f(x) \neq x$.

It follows that the ray from $f(x)$ to x on D^2 intersects the boundary S^1 at exactly one point. We could then define a continuous map $F : D^2 \to S^1$, where $F(x)$ is defined as the point found with the process above.

For all $x \in S^1$, it follows that $F(x) = x$. F is also continuous (small movements in x correspond to small movements in $F(x)$). Thus, this map F is a continuous retraction from D^2 to S^1, which implies a contradiction as follows.

Let f_0 be any loop on S^1. In D^2, it follows that we can find a homotopy $\{f_t\}$ between f_0 and constant loop (a point) x_0 on the loop f_0. To do this, we can define the homotopy linearly as $f_t(s) = (1 - t)f_0(s) + tx_0$.

From here, we can see that the homotopy defined by the composition $F \circ f_t$ is also a homotopy over S^1, where $F_\circ f_0 = f_0$ because F is a retraction onto S^1, and $F_\circ f_1 = x_0$. This implies that any loop $f_0 \in \pi_1 S^1$ is homotopic to a point and that $\pi_1 S^1$ is trivial, which is a contradiction of theorem 1.9.

Thus, by contradiction, f must have a fixed point. □

When further generalizing this result to n-dimensions, we need to develop further machinery in homology groups, chain complexes, and excision. While we won't focus on these concepts in detail, we will provide some intuition, generalizations, and theorems without proof that will give insight on how homology groups can be used to prove Brouwer's fixed point theorem in n dimensions.

For a given space X, let H_n send this space X to a corresponding free abelian group $H_n(X)$. In other words, $H_n(X)$ is a set equipped with an operation that is commutative. We also call H_n a functor, which in this case sends spaces to free abelian groups. These free abelian groups will act as a stand in for homology groups that have been reduced such that $H_0(X) = 0$.

The following are several properties, without proof, of H.

alternate: For any space X, let $H_n(X)$ be its n-dimensional homology group with integer coefficients. Then H_n can be viewed as a functor from the category of topological spaces to the category of abelian groups, with the following properties:

Proposition 1.11. *If two maps, $f, g : X \to Y$ are homotopic to one another, then they induce the same homomorphism $f_* = g_* : H_n(X) \to H_n(Y)$ for all $n \in \mathbb{N} \cup \{0\}$.*

Theorem 1.12. *The homomorphisms $f_* : H_n(X) \to H(Y)$ induced by a homotopy equivalence with $f : X \to Y$ are isomorphisms for all n.*

Theorem 1.13. *The homology groups of S^n and D^n can be calculated as follows:*

- *for all $i \neq n$, $H_i(S^n) = 0$, and $H_n(S^n) \cong \mathbb{Z}$.*
- *for all $j \in \mathbb{N}$, $H_j(D^n) = 0$.*

We can then generalize our previous proof for Brouwer's Fixed Point Theorem to any arbitrary D^n.

Proof. We suppose by contradiction that f does not have a fixed point. We can then define, in the same way as the previous proof, the retraction $F : D^n \to S^{n-1}$ and the inclusion map $i : S^{n-1} \to D^n$ where $i(x) = x$ for all $x \in S^{n-1}$.

It follows that $F \circ i : S^{n-1} \to S^{n-1}$ is the identity on S^{n-1}. We can then write that

$$H_{n-1}(S^{n-1}) \longrightarrow_{i_*} H_{n-1}(D^n) \longrightarrow_{r_*} H_{n-1}(S^{n-1})$$

is the identity map for $H_{n-1}(S^{n-1}) \cong 0$. This is a clear contradiction, as $H_{n-1}(S^{n-1}) \cong \mathbb{Z}$, and we cannot possibly have $r_* \circ i_*$ be the identity over $\mathbb{Z}$ if the image of r_* is trivial (one point).

Thus, by contradiction, f must have a fixed point. $\square$

2. Kakutani's Fixed Point Theorem

We will then proceed to prove a generalization of Brouwer's Fixed Point Theorem for set-valued functions known as Kakutani's Fixed Point Theorem.

Definition 2.1. A set X is **convex** if for all $x, y \in X$, the set of points on the line segment between x and y, $\{tx + (1-t)y \mid t \in [0, 1]\}$, is a subset of X.

Definition 2.2. An **n-simplex** is the smallest convex set in $\mathbb{R}^m$ containing $n+1$ points $\{v_0, v_1, \ldots, v_n\}$ that do not lie in a hyperplane of dimension less than n. The points v_i are known as the **vertices** of the simplex, which is denoted as $[v_0, \ldots, v_n]$.

Definition 2.3. The **standard n-simplex,** Δ^n, is defined as

$$\Delta^n = \{(t_0, \ldots, t^n) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n} t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i\}.$$

Definition 2.4. An arbitrary **n-simplex** S with the set of $n+1$ vertices $\{v_0, v_1, \ldots v_n\} \subset \mathbb{R}^{n+1}$, can be written as

$$S = \{\sum_{i=0}^{n} \theta_i v_i \mid \sum_{i=1}^{n} \theta_i = 1 \text{ and } \theta_i \geq 0 \text{ for all } i \in [n] = \{1, 2, \ldots, n\}\}.$$

Definition 2.5. The **power set** of a set X is the set $2^X = \{A \mid A \subset X\}$ (i.e. it is the set of subsets of X).

Definition 2.6. A **set-valued function** Φ from the set X to Y can be defined as $\Phi : X \to 2^Y$, where $\Phi(x) \neq \emptyset$ for all $x \in X$.

Definition 2.7. A set-valued function $\Phi : X \to 2^Y$ has a **closed graph** if the set $\{(x, y) \mid y \in \Phi(x)\}$ is a closed subset of $X \times Y$. In other words, this can be described as the following: for all sequences (x_n) and (y_n) such that x_n converges to x and y_n converges to y and $y_n \in \Phi(x_n)$ for all $n \in \mathbb{N}$, then $y \in \Phi(x)$.

Definition 2.8. A **Barycentric subdivision** of an n-simplex S, with the set of vertices P, divides S into $(n+1)!$ sub-simplices that each correspond to exactly one unique permutation of P, written as $\{p_i \in P \mid i \in [n] \cup \{0\}\}$. Each sub-simplex of S has a set of vertices $\{v_i \mid i \in [n] \cup \{0\}\}$, where v_i is the centroid of $p_0, p_1, \ldots, p_i$.

Note: Subsequent subdivisions can be further applied to S by subdividing every existing sub-simplex.

We will first prove Kakutani's Fixed Point Theorem for all sets S that are n-simplices, and use this result to generalize the theorem to any arbitrary compact, convex S.

Lemma 2.9. *Kakutani's Fixed Point Theorem for Simplices*

Let S be a closed simplex in $\mathbb{R}^n$. If the set-valued function $\Phi : S \to 2^S$ has a closed graph, and $\Phi(x)$ is non-empty and convex for all $x \in S$, then there exists a fixed point x^* of Φ.

Proof. Let S be an n-simplex. We wish to consider repeated Barycentric subdivisions of S.

Consider the mth Barycentric subdivision of S. It follows that this subdivision has, for some $k \in \mathbb{N}$, k subsimplices of S, where each subsimplex is an n-simplex with a set of $n + 1$ vertices.

Let us consider one such subsimplex of the mth Barycentric subdivision of S and denote its set of vertices as $\{v_0, v_1, \ldots, v_n\}$. For each $i \in \{0, 1, \ldots, n\} = [n] \cup \{0\}$, let y_i be some point such that $y_i \in \Phi(v_i)$. The set of points $\{y_i \mid i \in [n] \cup \{0\}\}$ then form the vertices of another subsimplex of S.

Given the properties of this mth subdivision of S, we can then define a continuous point-to-point map, f_m, from S to itself, and then apply Brouwer's fixed point theorem on this map f_m.

We define the map $f_m : S \to S$ as follows.

Let V be the set of distinct vertices of the subsimplices in the mth subdivision of S. For each $v \in V$, we pick some $y_v \in \Phi(v)$. To be clear, a point v in S that is a vertex of multiple subsimplices in the subdivision will be assigned to exactly one $y_v \in \Phi(v)$.

For all $s \in S$, it follows that s is contained in some closed sub-simplex with the set of vertices $\{v_0, v_1, \ldots v_n\}$. As such, by definition, it can be written as $s = \sum_{i=0}^{n} \theta_i v_i$, where $\theta_i \geq 0$ for all $i \in [n] \cup \{0\}$, and $\sum_{i=0}^{n} \theta_i = 1$. We then define $f(s)$ as $f(s) = \sum_{i=0}^{n} \theta_i y_i$. Intuitively, we are mapping all subsimplices in the subdivision of S to corresponding subsimplices in S.

It follows that f_m is a continuous map from the convex, compact set S to itself, and thus, by Brouwer's fixed point theorem, it follows that there must exist a fixed point x_m^* of f_m.

It then follows that we can construct a sequence of these fixed points (x_m^*), where x_m^* is defined as above for the mth barycentric subdivision of S. It follows that because (x_m^*) is defined on the compact set S that, by Bolzano-Wirestrauss, (x_m^*) has some subsequence that converges to some x^*.

For any $m \in \mathbb{N}$, it follows that the mth subdivision contains at least one subsimplex that contains x^*. Pick one of these subsimplices. Let us denote the set of its vertices as $\{v_0^m, v_1^m, \ldots, v_n^m\}$. We note that with each subdivision, the maximum distance between x^* and the vertices approaches 0. It then follows that for any $i \in [n] \cup \{0\}$, the sequence $(v_i^m)_{m\in\mathbb{N}}$ converges to x^*, as, with each subdivision, the distances between the vertices and x^* are continuously decreasing with 0 being their greatest lower bound.

We then consider the sequence $(y_i^m)_{m\in\mathbb{N}}$, where $y_i^m = f_m(v_i^m)$. It follows that because $(y_i^m)_{m\in\mathbb{N}}$ is defined on the compact set S, it must have a subsequence that converges to some y_i^*.

Because $y_i^m = f_m(v_i^m) \in \Phi(v_i^m)$ for all $m \in \mathbb{N}$ by definition of f_m, and that the sequences converge ($(v_i) \to x^*$ and $(y_i) \to y_i^*$), it follows from the definition of a closed graph that $y_i^* \in \Phi(x^*)$ for all $i \in [n] \cup \{0\}$.

Finally, because x^* is contained in the simplex formed by the vertices $\{y_i^*\}$ and the image $\Phi(x^*)$ is convex by definition, it follows that $x^* \in \Phi(x^*)$. $\square$

We then use this result to generalize Kakutani's fixed point theorem to any arbitrary, compact, convex set X.

Theorem 2.10. *Kakutani's Fixed Point Theorem*

Let S be a non-empty, compact, and convex set in $\mathbb{R}^n$. If the set-valued function $\Phi : S \to 2^S$ has a closed graph, and $\Phi(x)$ is non-empty and convex for all $x \in S$, then there exists a fixed point x^* of Φ.

Proof. For any compact, convex set X, there must exist some closed simplex S such that $X \subset S$. It follows that we can define some continuous retraction $\psi : S \to X$. It then follows that $\Phi \circ \psi : S \to 2^X \subset 2^S$ is an upper semi-continuous map from the closed simplex S into 2^S. Applying Kakutani's Fixed Point Theorem for simplices, we can write that there must exist a fixed point $x^* \in S$ such that $x^* \in \Phi \circ \psi(x^*) \subset X$. Thus, it follows that $x^* \in X$ and by the definition of a retraction that $\varphi(x^*) = x^*$. Finally, we can write that $x^* \in \Phi \circ \psi(x^*) = \Phi(x^*)$, and that x^* is a fixed point of Φ. $\square$

3. Existence of General Equilibrium

Both Brouwer's and Kakutani's fixed point theorems have multiple applications in game theory and mathematical economics. Here, we'll briefly introduce some of the concepts involved in proving one particular application: the existence of a general equilibrium.

Definition 3.1. An **economy** consists of N commodities, L producers, and M consumers.

Definition 3.2. A **price vector**, $\mathbf{p}$, for a market is a vector describing the prices for each of the N goods.

- All prices are non-negative. That is, for all $i \in [N]$, $p_i \geq 0$.

- Homogeneity of degree 0 in demand functions (see definition 3.4) with respect to prices allows us to normalize all price vectors such that:

$$\sum_{i=1}^{N} p_i = 1.$$

Note that this implies that $\mathbf{p}$ lies in the unit $N-1$-simplex (which we write as P), otherwise known as the price simplex.

Definition 3.3. An economy's **endowment** is given by an N-dimensional vector W, where for $i \in [N]$, $W_i \geq 0$ represents the non-negative, initial endowment of commodity i shared by all consumers in the economy. It is also assumed that $W \neq \mathbf{0}$, and that $W_i > 0$ for at least one $i \in [N]$.

Definition 3.4. The **market demand function** for the ith good in an economy with an endowment W, is given by a function of price, $\mathcal{E}_i(\mathbf{p})$, which is non-negative, continuous, and homogenous of degree 0 in prices and endowments. That is, if the unit of measurement of wealth is changed (e.g. dollars versus euros), and there is instead an economy with an endowment of kW, for some constant k, and a demand function $\mathcal{E}_i^k$, then it follows that

$$\mathcal{E}_i(\mathbf{p}) = \mathcal{E}_i^k(k\mathbf{p}).$$

Definition 3.5. An **equilibrium** for an economy is a price vector $\mathbf{p}^*$ such that $\mathcal{E}_i(\mathbf{p}^*) - W_i \leq 0$ for all $i \in [N]$, with strict equality holding if $p_i > 0$. We define the function $g_i : P \to P$ to be the **excess demand function**, where $g_i(p) = \mathcal{E}_i(\mathbf{p}^*) - W_i$.

Theorem 3.6. ***Walras's Law***

For an exchange economy with M consumers, and N goods, we define the excess demand function $z : S \to \mathbb{R}^+$ by, (note that "$\cdot$" denotes dot product)

$$z(p) = \sum_{i=1}^{M} (p \cdot x^m(p, p \cdot W_i) - W_i),$$

where $x^m(p, p \cdot W_i)$ is the marshallian demand for the ith consumer's endowment. In other words, $x^m(p, p \cdot W_i)$ is the level of consumption at which the consumer maximizes their utility function U^m, given the set of prices p and an income equal to the value of their endowment, $p \cdot W_i$.

It then follows that $p \cdot z(p) = 0$.

Proof. The budget constraint for marshallian demand requires that $p \cdot x_m(p, p \cdot W_i) = p \cdot W_i$, which directly implies that $(p \cdot x^m(p, p \cdot W_i) - W_i) = 0$, and thus $p \cdot z(p)$ is equal to 0. □

We can now prove that, in a economy where there are decreasing returns to scale and technology, where there is a unique profit maximizing output for producers given a certain price vector, we can find a general equilibrium.

Theorem 3.7. *For a model with one unique profit maximizing output for a set of prices, and continuous demand functions $\mathcal{E}_i$, there must exist an equilibrium price.*

Proof. We will define a continuous map from the price simplex P to itself, known as the Gale-Nikaido Mapping, whose fixed point is an equilibrium price.

We define the map $y : P \to P$ such that for $p \in P$ and $i \in [N]$,

$$y_i(p) = \frac{p_i + \max[0, g_i(\mathbf{p})]|}{1 + \sum_{j=1}^{N} \max[0, g_j(\mathbf{p})]}.$$

We also know that $y_i \geq 0$ for all $i\imath[N]$, and $\sum_{i=1}^{N} y_i = 1$, implying that $y(p) \in P$, and thus y is a map from P to itself.

We've also defined $\mathcal{E}_i$ to be continuous for all $i \in [N]$, and thus it follows that y is also continuous.

Thus, because P is non-empty, compact, and convex, it follows from Brouwer's Fixed Point Theorem that there exists a fixed point $p^* \in P$ such that $y(p^*) = p^*$.

We can they show that p^* is an equilibrium price for the economy. Consider any $i \in [N]$.

Because p^* is a fixed point, we can write that

$$p_i^* = y_i(p^*) = \frac{p_i^* + \max[0, g_i(\mathbf{p}^*)]|}{1 + \sum_{j=1}^{N} \max[0, g_j(\mathbf{p}^*)]}.$$

Let the denominator of this fraction be $c = 1 + \sum_{j=1}^{N} \max[0, g_j(\mathbf{p}^*)]$. It follows by definition that $c \geq 1$.

Suppose that $c > 1$. Rewriting the equation, we get that

$$p_i^*(c-1) = \max[0, g_i(p_u^*)].$$

It follows that $g_i(p^*) > 0$ if and only if $p_i^* > 0$.

Suppose $g_i(p^*) > 0$ and $p_i^* > 0$. The sum $\sum_{i\in[N]} p_i^* g_i(p^*)$ is the sum of only positive terms, as p_i^* and $g_i(p^*)$ have the same sign. However, Walras' Law implies that $\sum_{i\in[N]} p_i^* g_i(p^*) = 0$, which is a contradiction. Thus, it follows that $c \not> 1$ and that $c = 1$.

Because $c = 1$, it follows that $g_i(p^*) \leq 0$ for all $i \in [N]$. We also know from Walras' law that if $p_i^* > 0$, it must follow that $g_i(p^*) = 0$. Thus, we have an equilibrium. □

However, if producers receive constant returns from scale and there are multiple profit maximizing response from producers, we can set up more framework in order for us to apply Kakutani's Fixed Point Theorem in a similar way.

The following are some concepts regarding how consumers and producers make economic decisions.

Definition 3.8. A **consumption bundle** is a N-dimensional vector c, where for $c \in [N]$, $c_i \geq 0$ is the quantity of commodity i in the bundle. Let the set containing all consumption bundles be C.

Definition 3.9. A **preference relation** is a binary relation over the set of consumption bundles C that reads as the following:

- To write $a \succ b$ means that the individual strictly prefers bundle a to bundle b.
- To write $a \sim b$ means that the individual is indifferent between bundle a. and b
- To write $a \succeq b$ means that the individual prefers bundle a at least as much as they prefer bundle b.

Definition 3.10. An individual's **utility function** is a function $U : C \to \mathbb{R}$ defined in a way that reflects the individual's preferences. That is, for $c_1, c_2 \in C$, it follows that:

- if $c_1 \sim c_2$, then $U(c_1) = U(c_2)$.
- if $c_1 \succ c_2$, then $U(c_1) > U(c_2)$.

Definition 3.11. An individual's utility function is said to be **quasiconcave** if, for any $c_1 \in C$, the set of bundles weakly preferred to c_1, $\{c \in C \mid U(c) \geq U(c_1)\}$ is convex.

Definition 3.12. For all $m \in [M]$, we define $X_m \subset \mathbb{R}^N$ to be the feasible production set for the mth consumer. We will assume that X_m satisfies the following:

- X_m is closed and convex.
- $U^m(x_m)$ is continuous and semistrictly quasiconcave.
- The mth individual's income function is given by the sum of the value of their endowment and their share of profits:

$$I^m(\mathbf{p}) = (\sum_{i=1}^{N} p_i w_i) + \mu^m(\mathbf{p}).$$

Definition 3.13. We can define the point to set map $X_m : S \to (X^m)^2$ as the mth utility-maximizing consumer's response to a set of prices, where

$$X^m(\mathbf{p}) = \{x^m \in X^m \mid x^m \text{ maximizes } U^m(x^m) \text{ with } C^m(x^m, \mathbf{p}) \leq I^m(\mathbf{p}\},$$

and the expense of their consumption is given by $C^m(x^m, \mathbf{p}) = \sum_{i=1}^{N} p_i x_i^m$.

Definition 3.14. The market demand response, $X(p)$, is given as the sum of the individual demand responses in the economy. In other words,

$$X(\mathbf{p}) = \{\sum_{m=1}^{M} x^m \mid x^m \in X^m(\mathbf{p}) \text{ for all } m \in [M]\}.$$

Definition 3.15. For all $l \in [L]$, we define $Y_l \subset \mathbb{R}^N$ to be the feasible production set for the lth producer. We will assume that Y_l satisfies the following:

- Y_l is convex.
- $0 \in Y_l$.
- Y_l is closed.
- Y_l is bounded above.

Definition 3.16. We define the point to set map $Y_l : S \to {Y^l}^2$, to be the response that the lth profit-maximizing producer has to a set of prices, where

$$Y^l(\mathbf{p}) = \{\mathbf{y}^* \in Y \mid \sum_{i=1}^{N} p_i y_i^* \geq \sum_{i=1}^{N} p_i y_i \text{ for all } y \in Y^l\}.$$

Definition 3.17. The market supply response, $Y(p)$, is given as the sum of the individual supply responses in the economy. In other words,

$$Y(\mathbf{p}) = \{\sum_{l=1}^{L} y^l \mid y^l \in Y^l(\mathbf{p}) \text{ for all } l \in [L]\}.$$

Definition 3.18. We define the **market excess demand map** as

$$Z(\mathbf{p}) = \{x - y - W \mid x \in X(\mathbf{p}), y \in Y(\mathbf{p})\},$$

where W is the endowment.

With this set valued map Z, we can then apply Kakutani's Fixed Point Theorem in a way similar to how we applied Brouwer's Fixed Point Theorem in theorem 3.7.

Theorem 3.19. *In a model with multiple profit maximizing outputs for a set of prices, there exists an equilibrium price.*

Proof. Let Z be some convex, compact set such that $Z(p) \in Z$ for all $p \in S$. We define a continuous map $f : S \times Z \to (S \times Z)^2$ as follows.

For any $(p, z) \in S \times Z$, we define $f((p, z)) = \{(p', z') \mid p' \in p(z), z' \in Z(p)\}$, where $Z(p)$ is the map defined in 3.18, and $p(z)$ is defined as the set of prices that maximize the value of z. In other words, $p(z) = \{p \in S \mid \sum_{i=1}^{N} p_i z_i \text{ is maximized}\}$

We can apply Kakutani's fixed point theorem to f (it has a closed graph), and thus there exists $(p^*, z^*) \in S \times Z$, where $p^* \in p(z^*)$ and $z^* \in z(p^*)$.

We can then show that p^* is an equilibrium price.

It follows from Walras's Law that $\sum_{i=1}^{N} p_i^* z_i^* = 0$. It also follows that because $p^* \in p(z^*)$, we can write that, for any $p \in S$, $\sum_{i=1}^{N} p_i^* z_i^* \geq \sum_{i=1}^{N} p_i z_i^*$.

For any $k \in [N]$, let $p^k \in S$ be the price vector whose kth component is equal to 1, and all other components are equal to 0. It follows that we can write that

$$\sum_{i=1}^{N} p_i^k z_i^* = z_k^* \leq \sum_{i=1}^{N} p_i^* z_i^* = 0.$$

Because $z_k^* \leq 0$ for any $k \in [N]$, and Walras's law implies that strict equality holds when $p_k^* > 0$, we see that p^* is an equilibrium price. □

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