UNSTABLE CHROMATIC HOMOTOPY

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Abstract. In this paper, we discuss some basic notions in unstable chromatic homotopy theory. We focus primarily on the v_n-periodic unstable category $\mathcal{S}_*^{v_n}$, the Bousfield-Kuhn functor Φ, and the realization of $\mathcal{S}_*^{v_n}$ as a full subcategory of the category $\mathcal{S}_*$ of based spaces. Though we will need some results from stable chromatic homotopy theory, this material relies on relatively few stable results. As such, we will briefly summarize any results of chromatic homotopy theory used, although we will assume a great deal of familiarity with spectra. As a sample application of these ideas, we briefly discuss the 2017 work of Heuts proving that Φ lifts to an equivalence $\mathcal{S}_*^{v_n} \simeq \mathrm{Lie}\left(\mathcal{S}p_{T(n)}\right)$ of the v_n-periodic unstable category with the Lie algebras in the $T(n)$-local category of spectra.

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1. Introduction

One of the fundamental aspects of chromatic homotopy theory is the notion of v_n-periodicity. To each p-local finite spectrum X we associate a natural number n, known as its type. Then the celebrated periodicity theorem, conjectured in [2] and proven in [1], asserts that for any type n spectrum X, there exists some integer t and a map $v_n : \Sigma^t X \to X$ with some good properties to be discussed later. As we will discuss in section 3, the periodicity theorem admits a natural interpretation

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(in part) as an existence theorem asserting that there exist spectra of every type: the properties of v_n ensure that the cofiber of v_n is of type $n+1$, and the p-local sphere is of type 0.

This perspective is inextricably linked to the thick subcategory theorem of [1], and more specifically its consequence theorem 2.9 (known sometimes as class invariance). This asserts that the type of a finite p-local spectrum determines (and is determined by) its Bousfield class. There is hence a "filtration" of the category $f\mathcal{S}p_{(p)}$ of finite p-local spectra by the subcategories $\mathcal{T}yp_{\geq n}$ of spectra of type $\geq n$. The thick subcategory theorem implies it is "maximal" in some appropriate sense. Further, localizing away from these categories gives a dual filtration by categories $L_n^f\mathcal{S}p$ of L_n^f-local spectra. The successive "layers" of this dual filtration, the categories $\mathcal{S}p_{T(n)}$ of $T(n)$-local spectra, should then be roughly dual to the categories of finite p-local type n spectra. Furthermore, the $v_n : \Sigma^t X \to X$ induce maps on many interesting invariants, such as the homotopy groups with coefficients in X, defined by $\pi_*(Y;X) = [\Sigma^* X,\ Y]$. By formally inverting these maps, we obtain versions of these invariants which are invariant under $T(n)$-equivalences; for instance, if we invert v_n in the homotopy groups with coefficients in X we get the "v_n-periodic" homotopy groups $v_n^{-1}\pi_*(Y;X)$ that turn out to contain a surprisingly large amount of information about Y, despite being periodic (with period t, which is nonzero if $n > 0$).

Now, recall that the finite p-local type n spectra are precisely those which have nontrivial $K(n)$-homology but trivial $K(m)$-homology for all $m < n$, where $K(n)$ is the n^{th} (p-local) Morava K-theory (c.f. section 2). Then, we see that $T(n)$ should be a "finitary approximation" to $K(n)$, and can show that $\langle T(n)\rangle \leq \langle K(n)\rangle$. This gives a map $L_{T(n)} \to L_{K(n)}$, and the telescope conjecture of [2] asserts that this map is an equivalence. It is still open (and probably false), and is among the most famous open questions in chromatic homotopy theory. Two specific cases are known: if $n = 0$, it can be shown without too much work that $T(0) \simeq H\mathbb{Q} \simeq K(0)$. The case $n = 1$ is also known to be true (by Mahowald for $p = 2$ and Miller for p odd, c.f. [4]), but even this is highly nontrivial. For our purposes, we will mostly be interested in $T(n)$ and will generally ignore $K(n)$, since the map $L_{T(n)} \to L_{K(n)}$ can be used to deduce $K(n)$-analogs of most of our $T(n)$-centric results.

The goal of this paper is to sketch certain key ideas relating to the *unstable* (i.e. space-level) analog of $\mathcal{S}p_{T(n)}$, which we denote $\mathcal{S}_*^{v_n}$. We will not define it yet, but we make the following observation: in the case $n = 0$, $T(0) \simeq H\mathbb{Q}$ and so $\mathcal{S}p_{T(0)} \simeq \mathcal{S}p_{\mathbb{Q}}$ is the category of $\mathbb{Q}$-local spectra. Hence, we expect that $\mathcal{S}_*^{v_0}$ will be the category $\mathcal{S}_{\mathbb{Q}}$ of $\mathbb{Q}$-local spaces[1]. There are some issues in making this precise, including issues in defining $\mathcal{S}_*^{v_0}$, but the root cause of these issues is more technical and so we will gloss over them for now. Hence $\mathcal{S}_*^{v_0}$ is decently well-understood modulo some connectivity assumptions, and its study is essentially purely algebraic in nature. Indeed, the Quillen and Sullivan models ([6], [7]) give an explicit model for this category. Although this model is significantly more complicated than the algebraic models of $\mathcal{S}p_{T(0)}$, there is a serious link between the two so that one can think of a rational space as a rational spectrum with additional data; one of the

[1] Here and throughout this paper, we work exclusively with based spaces and maps unless otherwise mentioned.

major goals in this paper is to set up some notions relating $\mathcal{S}_*^{v_n}$ and $\mathcal{S}p_{T(n)}$ in a similar way for $n > 0$.

The (unstable) v_n-periodic homotopy groups will play a central role in the v_n-periodic unstable category. As with spectra, we expect them to be periodic, and hence one almost expects the unstable v_n-periodic category to trivially reduce to the $T(n)$-local category of spectra.

Motivated by this, we will be concerned the Bousfield-Kuhn functor Φ, which relates the v_n-periodic homotopy groups of a space X to the homotopy groups of a $T(n)$-local spectrum $\Phi(X)$. Unfortunately, as the unstable rational category is much more complicated than the stable rational category, we might expect that this information is insufficient to recover the v_n-periodic homotopy type of a space. The definition and basic properties of this functor nevertheless form the main goal of the first section of this paper. Of particular interest is the adjunction

$$\Theta : \mathcal{S}p_{T(n)} \rightleftarrows \mathcal{S}_*^{v_n} : \Phi.$$

The proof of this relies on several intermediary results concerning $\mathcal{S}_*^{v_n}$ which are proven by realizing $\mathcal{S}_*^{v_n}$ in a concrete way as a localization of the category $\mathcal{S}_*$ of based spaces. Rather than make a lengthy detour to prove these intermediary results, we will assume those results temporarily in order to give the proof. We will then conclude the first section by discussing the main theorem of [12], which states that the adjunction is monadic, i.e. that Φ defines an equivalence between $\mathcal{S}_*^{v_n}$ and the category $\mathcal{A}lg_M\left(\mathcal{S}p_{T(n)}\right)$ of algebras for some monad M on $\mathcal{S}p_{T(n)}$.

Next, in the second section, we discuss the embedding $\mathcal{S}_*^{v_n} \hookrightarrow \mathcal{S}_*$ and use it to give the proofs of the intermediary results assumed in the first section. This turns out to be extremely technical, and requires setting up the machinery of Bousfield classes in the unstable setting.

In the final section, we discuss a few related topics of interest to unstable chromatic homotopy. In particular, we give a rough idea of the connection to Bousfield calculus. In particular, we mention a result of Heuts that identifies the monad M mentioned previously. In [8], he proves in fact that M is the Lie algebra monad, so that the v_n-periodic unstable category is equivalent to the category of Lie algebras over $\mathcal{S}p_{T(n)}$ in a suitable sense. As a formal consequence, he also derives an analog for $K(n)$ ([8]). The proof is well beyond the scope of this paper, but the idea is to show that M is coanalytic, so that it corresponds to the algebras of some operad. By definition, the Lie algebra monad is produced in this way from the derivatives of the identity, and so Heuts' result comes from relating $M = \Phi \circ \Theta$ to the derivatives of the identity. Behrens and Rezk previously proved a result of this type in 2012 ([9]), but this result requires more work to formally state. For a more comprehensive sketch of this consequence, along with an approach of Arone and Ching, see [11]; both Heuts and Arone-Ching have additionally published chapters on this subject as part of the Handbook of Homotopy Theory.

2. Fundamentals of v_n-Periodic Homotopy

2.1. **The $T(n)$-Local and v_n-Periodic Categories.** We begin by recalling a few standard notions of chromatic homotopy theory. They may be found in any standard reference on the subject, e.g. [14], [15], or [16].

First, recall:

Definition 2.1. Let E be a spectrum. We write $\langle E \rangle$ for the Bousfield class of E, i.e. the class of spectra X with $E \wedge X \not\simeq *$. These have the usual ordering by inclusion.

Our first goal is to define a certain Bousfield class $\langle T(n) \rangle$. To do this we need the following input:

Definition 2.2. We let the spectra $E(n)$ and $K(n)$ be Lubin-Tate theory and Morava K-theory, respectively. In addition to the natural number n, they implicitly depend on a prime p, and satisfy:

$$\langle E(n) \rangle = \bigvee_{m \leq n} \langle K(m) \rangle$$

[2]. We follow the conventions that $E(0) = K(0) = H\mathbb{Q}$, $E(\infty) = H\mathbb{Z}_{(p)}$, and $K(\infty) = H\mathbb{F}_p$.

Observe that no nontrivial p-local spectrum is $E(\infty)$-acyclic, and that any $E(n)$-local spectrum is $E(m)$-local for $m \geq n$. Hence, the $E(n)$-local categories define a stratification of the category $\mathcal{S}p_{(p)}$ of p-local spectra. This stratification is not easy to understand, but behaves in a well-understood way with regards to finite spectra. Indeed, a classic result of chromatic homotopy theory states that a finite $K(n)$-acyclic p-local spectrum is $K(m)$-acyclic if $m < n$. As a result, we may assign to each finite p-local spectrum a type $\mathrm{typ}(X)$ given by the least n for which $K(n) \wedge X \not\simeq 0$. While this notion technically makes sense for infinite spectra, we will say that a spectrum has a certain type (or range of types) only if it is finite. We will also say that a finite space X has type equal to that of its suspension spectrum; since the usual definition of the E-homology and E-cohomology of a space agree with those of its suspension spectrum, this is equivalent to the vanishing of $K(m)_*(X)$ for $m < \mathrm{typ}(X)$ and nonvanishing of $K(\mathrm{typ}(X))_*(X)$.

Definition 2.3. We say that a self-map on a spectrum X is a map $f : \Sigma^k X \to X$ for some $k \geq 0$. Its r^{th} iterate $f^r : \Sigma^{kr} X \to X$ is the natural composite of r suspensions of f. A self-map is said to be a v_n-self map if it induces isomorphisms on $K(n)$-homology and nilpotent maps on $K(m)$-homology for $m \neq n$.

Remark 2.4. If X is a $K(n)$-acyclic spectrum, the 0 map is a v_n-self map.

It turns out that the theory of v_n-self maps is generally better-behaved if X is finite, so we will exclusively use the term for finite spectra.

Theorem 2.5 (Periodicity, [1])**.** *Every type $\geq n$ finite p-local spectrum admits a v_n-self map.*

A corollary of the nilpotence theorem also gives

Corollary 2.6 (Uniqueness of v_n-Self Maps)**.** *Let X be a finite p-local spectrum with v_n-self maps $v : \Sigma^s X \to X$ and $w : \Sigma^t X \to X$. Then, there exists some $a, b \gg 0$ for which $v^a = w^b$.*

Hence, if X is a type n spectrum, there is a well-defined spectrum $v_n^{-1}X$ defined by taking the mapping telescope $v^{-1}X$ for a v_n-self map v on X. This spectrum will be called $T(n)$.

[2] That is, a spectrum is $E(n)$-acyclic if and only if it is $K(m)$-acyclic for all $m \leq n$. Morally, the $K(n)$-local categories are the successive layers between the $E(n)$-local categories.

Of course, there is no canonical choice of X with which to define $T(n)$, so that there is no well-defined spectrum $T(n)$. However, we have:

Lemma 2.7. *The Bousfield class $\langle T(n)\rangle$ is independent of the choice of a finite p-local type n spectrum X.*

Before giving the proof, we recall a useful tool in Bousfield class computations.

Theorem 2.8 (Thick Subcategory Theorem)**.** *Let $\mathcal{C}$ be a thick subcategory of the category $\mathcal{fSp}_{(p)}$ of finite p-local spectra. That is, suppose $\mathcal{C}$ is a subcategory of $\mathcal{fSp}_{(p)}$ closed under cofibers, extensions, and shifts. Then, for some n, $\mathcal{C} = \mathcal{Typ}_{\geq n}$ is the category of finite p-local spectra of type at least n.*

From this, we deduce

Theorem 2.9 (Class Invariance)**.** *Let A and B be finite p-local spectra. Then, $\langle A\rangle \leq \langle B\rangle$ if and only if A has at least as large a type as B.*

Proof. By definition, A has type $\leq n$ if and only if $K(n) \in \langle A\rangle$. Taking n to be the type of B, the "only if" direction follows.

Now, let the complement of $\langle A\rangle$ be $\mathcal{Ann}_A$. This consists of those spectra X such that $A \wedge X \simeq *$; by symmetry of the smash product, this is the same as the class of spectra X for which $A \in \mathcal{Ann}_X$. For any fixed X, the class $\mathcal{Ann}_X^{\mathrm{fin}}$ of finite p-local spectra $Y \in \mathcal{Ann}_X$ is a thick subcategory of the category $\mathcal{fSp}_{(p)}$ of finite p-local spectra. Hence, $\mathcal{Ann}_X^{\mathrm{fin}} = \mathcal{Typ}_{\geq m}$ for some $m = \mathrm{cotyp}(X)$. Now, suppose $\mathrm{typ}(A) \leq \mathrm{typ}(B)$. Then, if $X \notin \langle A\rangle$, we wish to show $X \notin \langle B\rangle$. Since A is finite, $X \notin \langle A\rangle$ if and only if $\mathrm{cotyp}(X) \leq \mathrm{typ}(A)$. Since $\mathrm{typ}(A) \leq \mathrm{typ}(B)$, the conclusion follows. □

This gives the following consequence:

Lemma 2.10. *Let v be a self-map on X. Then*

$$\langle X\rangle = \langle \mathrm{cof}(v)\rangle \vee \langle v^{-1}X\rangle.$$

Proof. We first show that the complement of the left is contained in the complement of the right. Suppose $X \wedge Y \simeq *$. Then, $v \wedge Y : \Sigma^t X \wedge Y \to X \wedge Y$ is automatically $0 : * \to *$. Hence, $\mathrm{cof}(v) \wedge Y \simeq \mathrm{cof}(v \wedge Y)$ is trivial and so is

$$v^{-1}X \wedge Y \simeq (v \wedge Y)^{-1}(X \wedge Y).$$

This proves one direction. Conversely, suppose $\mathrm{cof}(v) \wedge Y \simeq *$ and $v^{-1}X \wedge Y \simeq *$. Then, the cofiber sequence

$$\Sigma^t X \wedge Y \to X \wedge Y \to \mathrm{cof}(v) \wedge Y$$

implies that $v \wedge Y$ is an isomorphism, so that

$$v^{-1}X \wedge Y \simeq (v \wedge Y)^{-1}(X \wedge Y) \simeq X \wedge Y$$

and so $X \wedge Y \simeq 0$. □

Now, we prove lemma 2.7, following [13, lec. 2].

Proof. We claim that

$$\langle X\rangle = \langle \mathrm{cof}(v)\rangle \vee \langle T(n)\rangle$$

is an orthogonal decomposition for $v : \Sigma^t X \to X$ a v_n-self map. Since $\operatorname{cof}(v)$ is finite p-local of type exactly $n+1$, both it and $\langle X \rangle$ only depend on n, and the claim will follow.

We must show that $T(n) \wedge \operatorname{cof}(v) \simeq *$. Replace the v_n-self map by a large power so that it induces the 0 map on all $K(m)$ with $m \neq 0$. Then

$$K(m)_* X \wedge \operatorname{cof}(v) \simeq K(m)_* X \otimes K(m)_* \operatorname{cof}(v) \simeq 0$$

for all $m \leq n$. Furthermore, the map on $K(m)$-homology induced by $v \wedge \operatorname{cof}(v)$ is zero for $m \neq n$ by assumption, and so $v \wedge \operatorname{cof}(v) : \Sigma^t X \wedge \operatorname{cof}(v) \to X \wedge \operatorname{cof}(f)$ is nilpotent by the nilpotence theorem. In particular,

$$(v \wedge \operatorname{cof}(v))^{-1}(X \wedge \operatorname{cof}(v)) = v^{-1} X \wedge \operatorname{cof}(v) \simeq *$$

$\square$

In particular, we have a well-defined $T(n)$-local category $\mathcal{S}p_{T(n)}$ and localization functor $L_{T(n)} : \mathcal{S}p \to \mathcal{S}p_{T(n)}$.

Next, we need the notion of the v_n-periodic homotopy groups. In order to properly motivate this, we give a rough spectrum-level sketch. Throughout this discussion we assume $n > 0$.

Let V be finite p-local and note that the Spanier-Whitehead dual $\mathfrak{D}\, V = \operatorname{Map}(V,\ S)$ is also finite p-local. In fact, we claim it has type $n = \operatorname{typ}(V)$, with a v_n self map $\mathfrak{D}\, v$ for v the v_n self-map $\Sigma^t V \to V$ on V. To see this, recall that $K(m)$ is a field, so that

$$(K(m))_* \mathfrak{D}\, V \simeq \operatorname{Map}\left((K(m))_* V,\ \mathbb{F}_p\right).$$

Furthermore, $K(m)_*(\mathfrak{D}\, v)$ is the dual map of $K(m)_*(v)$ hence an isomorphism if $m = n$ and a nilpotent map if $m \neq n$. As a result, we may take $T(n) = (\mathfrak{D}\, v)^{-1} \mathfrak{D}\, V$. Since $T(n)$-homology is a complete invariant in the $T(n)$-local stable category, we get that

$$(T(n))_* X \simeq \pi_* \Big((\mathfrak{D}\, v)^{-1} \mathfrak{D}\, V \wedge X \Big)$$

is a complete invariant, and this is $v^{-1} \pi_* \operatorname{Map}(V,\ X)$ in an appropriate sense. In the unstable world, we can do something similar.

Definition 2.11. Let V be a finite p-local space with a v_n-self map $v : \Sigma^t V \to V$. We define (following [13, lec. 5])

$$\Phi_V(X) = \operatorname{hocolim} \operatorname{Map}\left(\Sigma^{kt} V,\ X\right)$$

with the transition maps given by precomposition by v; by corollary 2.6, this is indeed independent of v. Then, we define the v_n-periodic homotopy groups $v_n^{-1} \pi_*(X; V)$ of X to be $\pi_* \Phi_V(X)$.

Remark 2.12. We can improve the definition somewhat to allow a broader class of V. Let $v : \Sigma^t V \to V$ be a v_n-self map on a finite p-local space V. Observe that $\Omega^t \Phi_V(X) \simeq \Phi_V(X)$, and use this to identify $\Phi_V(X)$ with a periodic spectrum. Then, Φ_V is stable in the sense that $\Phi_{\Sigma V}(X) \simeq \Omega \Phi_V(X)$. One can use these two properties to define $\Phi_E(X)$ for E a finite p-local spectrum equipped with a v_n-self map in such a way that

$$\Phi_{\Sigma^\infty V}(X) \simeq \Phi_V(X).$$

In fact, the formula is essentially the same as the previous one. We then can define $v_n^{-1}\pi_*(X;E) = \Phi_E(X)$ for E a finite p-local spectrum of type n; all the interesting properties of v_n-periodic homotopy groups will hold true in this extra generality.

Remark 2.13. One should also beware that the v_n-periodic homotopy groups of X depend on the choice of V. However, most interesting questions regarding them are independent of V. For instance, let $f : X \to Y$ be a map inducing an isomorphism on $v_n^{-1}\pi_*(-;E)$ for some finite p-local type n spectrum E such as the suspension spectrum of a finite p-local type n space V with a v_n-self map. Consider the class $\mathcal{C}$ of finite type n spectra W for which there exists a v_n-self map w on W such that $v_n^{-1}(f;W)$ is an isomorphism. Since $\Phi_{\Sigma W}(X) = \Omega\Phi_W(X)$, it is clear that this class is closed under shifts. Given a diagram $W \to Z$ in $\mathcal{C}$, let $w : \Sigma^a W \to W$ and $z : \Sigma^b Z \to Z$ be v_n-self maps. By iterating w and z, assume $a = b = t$, and then one can check that the natural map $\Sigma^t \operatorname{cof}(W \to Z) \to \operatorname{cof}(W \to Z)$ is a v_n-self map for which $v_n^{-1}(f;\operatorname{cof}(W \to Z))$ is an isomorphism. Hence $\mathcal{C}$ is thick, contains a type n spectrum E, and so contains all finite p-local type n spectra.

Thus, the condition that $v_n^{-1}\pi_*(f;E)$ be an isomorphism does not depend on the choice of E.

Now, given that Φ_V is a spectrum calculating $T(n)$-local information, one might anticipate the following:

Lemma 2.14. *For any X and any V of type n, $\Phi_V(X)$ is $T(n)$-local.*

Proof. In order to see that it is L_n^f-local, we must show that $\operatorname{Map}(W,\ \Phi_V(X))$ is trivial if W has type $> n$. Indeed, by passing to colimits in the usual smash-hom adjunction, we find $\operatorname{Map}(W,\ \Phi_V(X)) \simeq \Phi_{V\wedge W}(X)$. This latter is trivial since $V \wedge W$ has type $> n$. To finish, we wish to show $\operatorname{Map}\left(w^{-1}W,\ \Phi_V(X)\right) \simeq *$ for w a v_k-self map on W with $k < n$; inverting w corresponds with inverting the v_k-self map $\mathbb{1} \wedge w$ on $V \wedge W$, but this is a type n space so that $\mathbb{1} \wedge w$ is nilpotent. $\square$

This motivates the following definition of $\mathcal{S}_*^{v_n}$:

Definition 2.15 (Preliminary)**.** A map f is said to be a v_n-periodic equivalence if it induces isomorphisms on $v_n^{-1}\pi_*(X;V)$ for some V (equivalently all V by remark 2.13). The category $\mathcal{S}_*^{v_n}$ is the localization of $\mathcal{S}_*$ inverting all v_n-periodic equivalences.

This category was first constructed as a subcategory of $\mathcal{S}_*$ by Bousfield in [17]. We will later redefine $\mathcal{S}_*^{v_n}$ via this embedding, and it is for this reason that we use the word "preliminary."

Remark 2.16. This definition is subtler than it looks. In the category of spectra, we have two equivalent categories which are occasionally thought of as $\mathcal{S}p_{T(n)}$. The first is the category of $T(n)$-local spectra, defined in the usual way. The second is the category of L_n^f-local, L_{n-1}^f-acyclic spectra, sometimes denoted M_n^f; the equivalence is given $L_{T(n)}$ one way and L_n^f in the other. In the world of spaces, this fails. As we will show in section 3, $\mathcal{S}_*^{v_n}$ is essentially an unstable analog of M_n^f, and indeed one can show that its stabilization is M_n^f. However, the category of $T(n)$-local spaces is not equivalent, even after stabilization. That is, the stabilization of the category of $T(n)$-local spaces is not the category of $T(n)$-local spectra.

As suggested by the remark, the $\mathcal{S}_*^{v_n}$ form the "layers" in a filtration:

Definition 2.17. We define $L_n^f\mathcal{S}_*^{\langle d\rangle}$ similarly to definition 2.15 by formally inverting those f which are simultaneously v_m-periodic equivalences for every $0 \leq m \leq n$. It will turn out to be a localization of the category $\mathcal{S}_*^{\langle d\rangle}$ of d-connected spaces.

In order to make sense of this, we need to define what a v_0-periodic equivalence is (since we expressly did not define v_0-periodic homotopy groups). Morally, a v_0-periodic equivalence should just be a rational equivalence since $T(0) \simeq H\mathbb{Q}$. When we redefine $L_n^f\mathcal{S}_*^{\langle d\rangle}$ via the embedding into $\mathcal{S}_*$, we will see that there are issues of connectivity at play not present in the classic presentation of $L_n^f\mathcal{S}p$. For this reason, this moral definition is wrong[3]. We will revisit this in the next section; for now we just say that f is a v_0-periodic equivalence if it induces isomorphisms on the rational homotopy groups $\pi_* \otimes \mathbb{Q}$ if $* > d$. One intuitive reason that we might guess that connectivity is tricky for v_0-periodic dealings is that the standard models of rational homotopy theory requires the use of simply-connected spaces rather than all spaces. As we involve more telescopic layers, more subtle issues can arise, and so we might guess that d should get large as n does[4]. In any case, since $L_n^f\mathcal{S}_*^{\langle d\rangle}$ is mostly independent of $d \gg 0$, we write L_n^f for the localization functor.

It turns out that the tower used to define $\Phi_V(X)$ is essentially constant if X is L_n^f-local, so that for the most part, $\Phi_V(X)$ is just calculating $T(n)$-local information. More precisely, the tower is constant in that $\pi_* \operatorname{Map}(V,\ X) \to v_n^{-1}\pi_*(X;V)$ is an isomorphism for $* > d$. The proof has to do with the embedding $\mathcal{S}_*^{v_n} \hookrightarrow \mathcal{S}_*$, so we will revisit this later. Since $v_n^{-1}\pi_*(X;V)$ are determined by the values with $* \gg 0$, it follows that $\Phi_V(X)$ is "almost" $\operatorname{Map}(V,\ X)$. We make this more precise now.

2.2. **The Bousfield-Kuhn Functor.** In addition to almost having the same homotopy groups as $\operatorname{Map}(V,\ X)$, $\Phi_V(X)$ satisfies many of the same properties as $\operatorname{Map}(V,\ X)$. For instance, we previously used that $\operatorname{Map}(W,\ \Phi_V(X)) \simeq \Phi_{V\wedge W}(X)$, and this strongly resembles the smash-hom adjunction; likewise, $\Phi_V(X)$ sends cofiber sequences in the V argument to fiber sequences. Since we also are primarily interested in behavior (e.g. v_n-periodic equivalences) independent of V, we might even hope that there should be some description of $v_n^{-1}\pi_*(X;V)$ in which V is more-or-less irrelevant (such as the coefficient group in singular homology). Putting these observations together, one therefore posits that $\Phi_V(X) \simeq \operatorname{Map}(V,\ \Phi(X))$ for some $\Phi : \mathcal{S}_* \to \mathcal{S}p_{T(n)}$.

Remarkably, this is true. As the notation suggests, Φ is the Bousfield-Kuhn functor, constructed in [18, 19]. The construction is essentially given by a right Kan extension. Recall that the right Kan extension $\operatorname{Ran}_\iota F$ of a functor $F : \mathcal{C} \to \mathcal{D}$ along a fully faithful functor $\iota : \mathcal{C} \to \mathcal{E}$ agrees with F (up to equivalence) on objects of $\mathcal{C}$. We proceed as follows:

Construction 2.18 (The Bousfield-Kuhn Functor)**.** Let $\iota : \mathcal{T}yp_n \hookrightarrow f\mathcal{S}p_{(p)}$ embed the category of type n spectra into the category of finite p-local spectra. Since ι is fully-faithful, it satisfies the property just discussed of Kan extensions. Take F to

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[3] Some authors such as [13], do not use the term "v_0-periodic equivalence" for this reason at the expense of significantly wordier results and definitions. While we use it, this is a point worth keeping in mind when reading other papers on this subject.

[4] This turns out to be true. However, even with the embedding known, it is not obvious that d must grow without bound, let alone what d is "optimal."

be the functor

$$\Phi_* : \mathcal{T}yp_n \to \operatorname{Fun}\left(\mathcal{S}_*,\ \mathcal{S}p_{T(n)}\right)$$

defined by $E \mapsto \Phi_E$. Then, let the right Kan extension be

$$\operatorname*{Ran}_{\iota} \Phi_* = R : f\mathcal{S}p_{(p)} \to \operatorname{Fun}\left(\mathcal{S}_*,\ \mathcal{S}p_{T(n)}\right).$$

By the previous remark on Kan extensions, we have that $R(V) \simeq F(V) = \Phi_V$ for any V of type n. Let $\Phi = R(S)$, where S is the p-local sphere.

Remark 2.19. Observe that the Kan extension defining R above may be computed pointwise, so that $R(T)(X) \simeq \operatorname{holim}_{F\to T} \Phi_F(X)$, where F ranges over all finite type n spectra with a map to T. In particular, $\Phi(X) \simeq \operatorname{holim}_{F\to S} \Phi_F(X)$

Remark 2.20. Note that $\operatorname{Map}(V,\ \Phi(-))$ is an exact functor of a finite p-local spectrum V. Now, Φ_V is exact as a functor $F(V)$ and thus, by the above formula for $R(V)(-)$, $R(V)(-)$ is exact as a functor of V. It follows that the category of V for which $\operatorname{Map}(V,\ \Phi(-)) \simeq R(V)(-)$ is thick; it contains S since $\operatorname{Map}(S, \Phi(-)) \simeq \Phi(-) = R(S)(-)$, and hence is all of $f\mathcal{S}p_{(p)}$. It follows that $\Phi_V(X) \simeq \operatorname{Map}(V,\ \Phi(X))$.

Remark 2.21. Given any other $\Psi : f\mathcal{S}p_{(p)} \to \mathcal{S}p_{T(n)}$ such that $\operatorname{Map}(V,\ \Psi(X)) \simeq \Phi_V(X)$ for all finite p-local type n spectra V, a similar argument with the thick subcategory theorem and with the universal property of Kan extensions can be used to show that $\Psi \simeq \Phi$. Hence, Φ is the unique functor such that $\operatorname{Map}(V,\ \Phi(X)) \simeq \Phi_V(X)$.

Bousfield and Kuhn were initially motivated to construct this functor by the problem of delooping $K(n)$ (and $T(n)$)-local spectra. Indeed, we have the following alternate characterization of Φ, proven for $n = 1$ by Bousfield ([18]) and in the general case by Kuhn ([19]) once the nilpotence and periodicity theorems became available.

Theorem 2.22 (Bousfield,Kuhn)**.** *Any $T(n)$-local spectrum has a unique functorial delooping. That is, $L_{T(n)} \simeq \Phi \circ \Omega^\infty$ for some Φ, and in fact Φ is the functor constructed above. An analogous statement holds for the $K(n)$-local spectra.*

Proof. We follow [13, lec. 8]. The claim trivially implies that, for V a type n spectrum with a v_n-self map $v : \Sigma^t V \to V$,

$$\Phi_V(\Omega^\infty X) \simeq \operatorname{Map}(V,\ \Phi(\Omega^\infty X)) \simeq \operatorname{Map}\left(V,\ L_{T(n)}(X)\right),$$

and we prove this equivalence as a first step towards the factorization. By construction, the spectrum Φ_V was constructed by iterated delooping of its zeroth space, so we might as well prove the isomorphism at the level of zeroth spaces. That is, we have reduced the intermediate claim to the assertion that

$$\Omega^\infty\Phi_V(\Omega^\infty X) = \operatorname{hocolim} \Omega^\infty \operatorname{Map}\left(\Sigma^{kt}V,\ X\right) \simeq \Omega^\infty \operatorname{Map}\left(V,\ L_{T(n)}(X)\right).$$

But we argued earlier that the homotopy colimit is

$$\Omega^\infty\left((\mathfrak{D}\, v)^{-1}\, \mathfrak{D}\, V \wedge X\right)$$

and thus a model for

$$\Omega^\infty T(n) \wedge X$$

and so $T(n)$-localization, by definition, preserves this. Hence, replacing X by $L_{T(n)}X$, we wish to show

$$\Omega^\infty \Phi_V(\Omega^\infty X) = \operatorname{hocolim} \Omega^\infty \operatorname{Map}\left(\Sigma^{kt}V,\ X\right) \simeq \Omega^\infty \operatorname{Map}(V,\ X).$$

for X a $T(n)$-local spectrum. We claim that the tower in the homotopy colimit is in fact constant, so that this is immediate. To see this, note that X is $T(n)$-local, hence L_n^f-local, so that $\operatorname{Map}(f,\ X)$ is an equivalence for any $K(n)$-equivalence f of finite spectra. Then, the maps in the towers are of the form $\operatorname{Map}(f,\ X)$ for v_n-self maps f, hence with f a $K(n)$-equivalence, and this proves the claim.

Now the goal is to go from

$$\Phi_V(\Omega^\infty X) \simeq \operatorname{Map}(V,\ \Phi(\Omega^\infty X)) \simeq \operatorname{Map}\left(V,\ L_{T(n)}(X)\right)$$

to

$$\Phi(\Omega^\infty X) \simeq L_{T(n)}X \simeq \operatorname{Map}\left(S, L_{T(n)}X\right) \simeq \operatorname{Map}\left(L_{T(n)}S,\ L_{T(n)}X\right).$$

This follows as the left hand side is a homotopy limit of $\Phi_V(\Omega^\infty X)$ over maps $V \to S$. By the intermediate claim, this is the homotopy limit of

$$\operatorname{Map}\left(V,\ L_{T(n)}X\right)$$

and so it suffices to check that $L_{T(n)}S$ is a filtered colimit of type $\geq n$ spectra. In fact this follows for any $T(n)$-local spectrum Y: any $T(n)$-local spectrum Y is L_{n-1}^f-acyclic, and $L_{n-1}^f Y \simeq *$ is the cofiber of

$$\operatorname*{hocolim}_{E \to Y} E \to Y$$

for E ranging over all spectra of type $\geq n$, so that Y is equivalent to the homotopy colimit. This proves the theorem for $T(n)$.

Fortunately, the case of $K(n)$-local spectra is much less complicated, as we can just compute

$$L_{K(n)} \simeq L_{K(n)}L_{T(n)} \simeq L_{K(n)} \circ \Phi \circ \Omega^\infty.$$

□

This is a remarkably strong result; even with just the definition of Φ for $n = 1$, Bousfield was able to reprove and strengthen results of Kahn and Priddy ([18]). We will not need this in any essential way, and thus will not go into detail, but we remark on one particular concrete application. Let LX^{tG} be the L-local Tate construction for an L-local G-spectrum for some finite group G. The Kahn-Priddy theorem (c.f. [21]) implies that the transfer map $\Omega^\infty EG_+ \to \Omega^\infty S$ admits a section at least for $G = \mathbb{Z}/p\mathbb{Z}$; hence one finds that the L-local transfer $LEG_+ \to LS$ admits a section for any localization L factoring through Ω^∞. By our above work, this includes $L_{T(n)}$ and $L_{K(n)}$. Remarkably, the existence of such L-local splittings turns out to imply that $LX^{tG} \simeq *$ for all X and G, and so $L_{K(n)}X^{tG} \simeq L_{T(n)}X^{tG} \simeq *$ (c.f. [20]). This is proven by reducing to $G = \mathbb{Z}/p\mathbb{Z}$; the original proof in that case required computing $K(n)$ and $T(n)$ homology, but the existence of these sections turns out to significantly reduce the difficulty of the problem.

Now, we claimed in the introduction that Φ should admit a left adjoint. This is false, as Φ does not preserve products, let alone limits. However, we claim the following:

Theorem 2.23 (Construction of Θ). *The Bousfield-Kuhn functor Φ sends v_n-periodic equivalences to equivalences, hence factors as a composite $\mathcal{S}_* \to \mathcal{S}_*^{v_n} \to \mathcal{Sp}_{T(n)}$. Furthermore, the functor $\Phi : \mathcal{S}_*^{v_n} \to \mathcal{Sp}_{T(n)}$ is conservative and admits a left adjoint.*

The following proof, unfortunately, does require some results requiring knowledge of the embedding $\mathcal{S}_*^{v_n} \hookrightarrow \mathcal{S}_*$. We thus postpone some intermediary results to section 3.

Proof. Let $f : X \to Y$ be given. We note that, for any V of type n, $\Phi_V(f)$ is an equivalence if and only if $\mathrm{Map}(V,\ \Phi(f))$ is an equivalence, and this clearly happens if $\Phi(f)$ itself is an equivalence. This proves Φ is conservative. Conversely, we recall that a homotopy limit of equivalences is an equivalence. Hence, if $\Phi_F(f)$ is an equivalence for every finite p-local type n spectrum F $\Phi(f) = \mathrm{holim}_{F\to S}\,\Phi_F(f)$ is an equivalence. Hence $\Phi : \mathcal{S}_*^{v_n} \to \mathcal{Sp}_{T(n)}$ is a well-defined conservative functor.

To construct the left adjoint Θ, we use the explicit formula for Φ as a homotopy limit to reduce to the problem of finding left adjoints

$$\Theta_E : \mathcal{Sp}_{T(n)} \to \mathcal{S}_*^{v_n}$$

to Φ_E for each type n spectrum E. By writing $E = \Sigma^{\infty - k}V$, one can reduce to the case of E the suspension spectrum of a space V equipped with a v_n-self map $v : \Sigma^t V \to V$; since $\Phi_{\Sigma^\infty V}(X) \simeq \Phi_V(X)$, we can furthermore reduce to the problem of finding adjoints Θ_V to each Φ_V for V a finite p-local type n space with a v_n-self map. Then, we must show that the functor $X \mapsto \mathrm{Map}\left(Z,\ \Phi_V(X)\right)$ should be corepresentable for all Z. We note that this functor sends homotopy colimits in Z to homotopy limits. Hence, writing Z as a homotopy colimit of spheres, we reduce to $Z = S^m$; furthermore, each sphere is a colimit of spheres of dimension kt for some $k \in \mathbb{Z}$, so we might as well take $Z = S^{kt}$. Then, $\Phi_V(X) \simeq \Sigma^{kt}\Phi_V(X)$ and $\mathrm{Map}\left(S^{kt},\ \Phi_V(X)\right) \simeq \mathrm{Map}\left(S,\ \Phi_V(X)\right)$; it thus suffices to check $Z = S$, i.e. show that $\Omega^\infty \Phi_V(X)$ is corepresentable. This space is, by definition, the homotopy colimit of $\mathrm{Map}\left(\Sigma^{kt}V,\ X\right)$. Recall that we claimed that this tower stabilizes if X is L_n^f-local. The concrete embedding $\mathcal{S}_*^{v_n} \hookrightarrow \mathcal{S}_*$ will make clear that any $X \in \mathcal{S}_*^{v_n}$ is L_n^f-local, and so the tower stabilizes. Hence, it suffices to consider the problem of showing that $\mathrm{Map}\left(\Sigma^{kt}V,\ X\right)$ is representable for $k \gg 0$. Because X is L_n^f-local, this is represented by $L_n^f\Sigma^{kt}V$, and one can check from the embedding that this is actually in $\mathcal{S}_*^{v_n}$. $\square$

While we have not yet proven anything concrete about the nature of this adjunction, we have the following theorem:

Theorem 2.24 (Elred-Heuts-Mathew-Meier). *The adjunction is monadic. Hence, $\mathcal{S}_*^{v_n} \simeq \mathcal{Alg}_T\left(\mathcal{Sp}_{T(n)}\right)$ for some T.*

Proof. The claim follows from the ∞-categorical monadicity theorem. The following form of the statement may be found in [22, §4.7]: an adjunction

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

is monadic if and only if the right adjoint G is conservative and if every G-split augmented simplicial object in $\mathcal{D}$ admits a colimit which is preserved by G. In the case of interest, we already showed that $G = \Phi$ is conservative so we are mainly interested in the second condition.

The proof found in [12] is stronger, proving that Φ preserves all geometric realizations (and that $\mathcal{S}_*^{v_n}$ admits all geometric realizations), and though this extra strength is useful.

Thus, let X_* be a simplicial object of $\mathcal{S}_*^{v_n}$. There is a natural map

$$\eta : |\Phi(X_*)| \to \Phi|X_*|$$

and we claim that this is an equivalence. Similar to the proof of the last theorem, we consider

$$\eta_V = \mathrm{Map}\,(V,\ \eta);$$

if η is an equivalence, this clearly is one as well. Conversely, if we can show that this holds for V a finite spectrum of type n with a v_n-self map v, then $\eta_{L_{T(n)}S} = \mathrm{holim}_{V \to S}\, \eta_V$ is an equivalence. But η is a morphism in the $T(n)$-local category, and so

$$\eta_{L_{T(n)}S} = \mathrm{Map}\left(L_{T(n)}S,\ \eta\right) \simeq \mathrm{Map}\,(S,\ \eta) \simeq \eta;$$

hence, to show η is an equivalence, it suffices to show that η_V is for all finite p-local spectra of type n.

Thus we fix some V, and note that the domain of

$$\eta_V : \mathrm{Map}\,(V,\ |\Phi(X_*)|) \to \mathrm{Map}\,(V,\ \Phi|X_*|)$$

is equivalent to

$$|\mathrm{Map}\,(V,\ \Phi(X_*))| \simeq |\Phi_V(X_*)|$$

since mapping out of a finite spectrum commutes with geometric realization. Meanwhile, the codomain is equivalent to

$$\Phi_V|X_*|.$$

Hence, it suffices to show that Φ_V commutes with realizations.

We now, unfortunately, need two results about the embedding $\mathcal{S}_*^{v_n} \hookrightarrow \mathcal{S}_*$. First, as we briefly mentioned earlier, the embedding factors as a composite

$$\mathcal{S}_*^{v_n} \hookrightarrow \mathcal{S}_*^{\langle d\rangle} \hookrightarrow \mathcal{S}_*,$$

i.e. has image d-connected spaces. Furthermore, it will turn out that colimits in $\mathcal{S}_*^{v_n}$ agree with those in $L_n^f\mathcal{S}_*^{\langle d\rangle}$, and these are computed by applying L_n^f to the colimit in $\mathcal{S}_*^{\langle d\rangle}$. But L_n^f does not affect the value of Φ_V, so that we are equivalently trying to show that $\Phi_V : \mathcal{S}_*^{\langle d\rangle} \to \mathcal{S}p_{T(n)}$ commutes with geometric realization. As a spectrum, $\Phi_V(-)$ is a colimit of (suspensions of) mapping spectra; since (de)suspension and colimits commute with geometric realization, we therefore restrict our attention to $\mathrm{Map}(V,\ -)$. Now, since $\mathcal{S}_*^{v_n}$ does not depend on d, we may take d to be arbitrarily large. In particular, we may take d to be larger than the dimension of V.

We can then forget the other assumptions on V and prove the following: let V be a finite (p-local) space of dimension less than d and let X_* be a simplicial space with each X_* d-connected. Then, we wish to show

$$\mathrm{Map}\,(V,\ |X_*|) \simeq |\mathrm{Map}\,(V,\ X_*)|.$$

This follows by induction on the skeleton of V. $\square$

3. Bousfield's Theorem and $\mathcal{S}_*^{v_n}$ as a Subcategory of $\mathcal{S}_*$

3.1. **Overview.** In this section, we attempt to understand the categories $\mathcal{S}_*^{v_n}$ in a concrete way. In short, the problem with our previous definition is that it is overly abstract and makes it hard to represent the objects and morphisms of $\mathcal{S}_*^{v_n}$. As with most interesting localizations of $\mathcal{S}_*$ (and $\mathcal{S}p$), the solution is to realize $\mathcal{S}_*^{v_n}$ as the category of "L-local spaces" for some localization functor $L : \mathcal{S}_* \to \mathcal{S}_*$. That is, we wish to realize $\mathcal{S}_*^{v_n}$ as a subcategory of $\mathcal{S}_*$. To do this, (and keeping remark 2.16 in mind) we wish to come up with appropriate space-level analogs of $L_n^f : \mathcal{S}p \to L_n^f\mathcal{S}p$. A theorem that we will not prove (found in [13, lec. 2]) asserts that the spectrum-level L_n^f actually agrees with Dror nullification P_A for a spectrum A of type $n+1$. Recall that Dror nullification is essentially the "orthogonal complement" to the Bousfield localization L_A; it turns out to be easier to adapt than L_A, and so we will adapt this nullification-based approach to $L_n^f\mathcal{S}p$. We will do this shortly, but let us summarize the upshot first.

First, we will obtain a suitable notion of unstable Bousfield classes such that $\langle A\rangle$ represents enough data to reconstruct the nullification functor and such that the stable class $\langle \Sigma^\infty A\rangle$ looks something like a "stabilization" of the unstable class $\langle A\rangle$. Then, we wish to find some $\langle A\rangle$ with associated localization functor L_n^f playing a similar role to the localization L_n^f used to define $T(n)$-localization. Indeed, as the notation suggests, the L_n^f-local category of d-connected spaces will be denoted $L_n^f\mathcal{S}_*^{\langle d\rangle}$; for reasons that will be clear, the d-connectedness is actually necessary for the resulting category to satisfy the previous definition as the localization of $\mathcal{S}_*$ at those f which are simultaneously v_m-periodic equivalences for $0 \le m \le n$. From there, we apply a similar construction as in the stable case in order to achieve our desired embedding. Precisely, we will be able to define $\mathcal{S}_*^{v_n}$ as the category of d-connected, L_n^f-local, and $L_{n-1}^f\mathcal{S}_*$-acyclic spaces.

In order to see that we get a well-defined category, and also to understand why connectivity might be an issue, we invoke a theorem ([17, theorem $9.14-5$]) of Bousfield similar in spirit to class invariance (theorem 2.9). Namely:

Theorem 3.1 (Bousfield)**.** *Let A and B be suspensions of finite p-local spaces of positive type. Then, $\langle A\rangle \le \langle B\rangle$ if and only if the type of A is at least as large as that of B and the same is true of connectivity.*

The main difference with class invariance is that, in the unstable case, the connectivity of A and B matter. This is exactly the reason why the analog of $L_n^f\mathcal{S}p$ consists only of sufficiently-connected spaces. Unfortunately, this connectivity also makes Bousfield's theorem significantly harder than class invariance. We will ultimately need a hard theorem on a Postnikov tower-like gadget, proven with a great deal of rather technical Bousfield class calculations. We will attempt give concise arguments when discussing this theorem in order to keep the discussion to a manageable length, but is unclear to what degree a simple or concise proof is possible.

In any case, given these embeddings and especially Bousfield's theorem, we will be able to close the gaps in a few proofs from the last section. In brief, most of these have to do with proving that certain mapping spaces are either trivial or equivalent to certain other mapping spaces, and Bousfield classes generally contain exactly enough data to answer such questions. Hence, Bousfield's theorem, together with the explicit embedding, makes it easy to analyze relatively simple situations like those of interest.

3.2. **Unstable Bousfield Localization and Classes.** We now define unstable localization and Bousfield classes. It is initially tempting to try something like $L_A = L_{\Sigma^\infty A}$, but this is the wrong notion: then we would be led to define

$$\langle A \rangle = \langle \Sigma^\infty A \rangle,$$

and would be unable to learn anything new.

As we mentioned, L_E is not the right notion to adapt. Recall that the E-local spectra are defined essentially to be those X such that $\mathrm{Map}(f,\ X)$ is an equivalence for every E-equivalence f. This suggests the following definition ([13, lec. 2]):

Definition 3.2. Let $f : A \to B$ be an arbitrary map of either spectra or spaces and X be a spectrum or space, respectively. We say that X is L_f-local if the map $\mathrm{Map}(f,\ X)$ is an equivalence. In the case of topological spaces, we mean the *unbased* mapping space; the based mapping space will be denoted with Map_* for explicitness.

Remark 3.3. There is an associated localization functor L_f which takes a space to a L_f-local space and satisfies the usual universal property. If either A and B are disconnected, L_f is usually trivial, so we assume they are connected from now on (this isn't standard, but is sufficient for our purpose here).

Remark 3.4. The condition that a space X is f-local (for $f : A \to B$ a map of spaces) is not the same as the condition that $\Sigma^\infty X$ be $(\Sigma^\infty f)$-local. As we will see, there is a relation between the two, however, and which we will exploit this later.

Lemma 3.5. *Let $f : A \to B$ be a (based) map of connected spaces. Then, a space X is L_f-local if and only if $\mathrm{Map}_*(f,\ X)$ is an equivalence.*

Proof. We may write $\mathrm{Map}_*(B,\ X)$ as the homotopy fiber of the map $\mathrm{Map}\,(B,\ X) \to X$ given by evaluation at the basepoint of B, and likewise for $\mathrm{Map}(A,\ X)$. Since f is by assumption based, we get a map of fiber sequences

$$\mathrm{Map}_*(f,\ X) \to \mathrm{Map}(f,\ X) \to \mathbb{1}_X.$$

The last map is the identity, hence clearly an equivalence. It follows that the second is an equivalence if and only if the first is one as well. □

The realization of $L_n^f \mathcal{S}_*^{\langle d \rangle}$ and $\mathcal{S}_*^{v_n}$ as subcategories of $\mathcal{S}_*$ relies on a special case of L_f, which is better understood, hence more useful for our purposes. To motivate the definition, recall that the class of E-local spectra is the "orthogonal complement" of the E-null spectra, and have a localization functor P_E (Dror nullification). The unstable versions we call P_A-local spaces and P_A, respectively, following [13, lec. 2] [5]. To make this precise, we define:

[5] Intuitively, one might call P_A-local spaces A-null. However, one might also think that L_f is a relativized notion of this and hence call L_f-local spaces f-null. Of course, it is also perfectly reasonable to think of them as f-local spaces since these are spaces that treat f as an equivalence.

For similar reasons, the terminology for the unstable equivalents is more confusing than it ought to be, and some papers (such as [17]) call the functor P_A "A-periodization" or "A-localization". Rather than use this counterintuitive terminology or risk confusion by redefining them, we follow the guidance of [13, lec. 3] and use the most generic and unambiguous term possible.

Definition 3.6. We say that X is P_A-local if X is L_f-local for $f : A \to *$ the unique such map. More explicitly, X is A-null if the inclusion $X \simeq \mathrm{Map}\,(*,\ X) \to \mathrm{Map}\,(A,\ X)$ of the constant maps into the unbased mapping space $\mathrm{Map}(A,\ X)$ is a homotopy equivalence. By lemma 3.5, this is equivalent to the condition that $\mathrm{Map}_*(A,\ X) \simeq *$.

The localizations L_f and P_A may be constructed by a small-object argument as usual. We get maps $X \to L_f X$ inducing equivalences $\mathrm{Map}\,(L_f X,\ Y) \to \mathrm{Map}(X,\ Y)$ on the unbased mapping spaces for any L_f-local Y. Since the class of L_f-local spaces is closed under products, L_f is product-preserving. Next, observe the following:

Lemma 3.7 (Fibration)**.** *Let $g : X \to Y$ be any map with L_f-acyclic fibers. Then, g is a L_f-equivalence.*

Proof. Observe that the class of L_f-equivalences is closed under homotopy colimits. Now, g can be expressed as the homotopy colimit $g = \mathrm{hocolim}_{y\in Y}\,(Fy \to \{y\})$ of its fibers Fy. By assumption, the fibers are L_f-acyclic, and $\{y\} = *$ is clearly L_f-acyclic. Hence, g is a homotopy colimit of L_f-equivalences, and the result follows. $\square$

We have one last notion to introduce before we begin proving our desired results, namely the unstable Bousfield classes. Recall that, for spectra E and F, we say $\langle E\rangle \leq \langle F\rangle$ if every F-acyclic spectrum is E-acyclic. Likewise, we make the following definition:

Definition 3.8 (Unstable Bousfield Classes)**.** We define an ordering on the Bousfield classes $\langle A\rangle$ such that $\langle A\rangle \leq \langle B\rangle$ for spaces A, B if every P_B-local spectrum is P_A-local. One can check that this holds if and only if the reverse inclusion holds of the classes of P_A-acyclic and P_B-acyclic spaces (since these are, at least intuitively, orthogonal complements). Since A is P_A-acyclic, this implies that $P_B A \simeq *$; the converse is also true by a suitable transitivity theorem (c.f. [13, lec. 3]).

Any of these equivalent conditions then implies the existence of a natural transformation $P_A \to P_B$. If $\langle A\rangle = \langle B\rangle$ (defined in the obvious way so that the Bousfield classes form a poset), we find $P_A \simeq P_B$, and one checks readily that any such natural equivalence implies $\langle A\rangle = \langle B\rangle$.

Finally, we define (as with spectra)

$$\langle A\rangle \vee \langle B\rangle = \langle A \vee B\rangle$$

(which is readily checked to be the join of this poset) and

$$\langle A\rangle \otimes \langle B\rangle = \langle A \wedge B\rangle,$$

which is *not* necessarily the meet (recall that there are nonzero spectra E with $E \wedge E \simeq *$ even though $\langle E\rangle = \langle *\rangle$ implies $E \simeq *$) but is the meet on “reasonable” spaces. This is always a lower bound, however, i.e. $\langle A \wedge B\rangle \leq \langle A\rangle$; in particular $\langle \Sigma A\rangle \leq \langle A\rangle$.

Having now established a great deal of terminology, we make it useful by proving several practical theorems about unstable localizations.

3.3. **Generalized Postnikov Towers.** We begin by proving the “easy” direction of Bousfield’s theorem in order to develop an intuition for its meaning and thereby a roadmap to the proof.

Let $C = S^n$ be an arbitrary sphere. By lemma 3.5, a space X is P_C-local if and only if the based mapping space $\mathrm{Map}_*\,(S^n,\ X)$ is contractible, or equivalently if and only if $\pi_m X \simeq 0$ for $m \geq n$. It follows that P_C agrees with taking the n-coconnected cover, i.e. the $(n-1)^{\text{st}}$ Postnikov section. Dually, X is P_C-acyclic if and only if X is equivalent to the fiber of $X \to P_C$, i.e. if and only if X is $(n-1)$-connected (or n-connective).

We can use this to prove one direction of Bousfield's theorem. Suppose $\langle A \rangle \leq \langle B \rangle$. Then, consider the set of n for which A and B are P_C-acyclic. By the results of the last paragraph, we obtain the desired inequality of their connectivities. We can also use $\Omega^\infty K(n)$ instead of S^n, and then get the desired inequality on the types of A and B.

Bousfield's theorem, loosely speaking, states that there are *no* other restrictions. For some intuition, consider

$$\langle \Sigma^k A \rangle \approx \langle S^k \rangle \wedge \langle A \rangle.$$

This is not meant to be precise, but it suggests that $\langle \Sigma^k A \rangle$ is the class of k-connected P_A-acyclic spaces. Conversely, $\langle A \rangle$ is the class of $P_{\Sigma^k A}$-acyclic spaces "up to k connectivity". If we take this to the extreme, we might guess that the stable Bousfield class is just the unstable Bousfield class with the connectivity restriction somehow removed. This reasoning is further supported by the following fact, which can proven by explicitly constructing L_f and using a relative Hurewicz theorem:

Fact 3.9. *If f is a k-connected map, so is $X \to L_f X$.*

This suggests considering the generalized Postnikov tower

$$\ldots \to P_{\Sigma^{k+1} A} X \to P_{\Sigma^k A} X \to \ldots.$$

For $A = S^0$, this is the Postnikov tower of X by our previous discussion, and the consecutive fibers are Eilenberg-MacLane spaces. We will soon prove theorem 3.10, which generalizes this significantly. Furthermore, it asserts that the maps in this tower are not "too" far from being equivalences. This theorem appears as [17, theorem 7.2] and is stronger than what we need; we nevertheless state the full version for completeness:

Theorem 3.10 (Bousfield)**.** *Let $n \geq 1$ and $J \subseteq \mathbb{Z}$ be a set of primes. Suppose W is a space satisfying the d-supported J-torsion (resp. J-local) condition. That is, suppose*

(1) $\widetilde{H}_*(W)$ is J-torsion (resp. J-local),
(2) $\widetilde{H}_i(W) \simeq 0$ for $i < d$, and
(3) $H^d(W; \mathbb{Z}/p\mathbb{Z}) \neq 0$ for each $p \in J$ (resp. $H^d\big(W; \mathbb{Z}_{(J)}\big) \neq 0$).

Then for any space Y and integer $i \geq 1$, $P_{\Sigma^{i+1} W} Y \to P_{\Sigma^i W} Y$ has fiber $K(G,\ d+i)$ with G a J-torsion (resp. J-local) group.

Remark 3.11. We can reduce to the case $i = 1$ by replacing W by its $(i-1)$-fold suspension.

The case of interest for us will be the J-torsion case for $J = \{p\}$. As such, we will merely sketch the argument of [13, lec. 2-3] for this case. The general case is not much harder, but the bookkeeping involved is tedious, and the proof is already long and hard; the full argument in [17] is of similar flavor to this one.

Also, we make the following note: in our proof of this theorem and others in this section, we will not go into detail proving basic results on Bousfield classes. A list of some results (together with a couple proofs) may be found at [13, lec. 3]. Except when noted, we follow the proofs there.

The key idea in the proof is the following criterion:

Fact 3.12. *A space X is a generalized Eilenberg-MacLane space (i.e. is a product of Eilenberg-MacLane spaces) if and only if it is $\Omega^\infty M$ for some $H\mathbb{Z}$-module M*[6]*. In particular, retracts of generalized Eilenberg-MacLane spaces are generalized Eilenberg-MacLane spaces.*

The "if" direction follows from the fact that $H\mathbb{Z}$ represents homology; to prove the other, one can consider Moore spectra M_n with n^{th} homology isomorphic to $\pi_n X$ and then obtain an $H\mathbb{Z}$-module structure on $\bigvee M_n$.

We will use this criterion to show that the fibers of $P_{\Sigma^2 W} Y \to P_{\Sigma W} Y$ are generalized Eilenberg-MacLane spaces, say $\prod K(G_r,\ r)$. From there, we can do suitable analysis to eliminate the terms with $r \neq d+i$ and identify what G_{d+i} are possible. In order to apply this fact, it is necessary to understand the interaction between localization and infinite loop spaces. Since L_f preserves products and is suitably functorial, it sends Segal spaces to Segal spaces, Γ spaces to Γ spaces, and hence sends (infinite) loop spaces to (infinite) loop spaces. Hence, for any spectrum X, $L_f \Omega^\infty X$ is the zeroth space of some connective spectrum $\widetilde{L_f} X$. We have:

Lemma 3.13. *Let X and Z be connective spectra. Then, the obvious map*

$$\widetilde{L_f}(Z \wedge X) \to \widetilde{L_f}\Big(Z \wedge \widetilde{L_f} X\Big)$$

is an equivalence.

Proof. We pass to infinite loop spaces; it then suffices to show

$$L_f \Omega^\infty (Z \wedge X) \to L_f \Omega^\infty \Big(Z \wedge \widetilde{L_f} X\Big);$$

write Z as a filtered colimit of finite complexes to reduce to the case that Z is finite. Then, induct on a cellular decomposition of Z and use lemma 3.7 in order to reduce to the case $Z = S^n$. It would then suffice to show that L_f-equivalences are preserved upon delooping. To see this, consider an L_f-equivalence $\Omega g : \Omega X \to \Omega Y$ with X and Y connected. One has the canonical ΩX-bundle $\Omega X \to PX \to X$ and canonical ΩY-bundle $\Omega Y \to PY \to Y$ such that Ωg is a map of fibrations, with the map $Pg : PX \to PY$ of (contractible) total spaces an L_f-equivalence. One can express the map $X \to Y$ as a Bar construction of the map $\Omega X \to \Omega Y$ and $PX \to PY$. Since the class of L_f-equivalences is closed under colimits, the claim follows. $\square$

Now, suppose we are given any connective ring spectrum R, and a connective module M over R. Choose an inverse to the map of lemma 3.13 with $Z = R$ and $X = M$; we then find that $L_f \Omega^\infty M$ is the zeroth space of an R-module.

Remark 3.14. Taking $R = H\mathbb{Z}$, we find that L_f of a generalized Eilenberg-MacLane space is also a generalized Eilenberg-MacLane space.

We use this to prove the following more practical criterion for a space to be a generalized Eilenberg-MacLane space [7]:

[6] Here and elsewhere, we restrict our attention to generalized Eilenberg-MacLane spaces with abelian π_0 and π_1.

[7] Note the resemblance to theorem 3.10. That said, note also that this applies to the L_f scenario and not just P_A.

Lemma 3.15 (Improved Criterion)**.** *Let f be a map of connected spaces. and suppose X is a $L_{\Sigma f}$-acyclic space. Then $L_{\Sigma^2 f}X$ is a generalized Eilenberg-MacLane space.*

To prove this it is convenient to have the following:

Fact 3.16. *Let $QX = \Omega^\infty\Sigma^\infty X$. Then, QA is P_A-acyclic. More generally, if X is L_f-acyclic, so is QX.*

This is proven using the May model for QX. In this model, we identify QX as the homotopy colimit of spaces $\mathrm{Fil}_n X$ defined by induction from $\mathrm{Fil}_1 X$ via configuration spaces.

Now, we prove lemma 3.15.

Proof of lemma 3.15. We use fact 3.12. The natural way to obtain an $H\mathbb{Z}$-module from a space X is to try smashing a fiber sequence $Z \to S \to H\mathbb{Z}$ with $\Sigma^\infty X$ to produce a fiber sequence

$$Z \wedge X \to \Sigma^\infty X \to H\mathbb{Z} \wedge X.$$

Clearly, $H\mathbb{Z} \wedge X$ is an $H\mathbb{Z}$-module. If we take infinite loop spaces, we get a fiber sequence

$$\Omega^\infty Z \wedge X \to QX \to \Omega^\infty(H\mathbb{Z} \wedge X);$$

furthermore, since $S \to H\mathbb{Z}$ is 0-connected, Z is connected. We claim that $\Omega^\infty Z$ is $L_{\Sigma^2 f}$-acyclic,

Since X is $L_{\Sigma f}$-acyclic by assumption, one can check that $\Sigma^k X$ is $L_{\Sigma^2 f}$-acyclic for any $k \geq 1$. Because Z is connected, we can choose a cellular decomposition for Z with no nontrivial 0-cells. Hence, we may ensure that $\Omega^\infty(C \wedge X) \simeq Q\Sigma^k X$ is $L_{\Sigma^2 f}$-acyclic for any cell C of Z (by fact 3.16). By induction on a skeleton of Z to see that $\Omega^\infty W \wedge X$ is $L_{\Sigma^2 f}$-acyclic for any finite subcomplex W of Z. Passing to homotopy colimits, so is $\Omega^\infty Z \wedge X$.

As a result, we have that $QX \to \Omega^\infty(H\mathbb{Z} \wedge X)$ is a $L_{\Sigma^2 f}$-equivalence. Hence,

$$L_{\Sigma^2 f}QX \simeq L_{\Sigma^2 f}\Omega^\infty(H\mathbb{Z} \wedge X)$$

is the zeroth space of an $H\mathbb{Z}$-module, and so $L_{\Sigma^2 f}QX$ is a generalized Eilenberg-MacLane space by remark 3.14.

Of course, we were not interested in $L_{\Sigma^2 f}QX$, but in $L_{\Sigma^2 f}X$; we complete the proof by presenting the latter as a retract of the former. Since QX is the free infinite loop space on X, it suffices to show that $L_{\Sigma^2 f}X$ is an infinite loop space. This follows if we can construct suitable maps $L_{\Sigma^2 f}X^{\times k} \to L_{\Sigma^2 f}X$. We claim that the domain is equivalent to $L_{\Sigma^2 f}X^{\vee k}$. Then, we will be able to use the fold map to obtain the desired algebra structure. We will prove this by induction on k; the case $k = 1$ is trivial, so consider

$$r : L_{\Sigma^2 f}(X \vee Y) \to L_{\Sigma^2 f}(X \times Y).$$

We now show that this is an equivalence for a sufficiently general class of Y.

Write the fiber of r as $L_{\Sigma^2 f}\,\mathrm{hof}(X \vee Y \to X \times Y)$; we readily check that

$$\mathrm{hof}(X \vee Y \to X \times Y) \simeq \Sigma\Omega X \wedge \Omega Y \simeq \Omega X \wedge (\Sigma\Omega Y).$$

We wish to show that this is $L_{\Sigma^2 f}$-acyclic for sufficiently nice Y. Recall that we assumed X is $L_{\Sigma f}$-acyclic; one can use this to show that ΩX is L_f-acyclic.

Furthermore, a cellular decomposition trick along the lines of fact 3.9 proves that the smash of a k-connected space with a L_f-acyclic space is $L_{\Sigma^k f}$-acyclic.

It therefore suffices to show $\Sigma\Omega Y$ is simply connected. This will follow if Y is simply connected. In our case of interest, we claim X is simply connected so that $Y = X^{\times k}$ is also simply connected. By a classifying space argument of [17, theorem 3.1], which we will not prove, one finds that $L_f \Omega Y \simeq \Omega L_{\Sigma f} Y$ for any connected space Y. Now, f is a 1-connected map, so $Y \to L_f Y$ is 1-connected by fact 3.9, and thus any L_f-acyclic space is connected. In particular, X is connected and so $L_f \Omega X \simeq \Omega L_{\Sigma f} X \simeq *$ implies ΩX is connected, i.e. X is simply connected. This, at long last, concludes the proof. $\square$

We now return to the situation of theorem 3.10. We wish to apply this to the fiber F of

$$P_{\Sigma^2 W} X \to P_{\Sigma W} X \simeq P_{\Sigma W} P_{\Sigma^2 W} X$$

and hence prove that F is a generalized Eilenberg-MacLane space. Note that it is the fiber of the canonical map $\eta : Y \to P_A Y$, where $A = \Sigma W$ and $Y = P_{\Sigma^2 W} X$. We expect the following more general result:

Lemma 3.17 ([17, Cor. 4.8])**.** *The fibers Z of $Y \to P_A Y$ are P_A-acyclic for any A and Y.*

Proof. Consider the diagram of fiber sequences

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & P_A Y \arrow[d] \arrow[dl, dotted] \\ P_A Z \arrow[r] & Y' \arrow[r] & P_A Y \end{tikzcd}$$

where the bottom row is the fiberwise localization of the top row. We have not defined fiberwise localization, but the meaning should be clear given that it sends a fibration over B with fibers F to a new one whose fiber $P_A F$. Explicitly, the fiberwise localization of $S \to T$ may be constructed $\operatorname{hocolim}_{t \in T} P_A(Ft \to \{t\})$ where Ft is the fiber over at T. Now, the class of P_A-local spaces is closed under extensions by lemma 3.5[8], and so Y' is P_A-local. The map between these fiber sequences is given by $\operatorname{hocolim}_{y \in P_A Y} (Z \to P_A Z)$. Furthermore, the universal property of $P_A Y$ implies that the map $Y \to Y'$ factors as $Y \to P_A Y \to Y'$; this is the dashed arrow in the diagram, and it must make the diagram commute. But then $P_A Z \to Y'$ factors through $Z \to P_A Y$, which is the zero map essentially by definition. Furthermore, since the right-hand triangle commutes, the bottom fibration admits a section. It is a fact (which may be found in, e.g. [13, lec. 3]) that any fibration $F \to X \to B$ with $F \to X$ null and which admits a section necessarily has contractible fibers, and this evidently applies here to finish the proof. $\square$

Hence, the fiber F of

$$P_{\Sigma^2 W} X \to P_{\Sigma W} X \simeq P_{\Sigma W} P_{\Sigma^2 W} X$$

is $P_{\Sigma W}$-acyclic, and one can check that the fiber of a map of $P_{\Sigma^2 W}$-local spaces is $P_{\Sigma^2 W}$-local, so that $P_{\Sigma^2 W} F \simeq F$ is a generalized Eilenberg-MacLane space. Let $F = \prod K(G_m,\, m)$. We must now show that G_m is trivial unless $m = n + 1$, where W is n-connective by assumption. We recall from fact 3.9 that $X \to P_{\Sigma W} X$

[8] The analog for L_f is *not* true, and this is one of the main reasons P_A is better-behaved.

is $(n+1)$-connected since $\Sigma W \to *$ is, and thus F is n-connected. This proves that the G_m with $m \le n$ are trivial. Since each $K(G_m,\ m)$ is a retract of F, they are $P_{\Sigma W}$-acyclic and $P_{\Sigma^2 W}$-local. We now must analyze when an individual Eilenberg-MacLane space can be P_A-local or acyclic.

Take all homology to have coefficients $k = \mathbb{Z}/p\mathbb{Z}$ unless otherwise specified. Let $c(X) = \mathrm{conn}(X) + 1$ be the least n for which $\pi_n(X) \neq 0$, and let $d(X)$ be the least n for which $H_n(X) \neq 0$ (which agrees with $c(X)$ for X of type > 0). We will need a few lemmas.

Lemma 3.18. *For any A, $K(k,\ d(A))$ is P_A-acyclic. Hence, by induction, we have that $K(k,\ m)$ is P_A-acyclic for $m \ge d(A)$. By passage to colimits, the same is true with k replaced by any k-module or p-power torsion group.*

Dually, if A has p-torsion homology, $K(G,\ m)$ is P_A-local for any G on which p acts invertibly and any m.

Proof. Use fact 3.16. Any A and d for which $H_d(A) \neq 0$ give a retract $\Sigma^d Hk \hookrightarrow \Sigma^d Hk \wedge \Sigma^\infty A \twoheadrightarrow \Sigma^d Hk$ hence a retract

$$K(k,\ d) = \Omega^\infty \Sigma^d Hk \hookrightarrow \Omega^\infty \Sigma^d Hk \wedge \Sigma^\infty A \twoheadrightarrow \Omega^\infty \Sigma^d Hk = K(k,\ d).$$

We claim that the middle, being roughly a smash of A, is P_A-acyclic, in which case so is $K(k,\ d)$. This follows because

$$P_A \Omega^\infty \Sigma^d Hk \wedge \Sigma^\infty A = \Omega^\infty \widetilde{P_A}\big(\Sigma^d Hk \wedge \Sigma^\infty A\big) \simeq \Omega^\infty P_A \widetilde{\Sigma^d Hk \wedge {}_f \Sigma^\infty A}$$

by lemma 3.13. Finally, $\widetilde{P_A}\Sigma^\infty A$ is a connective spectrum with zeroth space $P_A QA \simeq *$, hence trivial.

Next, the induction step follows by noting that $d(\Sigma A) = d(A) + 1$, $K(k,\ m) \simeq K(k,\ m)$, then using [17, theorem 3.1].

In the other case, just note that A has trivial homology with coefficients in G, and this is represented by maps to $K(G,\ m)$. Hence, A has no nontrivial maps to $K(G,\ m)$ and this proves the claim. □

To see that the G_m from before are p-power torsion (this argument due to [13, lec. 4]), note that W has no rational homology by assumption. Hence, neither can F. If $m > n+1$, $K(G_m,\ m)$ are $P_{\Sigma^2 W}$-acyclic by the above result. We already showed that they were $P_{\Sigma^2 W}$-local, so they are trivial.

This proves theorem 3.10. Before continuing on, we note the following useful application:

Remark 3.19. Let A be a suspension, i.e. a space of the form ΣW. Let n be as before. Then, $P_{\Sigma^2 W} A \to P_W A \simeq *$ has fiber $P_{\Sigma^2 W} A = P_{\Sigma A} A$ necessarily an Eilenberg-MacLane space of the form $K(G,\ n)$ with G a p-power torsion group.

3.4. **Bousfield's Theorem on Unstable Bousfield Classes of Suspensions.**

We turn at long last to the proof of Bousfield's theorem. Thanks to the last section, the proof is relatively short. The key idea, as mentioned previously, is to use theorem 3.10 as a means of understanding how much information is lost in replacing unstable classes by stable ones. We begin with two simple lemmas along these lines (c.f. [13, lec. 3]).

Lemma 3.20. *Let A and B be finite p-local spaces. Note that*

$$\langle A \rangle \le \langle B \rangle$$

implies

$$\langle \Sigma^\infty A\rangle \le \langle \Sigma^\infty B\rangle.$$

The converse is (from one direction of Bousfield's theorem) false, but is true up to suspension (in a way which will be made precise in the proof).

Proof. Consider the subcategory $\mathcal{C}$ of $X \in f\mathcal{S}p_{(p)}$ with the property that there exists a $k \in \mathbb{Z}$ for which $\Sigma^k X = \Sigma^\infty Z$ is a suspension spectrum with $\langle Z\rangle \le \langle B\rangle$. Since ΣZ is P_Z-acyclic, we have that $\langle \Sigma Z\rangle \le \langle Z\rangle$, and we may take $k \ge 0$ without loss of generality. For $X = \Sigma^\infty B$, one can take $k = 0$ and so $\Sigma^\infty B \in \mathcal{C}$. It turns out that $\mathcal{C}$ is thick; it is clearly closed under shifts, but more work must be done to show that it is closed under taking cofibers and so we direct the reader to [13, lec. 3] for a proof of this fact.

Now, since $\langle \Sigma^\infty A\rangle \le \langle \Sigma^\infty B\rangle$, $\Sigma^\infty A$ has at least as large a type as $\Sigma^\infty B$, and hence lies in any thick subcategory of $f\mathcal{S}p_{(p)}$ containing $\Sigma^\infty B$. In particular, $\Sigma^\infty A \in \mathcal{C}$ so that $\langle \Sigma^k A\rangle \le \langle B\rangle$ for some $k \ge 0$. ☐

In the case of equality, a stronger result is true:

Lemma 3.21. *If $\langle \Sigma^\infty A\rangle = \langle \Sigma^\infty B\rangle$, with both A and B finite p-local spaces, the functors $\langle d\rangle \circ P_A$ and $\langle d\rangle \circ P_B$ agree for d sufficiently large. Here, $\langle d\rangle$ is the functor $X \mapsto X\langle d\rangle$ which takes d-connected covers.*

Proof. By the last lemma, we have that

$$\langle \Sigma A\rangle \ge \langle \Sigma^a B\rangle \ge \langle \Sigma^b A\rangle \ge \langle \Sigma^c B\rangle$$

and thereby get natural transformations

$$P_{\Sigma^c B} \to P_{\Sigma^b A} \to P_{\Sigma^a B} \to P_{\Sigma A}.$$

Of course, the composites of the form $\eta_Z(s,\ t) : P_{\Sigma^s Z} \to P_{\Sigma^t Z}$ are composites of maps from the generalized Postnikov tower. By theorem 3.10 any map of the form $P_{\Sigma^k A}X \to P_{\Sigma^\ell A}X$ has d-coconnected fibers for some d (depending on k, ℓ, and A, but not X), so that $\eta_A(s,\ t)$ and $\eta_B(s,\ t)$ are equivalences after passage to d-connected covers for some d and any $1 \le s, t \le a+b+c+1$; the claim then follows from some diagram chasing, spelled out in more detail in [13, lec. 3]. ☐

In fact, it suffices to take $\max(\mathrm{conn}(A),\ \mathrm{conn}(B))$-connected covers.

Now, we sketch the proof of Bousfield's theorem found in [13, lec. 4]. Suppose A and B are suspensions with A having at least as large a type and connectivity. By class invariance and lemma 3.20, we have that $\langle \Sigma^k A\rangle \le \langle B\rangle$ for some k. We therefore reduce to the following

Theorem 3.22 (Elimination of Suspensions)**.** *Suppose $\langle \Sigma A\rangle \le \langle B\rangle$ with both A and B suspensions, and suppose A is at least as connected as B. Then, $\langle A\rangle \le \langle B\rangle$.*

Proof. The fiber of $A \to P_{\Sigma A}A$ is $P_{\Sigma A}$-acyclic by lemma 3.17, hence P_B-acyclic by assumption. Thus, this map is a P_B-equivalence. It thus suffices to show $P_{\Sigma A}A$ is P_B-acyclic. To see this, simply apply remark 3.19 and lemma 3.18. ☐

3.5. **Constructing the Embedding.** Now that we have Bousfield's theorem, we can therefore make sense of the following redefinitions as being reasonably free of choices.

Definition 3.23 (The embedding $\mathcal{S}_*^{v_n} \hookrightarrow \mathcal{S}_*$)**.** Let A and B be suspensions of finite p-local spaces of type n and $n+1$, respectively. By suspending both enough, assume they both have connectivity d (i.e. are $(d-1)$-connected). Then, $L_n^f \mathcal{S}_*^{\langle d \rangle}$ is defined to be the category of d-connected P_A-local spaces, and $\mathcal{S}_*^{v_n}$ to be the category of d-connected, P_A-local spaces which are also P_B-acyclic.

Our next major goal is to prove the following:

Theorem 3.24 (Compatibility)**.** *These new definitions are equivalent to the old, up to equivalence. That is, $L_n^f \mathcal{S}_*^{\langle d \rangle}$ satisfies the universal property of the localization of $\mathcal{S}_*$ at those maps which are v_m-periodic equivalences for $0 \leq m \leq n$*[9]*, and likewise $\mathcal{S}_*^{v_n}$ satisfies the universal property of the localization of $\mathcal{S}_*$ at the v_n-periodic equivalences.*

Note that such A and B do not exist for every n and d. By the periodicity theorem, there do exist such spaces for any given n and d sufficiently large. It is unfortunately not easy to find any bound on how large we must take d to be, but it is thankfully irrelevant to our present purposes. For the purposes of this section, we treat A and B as fixed suspensions of spaces with the given type and connectivity. By the universal property of $\mathcal{S}_*^{v_n}$ asserted in the above theorem, $\mathcal{S}_*^{v_n}$ is independent of d up to equivalence; we have already seen how this explicit construction (and invariance under increasing d) can prove useful.

We begin by understanding $L_n^f \mathcal{S}_*^{\langle d \rangle}$. There is an explicit localization, which we denote $X \mapsto L_n^f X \langle d \rangle$ (or $X \mapsto L_n^f X$ if d is obvious), given by

$$X \mapsto P_A\Big((X\langle d \rangle)_{(p)}\Big).$$

Since A and B are d-connective, P_A and P_B preserve the property of being d-connected, so that the result is d-connected still. It is instructive (and useful, as we will see later) to analyze the role of d more precisely. For instance:

Lemma 3.25. *A space X is P_A-acyclic if and only if its d-connected cover is $P_{\Sigma A}$-acyclic and it is d-connective with π_d a p-power torsion group.*

Proof. The "if" direction follows by taking the fiber sequence $X\langle d \rangle \to X \to K(G,\, d)$ and using Bousfield's theorem on the first. Conversely, P_A is d-connected so that X must be d-connective, and it remains to check the p-power torsion. But we can apply theorem 3.10 to get a fiber sequence of the form $F \to X \to P_{\Sigma A} X \simeq K(G,\, d)$, with the first $P_{\Sigma A}$-acyclic, hence d-connected and thus equivalent to $X\langle d \rangle$; this proves the claim. □

Remark 3.26. In particular, if X has p-power torsion homotopy groups and is d connective, to show X is P_A-acyclic it suffices to check that $X\langle m \rangle$ is for $m \gg 0$, or even that Y is for some Y with $Y\langle m \rangle \simeq X\langle m \rangle$ for $m \gg 0$.

One can also relate the categories $L_n^f \mathcal{S}_*^{\langle d \rangle}$ as d varies, viz.:

---

[9] Recall that we defined v_0-periodic equivalences to be maps which give isomorphisms on rational homology above degree d; this is the concrete interpretation of that seemingly mysterious parameter d.

Theorem 3.27 (Image of P_A)**.** *The localization $P_A : L_n^f \mathcal{S}_*^{\langle d+1\rangle} \to L_n^f \mathcal{S}_*^{\langle d\rangle}$ is fully faithful with image the spaces with torsion π_{d+1}.*

Proof. First, we show that $\eta : Y \to P_A Y$ is a rational equivalence for any $(d+1)$-connected Y. Indeed, consider the P_A-acyclic fibers of $\eta : Y \to P_A Y$; if they were not rationally trivial, they would admit a $K\big(\mathbb{Z}_{(p)},\ m\big)$-retract for some $m \geq d$. Such a space is P_A-local, so cannot admit nontrivial maps from the fibers, contradiction. Note also that Y, being P_A-local, has $Y\langle d+1\rangle \simeq Y$ a $P_{\Sigma A}$-local space. The fiber sequence $F \to Y \to P_A Y$ thus has F an Eilenberg-MacLane space by theorem 3.10, and hence η presents Y as $(P_A Y)\langle d+1\rangle$ by delooping this.

That this restricted localization is fully faithful is not hard to check: indeed, the domain consists only of P_A-local spaces, and fully-faithfulness follows by expanding out the definitions.

Finally, suppose Y is d-connected, P_A-local, and has π_{d+1} torsion. Then, its $(d+1)$-connected cover X lives in $L_n^f \mathcal{S}_*^{\langle d+1\rangle}$ from the fiber sequence $X \to Y \to K(G,\ d+1)$ and the fact that G is p-power torsion. Together with our observations above, we can see that it would suffice to show that $X \to Y$ agrees with $\eta : X \to P_A X$, i.e. that the fibers are P_A-null. This follows from lemma 3.18 as the fibers are $K(G,\ d)$ with G p-power torsion. $\square$

We now show the following half of theorem 3.24:

Theorem 3.28 ([13, lec. 6])**.** *Let V be a finite space of type $n > 0$ which is equipped with a v_n-self map. Then, the spectrum $\Phi_V(X)$ is unaffected by replacing X with $L_m^f X$ for $m \geq n$*

Proof. We prove it in steps. Since Φ_V is periodic, $\Phi_V(X\langle d\rangle) \simeq \Phi_V(X)$. To show invariance under p-localization, write X as the pullback of $p^{-1}X$ and $X_{(p)}$ over $X_{\mathbb{Q}}$ and apply $\mathrm{Map}_*(V, -)$: since V has type > 0, $\mathrm{Map}_*\big(V,\ p^{-1}X\big)$ and $\mathrm{Map}_*(V,\ X_{\mathbb{Q}})$ are trivial. Hence, $\mathrm{Map}_*(V, -)$ is invariant under p-localization; we may then replace V by its suspensions and pass to colimits to show the same of Φ_V.

This leaves the task of showing that replacing X by $P_A X$ does not affect $\Phi_V(X)$ for $\mathrm{typ}(V) \leq n$. Reduce to $\Phi_V(X)\langle d\rangle$ to by periodicity of Φ_V. Explicitly, we wish to show that

$$\Omega^\infty \Phi_V(X) = \mathrm{hocolim}\,\mathrm{Map}_*\big(\Sigma^{kt}V,\ X\big) \to \mathrm{hocolim}\,\mathrm{Map}_*\big(\Sigma^{kt}V,\ P_A X\big)$$

is an equivalence up to d-connected covers, where V has a v_m-self map $v : \Sigma^t V \to V$. Now, $\Sigma^\infty A$ is $L_{T(m)}$-acyclic, so that $\mathrm{Map}_*(A,\ \Omega^\infty\Phi_V(X)) \simeq *$, i.e. $\Omega^\infty\Phi_V(X)$ is P_A-local. Hence, so is the d-connected cover. Since P_A commutes with filtered colimits, the map is equivalent to (again, suppressing connected covers for readability)

$$\mathrm{hocolim}\, P_A\big(\mathrm{Map}_*\big(\Sigma^{kt}V,\ X\big)\big) \to \mathrm{hocolim}\,\mathrm{Map}_*\big(\Sigma^{kt}V,\ P_A X\big).$$

It evidently suffices to show that this is a termwise equivalence, i.e. that $\mathrm{Map}_*\big(\Sigma^{kt}V,\ -\big)\langle d\rangle$ commutes with P_A. By writing $\Sigma^{kt}V$ as a finite hocolim S^r, we reduce to theorem 3.29, which we will now state and prove separately. $\square$

Theorem 3.29. *The functor $P_A : \mathcal{S}_*^{\langle d\rangle} \to \mathcal{S}_*^{\langle d\rangle}$ preserves finite homotopy limits, as does L_n^f.*

Remark 3.30. Since P_A is a left adjoint, we obtain an explicit way to compute homotopy colimits in $L_n^f\mathcal{S}_*^{\langle d\rangle}$: apply P_A and then compute the colimits in $\mathcal{S}_*^{\langle d\rangle}$. This issue arose, for instance, in c.f. theorem 2.24.

Limits are significantly more difficult to compute, and so the proof of theorem 3.29 is nontrivial.

Proof. We follow [13, lec. 4], and only discuss the first part; the second follows from the first by decomposing L_n^f into a composite of several functors as in the arguments above.

It is clear that P_A preserves the terminal object, so we must show that it preserves pullbacks in $\mathcal{S}_*^{\langle d\rangle}$. To do this, it suffices to show that, for every pullback diagram $X_0 \to X_{01} \leftarrow X_1$ in $\mathcal{S}_*^{\langle d\rangle}$, the fiber F of

$$(X_0 \times_{X_{01}} X_1)\langle d\rangle \to (P_A X_0 \times_{P_A X_{01}} P_A X_1)$$

is P_A-acyclic. It evidently is $(d-1)$-connected, and a bit of work yields that $\pi_* F$ is p-power torsion. We wish to replace F by something with "better" behavior in low connectivities. Indeed, let Y_* be the d-connected cover of the (necessarily P_A-acyclic) homotopy fiber of $X_* \to P_A X_*$, and consider $Y_0 \times_{Y_{01}} Y_1$. This is what F would be if X_* were replaced by Y_*. If we replace a single X_* by the corresponding Y_*, one can check that $F\langle m\rangle$ does not change for $m \gg 0$; in particular, we may replace the X_* by Y_*. In this way, we reduce to the case where X_* are P_A-acyclic and we wish to show that the pullback also is.

Recall that there is a criterion for a square to be Cartesian in terms of iterated fibers. Let $X = \operatorname{holim}(X_0 \to X_{01} \leftarrow X_1)$ be a pullback of d-connected P_A-acyclic spaces and now let F be the fiber of $X_1 \to X_{01}$: the square is a pullback if and only if F is also the fiber of $X \to X_0$. Then, it suffices to show that F is P_A-acyclic. By considering $\Omega X_{01} \to F \to X_1$, we reduce to showing that ΩX_{01} is P_A-acyclic. But by some classifying space argument, we can show

$$P_A \Omega X_{01} \simeq \Omega P_{\Sigma A} X_{01},$$

and X_{01} is $P_{\Sigma A}$-acyclic by lemma 3.25. $\square$

Earlier, we claimed that for any type $m \le n$ V with a v_m-self map $v : \Sigma^t V \to V$ and any L_n^f-local space X, $\operatorname{Map}\left(\Sigma^{kt} V,\ X\right)$ is eventually almost constant: that is, after replacing V by $\Sigma^{kt}V$ for $k \gg 0$ and taking connected covers, the cofibers $\operatorname{Map}_*(\operatorname{cof}(v),\ X)\langle d\rangle$ are trivial. Equivalently, the cofibers are d-truncated, or equivalently d-fold loop space is contractible. This follows as this is $\operatorname{Map}_*\left(\Sigma^d \operatorname{cof}(v),\ X\right)$ and we can take $B = \Sigma^d \operatorname{cof}(f)$ and $A = X$ in Bousfield's theorem; the type and connectivity requirements follow without much difficulty.

We can now give the proof of theorem 3.24 in the $L_n^f\mathcal{S}_*^{\langle d\rangle}$ case. More explicitly, we now prove the following:

Theorem 3.31 (Embedding $L_n^f\mathcal{S}_*^{\langle d\rangle}$)**.** *A map $u : X \to Y$ is an L_n^f-equivalence if and only if it is a v_m-periodic equivalence for $0 \le m \le n$, where we use the convention that a v_0-periodic equivalence is a map inducing isomorphisms on rational homotopy groups of degree $> d$.*

Proof. Assume first that u is an L_n^f-equivalence. By theorem 3.28, $L_n^f u$ is a v_m-periodic equivalence and induces the same map on v_m-periodic homotopy groups as u, so that u must also be a v_m-periodic equivalence. With some work, one can also check the rational homotopy condition and the converse for $n = 0$. More explicitly, $L_0^f \mathcal{S}_*\langle d\rangle$ agrees with $X \mapsto (X\langle d\rangle)_{\mathbb{Q}}$, and so u is a v_0-periodic equivalence if and only if it is an L_0^f-equivalence.

To prove the converse, i.e. that maps which are v_m-periodic equivalences for $m \leq n$ are necessarily L_n^f-equivalences, we go by induction on n. We already discussed the case $n = 0$ above, so it suffices to consider $n > 0$. Additionally, replace everything by its image under L_n^f. Consider the (now P_A-local) fiber F of u; we wish to show that it is trivial, or equivalently P_A-acyclic. For any type n space V, the above applies to give

$$* \simeq \Omega^\infty \Phi_V(F)\langle d\rangle \simeq \mathrm{Map}(V,\ F)\langle d\rangle$$

and thus $\mathrm{Map}(V,\ F)\langle d\rangle \simeq *$, i.e. F is P_V-local after we suspend V enough; assume $e = \mathrm{conn}(V) + 1 \geq d$ as well. Then, $F\langle e\rangle \in L_{n-1}^f \mathcal{S}_*^{\langle e\rangle}$. The inductive hypothesis finishes the proof: u is a L_{n-1}^f-equivalence, and $L_n^f u\langle d\rangle$ therefore has fiber $L_n^f F\langle d\rangle \simeq F\langle e\rangle$ contractible. This proves that F is e-truncated. Since u is a v_0-periodic equivalence, $F\langle d\rangle$ is rationally trivial, hence has p-power torsion homotopy groups. But then $F\langle d\rangle$ is in the essential image of $P_A : L_n^f \mathcal{S}_*^{\langle e\rangle} \to L_n^f \mathcal{S}_*^{\langle d\rangle}$ by theorem 3.27. In particular, $F\langle d\rangle$ is the image of $F\langle e\rangle \simeq *$, so that F is d-truncated. Being the fiber of a map of d-connected spaces, it is furthermore $(d-1)$-connected, so an Eilenberg-MacLane space $K(G,\ d)$. Since F is rationally trivial and p-local, G is p-power torsion and F is P_A-acyclic by lemma 3.18. This proves the theorem. □

We can finally prove:

Corollary 3.32 (theorem 3.24, $\mathcal{S}_*^{v_n}$ case)**.** *There exist localization functors M_n^f presenting $\mathcal{S}_*^{v_n}$ as the localization of $\mathcal{S}_*$ at the v_n-periodic equivalences.*

Proof. Explicitly take

$$M_n^f(X) = \mathrm{hof}\left(L_n^f X \to L_{n=n+1}^f X\right)\langle d\rangle.$$

We have a few things to check. First, we must show that this is a v_n-periodic equivalence. Second, we must check that the image lies in $\mathcal{S}_*^{v_n}$. Third, we must prove that $M_n^f f$ is an equivalence if and only if f is a v_n-periodic equivalence. The first claim follows from:

Lemma 3.33. *Let X be P_A-local. Then X has no v_m-periodic homotopy for $m > n$.*

Proof. We must check that $\Phi_V(X) \simeq *$ for all, equivalently some, finite type m space V. We may suspend V enough that V is a suspension with greater connectivity than A, say with v_m-self map $v : \Sigma^t V \to V$; then $\langle V\rangle \leq \langle A\rangle$ by Bousfield's theorem and so X is P_V-local (and hence $P_{\Sigma^{kt}V}$-local for any k). Then, $\Phi_V(X) = \mathrm{hocolim}\ \mathrm{Map}_*\left(\Sigma^{kt}V,\ X\right) \simeq \mathrm{hocolim}\ *$ is trivial, as desired. □

As such, P_B kills all v_n-periodic homotopy and thus M_n^f preserves v_n-periodic homotopy, proving the first claim. The second is not hard, but tedious, so we omit it. Finally, for the third, note that P_B preserves v_m-periodic homotopy for $0 \leq m < n$, and so P_B-acyclic spaces such as those of $\mathcal{S}_*^{v_n}$ have trivial v_m-periodic homotopy for $0 \leq m < n$. Together with theorem 3.31, this proves the third claim and thus the theorem. □

4. Rapid-fire Tour of Goodwillie Calculus and Heuts' Theorem

4.1. **Goodwillie Calculus.** We now turn to some discussion of [8]. We give no rigorous proofs, and similarly minimize the proof sketches. The result is a summary of Goodwillie calculus, which is a deep subject of current research programs; we mention two (those of Arone and Ching and of Heuts) which have successfully been applied to chromatic homotopy theory, and give a sketch of the connection.

The core idea of Goodwillie calculus is the following: we fix a notion of n-polynomial (topological) functor between ∞-categories with some reasonable characteristics similar to polynomial functions (of degree n) in the ordinary sense. Choosing different definitions results in slightly different theories, e.g. manifold calculus, homotopy calculus (the one we focus on), and orthogonal calculus (c.f. [23, §7]).

Given such a notion, it makes sense to consider the following:

Definition 4.1. Let $F : \mathcal{C} \to \mathcal{D}$ be any (topological) functor. The universal n-polynomial approximation of F near X, denoted $F \to P_n^X F$, (morally speaking, the degree n Taylor polynomial near X), is the initial object among maps from F to an n-polynomial functor $\mathcal{C}/X \to \mathcal{D}$ (where $\mathcal{C}/X$ is the category over X, i.e. the category of maps in $\mathcal{C}$ with codomain X).

In the "dual" calculus, the universal n-polynomial approximation, denoted instead $P_X^n F$, is the terminal object among maps to F from an n-polynomial functor.

Remark 4.2. The polynomial approximations might not exist, but it turns out that they will if $\mathcal{C}$ and $\mathcal{D}$ are sufficiently complete and cocomplete, so we don't need to worry too much.

For any $X \in \mathcal{C}$ and $F : \mathcal{C} \to \mathcal{D}$, this gives a tower of functors $\ldots \to P_n^X F \to P_{n-1}^X F \to \ldots$ in $\mathrm{Fun}(\mathcal{C}/X,\ \mathcal{D})$, which we think of as the Taylor series of F at X. As in ordinary calculus, arguably the most important question is when the tower converges, i.e. has homotopy limit F. A refinement of this question is the following:

Question 4.3. *For what objects $(Y,\ f : Y \to X) \in \mathcal{C}/X$ does the series converge? That is, when is the canonical map*

$$F(Y) \to \operatorname{holim} P_n^X F(Y)$$

is an equivalence?

In order for the theory to be much use, this should happen whenever $f : Y \to X$ is nice. We note the following simplification: let $\pi : \mathcal{C}/X \to \mathcal{C}$ be the forgetful functor, and let $F^X = F \circ \pi$. Then, if $\mathcal{C}$ admits a terminal object $*$,

$$P_n^X F \simeq P_n^* F^X,$$

and so it is safe to restrict our attention to $P_n F = P_n^* F$.

In reasonable circumstances, we also have a well-behaved notion of homogeneous polynomials:

Definition 4.4. A homogeneous polynomial functor of degree n is an n-polynomial functor F with $P_{n-1}^X F$ trivial. The n^{th} homogeneous part of F is the homotopy fiber D_n of $P_n^X F \to P_{n-1}^X F$. In good circumstances, these D_n are given by some explicit formula of the derivatives of F, denoted $\partial_* F$.

Goodwillie set up this machinery in the case now known as the homotopy calculus, and proved that the theory is reasonable and reasonably understandable. In this theory, the meaning of n-polynomial is taken to be n-excisive. There are several good expositions which we recommend to the curious reader, e.g. [24], [23], and the original papers ([25], [26], and [27]) are certainly readable. We begin by defining n-excisive functors.

Definition 4.5. A functor is said to be n-excisive if it satisfies a "n-dimensional" excision-type axiom. Precisely, a functor is n-excisive if it sends strongly coCartesian n-cubes (cubes whose 2-faces are coCartesian) to Cartesian n-cubes.

Remark 4.6. For instance, 1-excisive functors $F : \mathcal{S}_* \to \mathcal{S}_*$ are roughly generalized homology theories; indeed, any 1-excisive F which is reduced ($F(*) \simeq *$) and finitary (preserves filtered colimits) is necessarily of the form $\Omega^\infty(E \wedge -)$ for some spectrum E, by a version of Brown representability.

Indeed, one model of the category of spectra (of an arbitrary ∞-category) is that of the 1-excisive functors for this reason, so that stabilization is almost literally linearization.

Definition 4.7. Similarly to remark 4.6, an n-homogeneous functor F is always of the form $\Omega^\infty\big(E \wedge (-)^{\wedge n}\big)_{h\Sigma n}$, where E is a spectrum with a Σ_n action and $(-)^{\wedge n}$ has the obvious Σ_n action permuting the coordinates. As suggested by the previous paragraph, E is then $\partial_n F$. More generally, we set $\partial_n F = \partial_n D_n F$.

Remark 4.8. A remarkable feature of the homotopy calculus is that the identity is not necessarily linear. Indeed, the identity functor $\mathbb{1} : \mathcal{S}p \to \mathcal{S}p$ is linear, but $\mathbb{1} : \mathcal{S}_* \to \mathcal{S}_*$ is not. As suggested by the naming, this has to do with the failure of homotopy excision to hold in every degree.

In a remarkable paper of Johnson ([28]), the derivatives of $\mathbb{1} : \mathcal{S}_* \to \mathcal{S}_*$ were computed explicitly; since we will not need the derivatives of identity functors other than this one (indeed, we only care about $\mathcal{S}_*$, $\mathcal{S}p$, and their localizations), we will use $\partial_k \mathbb{1}$ only to refer to the case of $\mathbb{1} : \mathcal{S}_* \to \mathcal{S}_*$.

Perhaps one of the simplest questions, and one which is remarkably subtle, is the following. We say that F is *coanalytic* if it is given by

$$X \mapsto F_{\mathcal{O}}(X) = \bigvee_k \left(\mathcal{O}(k) \wedge X^{\wedge n}\right)_{h\Sigma_k}$$

for $\mathcal{O}(n)$ a sequence of spectra with Σ_n-actions, also known as a *symmetric sequence*. In the case that X and $F(X)$ are supposed to be L-local, we mean the L-local orbit space, local smash product, and so on; explicitly, these are given from the usual orbit space and smash products by applying L.

Theorem 4.9. *One might naïvely hope that $F_{\mathcal{O}}$ has derivatives $\partial_k F_{\mathcal{O}} \simeq \mathcal{O}(k)$. This is not true in general (consider $\mathbb{1}$), but is true of coanalytic functors $\mathcal{S}p_{T(n)} \to \mathcal{S}p_{T(n)}$.*

This is proven via some clever reasoning and the telescopic Tate vanishing, so we do not even sketch the proof. One can show from this that $(\mathcal{O}(n)) \mapsto F_{\mathcal{O}}$ is fully faithful as a functor from the category of symmetric sequences to that of functors $\mathcal{S}p_{T(n)} \to \mathcal{S}p_{T(n)}$.

4.2. **Heuts' Theorem on Lie Algebras.** Recall that we proved that $\Theta \dashv \Phi$ is monadic, but that our proof was overkill. Specifically, we showed that Φ, and hence $\Phi\Theta$, not only preserves colimits of split simplicial objects, but in fact preserves all geometric realizations. One can check that $\Phi\Theta$ preserves filtered colimits as well. It is a general fact that any functor preserving geometric realizations and filtered colimits also preserves sifted colimits. Furthermore, a functor $\mathcal{S}p_{T(n)} \to \mathcal{S}p_{T(n)}$ is coanalytic if and only if it preserves sifted colimits. Thus, $\Phi\Theta$ is coanalytic, given by $F_{\mathcal{O}}$ for some $\mathcal{O}$.

When analyzing the relation between $\mathcal{S}p_{T(n)}$ and $\mathcal{S}_*^{v_n}$, there is an adjunction which is much more obvious than $\Theta \dashv \Phi$, namely $\Sigma^\infty_{T(n)} \dashv \Omega^\infty_{T(n)}$. It is not obvious, however, if this is (co)monadic, or even if this is related to $\Theta \dashv \Phi$. That said, since Φ is supposed to "undo" Ω^∞, it is plausible that there should be a link. As it happens, Kuhn proved ([29]) that the comonad $\Sigma^\infty_{T(n)}\Omega^\infty_{T(n)}$ is also coanalytic, given by $F_{\mathcal{O}'}$ for $\mathcal{O}'(k) = S$ (with the trivial action). That is,

$$\Sigma^\infty_{T(n)}\Omega^\infty_{T(n)} X \simeq \bigvee_k \left(X^{\wedge k}\right)_{h\Sigma_k}.$$

This functor can be thought of as the exponential map from a Lie algebra to its Lie group. Indeed:

Theorem 4.10 ([8, thm. 5.1])**.** $\Sigma^\infty_{T(n)}$ *gives an equivalence in* $\mathcal{S}p_{T(n)}$ *between* $\mathcal{P}_k\mathcal{S}_*^{v_n}$ *and the k-truncated commutative ind-coalgebras, i.e. commutative coalgebras only admitting the structure maps* $E \to \left(E^{\wedge j}\right)^{h\Sigma_j}$ *for* $j \leq k$ *(satisfying those usual relations which still make sense).*

Here, the tower

$$\ldots \to \mathcal{P}_k\mathcal{C} \to \mathcal{P}_k\mathcal{C} \ldots$$

for an ∞-category $\mathcal{C}$ is a variant of the Goodwillie tower such that the identity on $\mathcal{P}_k\mathcal{C}$ is k-excisive, introduced (in [30]) in order to study what ∞-categories have the same stabilization (which turns out to be $\mathcal{P}_1$). Unfortunately, this does not converge properly, so there is no untruncated version. That said, we can use this interpretation as a first step towards Heuts' theorem: the functor sending a spectrum to its trivial coalgebra has a right adjoint sending a coalgebra to its primitives, and this right adjoint turns out to be formally dual to something called topological André-Quillen homology (or TAQ for short). These constructions naturally pass to k-truncated versions. As a formal consequence of these truncated results, Heuts shows:

Corollary 4.11. *There exist natural equivalences from* $P_k\Phi(X)$ *to the primitives of the k-truncated part of the coalgebra* $\Sigma^\infty_{T(n)}X$*.*

This particular result works $K(n)$-locally and dualizes to TAQ and is then a theorem originally due to Behrens and Rezk (at least in the $K(n)$ setting, c.f. [9]), namely that the map from $\Phi(X)$ to the primitives of the coalgebra $\Sigma^\infty_{T(n)}X$ is an equivalence if and only if $P_n\Phi$ converges at X. Unfortunately, this does not happen with all X, and the class of such is difficult to understand since it does not even include wedges of spheres.

Meanwhile, the theory of symmetric sequences is naturally connected to the theory of operads (c.f. [31]). More precisely, one can check that the composite of coanalytic functors is coanalytic, and this defines a monoidal structure (the "composition product") on symmetric sequences such that monoids correspond to (nonunital) operads; in particular, coanlytic monads are equivalent to operads, and one can check that the categories of algebras of the monad and operad agree. Hence, we would like to identify the symmetric sequence E defining $\Phi\Theta$ as an operad.

One of the primary motivations for the theory of operads of spectra, as developed by Ching ([31, 32]) is the cobar construction for derivatives of $\Sigma^\infty\Omega^\infty$, which turns out to be the operad $(\partial_k\mathbb{1})$, and one can show that $(\partial_k\Sigma^\infty\Omega^\infty)$ represents the commutative coalgebra cooperad. Now, Koszul duality extends to operads as a duality between bar and cobar constructions ([32]), and so the "derivatives of the identity" operad is the Koszul dual of the commutative coalgebra cooperad; a standard result then identifies this operad as the Lie algebra operad. Hence, Heuts' theorem ([8]) is essentially contained in the explicit calculation that

$$\mathcal{O}(k) \simeq \partial_k\mathbb{1}$$

with the same monoidal structure as before (recall that we defined the $\mathcal{O}(k)$ by $F_{\mathcal{O}} \simeq \Phi\Theta$). Recall, furthermore, that we claimed $\mathcal{O}(k) \simeq \partial_k F_{\mathcal{O}}$. Hence, Heuts' theorem reduces essentially to the problem of computing $\partial_k\Phi$.

As a closing remark, one should keep in mind that Heuts' theorem has practical computational applications for v_n-periodic homotopy. For instance, Heuts uses his theorem to show:

Theorem 4.12. *There is an equivalence*

$$\Phi(W) \simeq \bigvee_k \left(\partial_k\mathbb{1} \wedge \Sigma^\infty W^{\wedge k}\right)_{h\Sigma_k}$$

provided W is a double suspension of a type n space admitting a v_n-self map. In this light, one can also understand $\Sigma^\infty_{T(n)}$ as calculating something called topological Quillen homology, which has a canonical filtration whose associated graded (nonunital) algebra turns out to be

$$\operatorname{Sym}^* \Phi(X) = \bigvee_k (\Phi X)^{\wedge k}_{h\Sigma_k}.$$

We do not have the space to explain the proof of Heuts' theorem, nor even these consequences, as they require understanding the explicit computation of $\partial_k\mathbb{1}$, so we simply recommend that the reader read [8], or an exposition such as [11] or [13].

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