AN INTRODUCTORY OVERVIEW TO CHARACTERISTIC CLASSES

JACOB LAXER

Abstract. In this paper we present the basic theory of characteristic classes. We will assume knowledge of homology and cohomology and will gloss over basic theory of smooth manifolds. While extensive theoretical work is needed to treat the topic rigorously, we seek to demonstrate why characteristic classes are interesting what they measure, and present some of the more elegant computational results. Any theorem that is presented without proof can be found in *Characteristic Classes*, by J. W. Milnor.

Contents

1. Introduction: Smooth Manifolds and Real Vector Bundles

We begin this paper with a brief refresher on manifolds and then introduce definitions of vector bundles and cross-sections. Generally, smoothness of both the manifolds and vector bundles under investigation is not required, but given that tangent and normal bundles are both amongst the most interesting and require smoothness, smoothness will sometimes be casually assumed. Moreover, the finer details of point set topology will generally be omitted, but all the spaces we deal with in this paper can be assumed to be Hausdorff and Normal when needed and all maps should be assumed to be continuous unless stated otherwise.

Definition 1.1. A function $f : U \to \mathbb{R}$ is said to be smooth if it is infinitely differentiable.

Definition 1.2. For $U \subset \mathbb{R}^n$, a function $f : U \to \mathbb{R}^{\mathbb{A}}$ is smooth if each $f_\alpha : U \to \mathbb{R}$ is smooth for all $\alpha \in A$.

Definition 1.3. A subset M of $\mathbb{R}^A$ is a smooth n dimensional manifold if for every $x \in M$ there is a smooth function h that maps an open set $U \subset \mathbb{R}^n$ to M such that h satisfies the following conditions:

Date: September 3rd.

(1) the function h maps U homeomorphically into an open subset V of M which contains x;
(2) for each $u \in U$, the matrix of partials of h at u has rank n.

This sort of notion of local 'rectifiablity' is one which re-appears in this paper: first with the local triviality condition on vector bundles, and again with the local compatibility of orientation of bundles. Central to most topics in algebraic topology is an investigation of how locally simple and well-behaved objects can have global or non-trivial defects. This paper will specifically investigate non-triviality of vector bundles as measured by characteristic classes.

Remark 1.4. For the purposes of this paper we will only be considering connected manifolds with constant dimension. Each of the disconnected components of a non-connected manifold can simply be treated as distinct connected manifolds.

Examples 1.5. The n-sphere S^n, n-torus $(S^1)^n$, and projective n-space $\mathbb{P}^n$ are all n-dimensional smooth manifolds.

The primary objects of study in this paper, characteristic classes, are designed to measure and characterize vector bundles on manifolds. Therefore, an examination of vector bundles themselves is called for.

Definition 1.6. Without further ado, a (real) vector bundle ζ over a manifold B, (also referred to as the base space), consists of the following components:
(1) a topological space $E = E(\zeta)$ known as the total space;
(2) a map $\pi : E \to B$, known as the projection map, which is continuous with respect to the topology of E and B;
(3) for each $b \in B$, a (real) vector space structure on the preimage $\pi^{-1}(b)$.

In addition, the vector bundle ζ must satisfy the condition of local triviality, defined below.

Definition 1.7. A bundle is locally trivial if, for every $b \in B$, there is an open neighborhood $U \subset B$, a non-negative integer n, and a homeomorphism,

$$h : U \times R^n \to \pi^{-1}(U),$$

such that for each $b \in U$, the function that takes x to $h(b, x)$ is an isomorphism between the vector space R^n and the preimage of b. If U can be chosen to be the entire base space then the vector bundle is called a trivial bundle. Naturally, all trivial bundles of the same fiber dimension over the same base space are homeomorphic. In this paper, we will omit proofs that vector bundles satisfy local triviality for the sake of brevity, but proving local triviality once or twice may serve as a useful exercise to the reader.

Just as a manifold must locally look like real space, a vector bundle must locally look like a real vector field on the manifold. Though we have introduced a preliminary method of determining whether a bundle is trivial, we will develop better techniques to determine triviality.

In addition, smooth vector bundle requires that B and E be smooth, that the projection map π is smooth, and that the function h is a diffeomorphism.

Definition 1.8. For a given $b \in B$, the fiber over b denoted $F_b(\zeta)$, is the set $\pi^{-1}(b)$ and has the structure of an n dimensional vector space. For all cases that we will consider, the fiber dimension n will be constant as a function over B. In this case we refer to ζ as an n-plane bundle. A 1-plan bundle we refer to as a line bundle.

Definition 1.9. Two vector bundles over the same base space are isomorphic if there is a function f which is a homeomorphism of the total spaces and carries the fiber over each point isomorphically.

Example 1.10. Two examples: the tangent bundle of a manifold M, denoted τ_M, consists of all pairs (x, v), such that $x \in M$ and v is tangent to M at x. The projection map would take (x, v) to x. Further, the tangent vectors at each point satisfy scalar multiplication as every tangent vector is a derivative of a path through x, so one can simply parameterize the path through x at a scalar speed.

A more obvious example of a vector bundle over a manifold M is the trivial n-bundle $M \times \mathbb{R}^n$ with projection map π that takes (a, b) to a for any $a \in M$ and any $b \in \mathbb{R}^n$. The vector space structure over each fiber is the canonical vector space structure of $\mathbb{R}^n$:

$$\alpha(a, b) + \beta(a, c) = (a, \alpha a + \beta c) \in \pi^{-1}(a)$$

Definition 1.11. If the tangent bundle τ_M of a manifold M is trivial, then M is *parallelizable.*

Example 1.12. An open set $U \subset \mathbb{R}^n$ is a parallelizable manifold since the tangent manifold is exactly $U \times \mathbb{R}^n$.

Example 1.13. As a result of the hairy ball theorem, $S^2 \subset \mathbb{R}^3$ is not parallelizable.

Definition 1.14. The projective space $\mathbb{P}^n$ is the quotient space of the spheres S^n. Specifically, for every n, $\mathbb{P}^n$ can be constructed as the set of all unordered pairs $\{x, -x\} \in S^n$ topologized as a quotient space of S^n.

Definition 1.15. Further, we will define a canonical line bundle $E(\gamma_n^1) \subset \mathbb{P}^n \times \mathbb{R}^{n+1}$ over $\mathbb{P}^n$ as follows: it will consist of all pairs $(\{\pm x\}, v)$ where $\{\pm x\} \in \mathbb{P}^n$ and $v = \lambda x$ for some $\lambda \in \mathbb{R}$. Here, the vector space structure is the usual one arising from $\mathbb{R}$.

Local triviality of this bundle and all of the previous examples is easy to verify. In the case of the canonical line bundle, the required homeomorphism can be constructed by choosing a sufficiently small open set U such that we can unambiguously make a choice for $x \in \{\pm(x)\}$ that is continuous with respect to the topology of $\pi^{-1}(U)$ in S^n. Then, the required homeomorphism is

$$h : U \times \mathbb{R} \to \pi^{-1}(U) \tag{1.16}$$

$$h(\{\pm(x)\}, \lambda) = (\{\pm(x)\}, \lambda x) \tag{1.17}$$

Here we see that our homeomorphism is dependent on a sufficiently small choice of open set. This is our first hint that γ_n^1 will not be globally trivial. However, we will hold off on proving its non-triviality until the notion of cross-section has been introduced.

Definition 1.18. A *cross-section*, often referred to simply as a section, of a vector bundle ζ over a manifold B is a continuous function

$$s : B \to E(\zeta)$$

such that for each $b \in B$, $s(b)$ is in the fiber over b, $F_b(\zeta)$. In particular, we are interested in cross section which are nowhere zero. A cross section is nowhere zero if and only if for all $b \in B$, $s(b)$ is not the zero vector of $F_b(\zeta)$. A cross section of the tangent bundle is often called a vector field over the base space.

Remark 1.19. A trivial n-plane bundle $B \times \mathbb{R}^n$ over any manifold admits a nowhere-zero cross section given by:

$$s(b) = (b, (1, 0..., 0))$$

Theorem 1.20. *The canonical bundle over projective n-space, $\mathbb{P}^n$ has no nowhere zero section.*

Proof. Suppose that $s : \mathbb{P}^n \to E(\gamma_n^1)$ is a nowhere zero cross section. Let $f : S^n \to \mathbb{P}^n$ be the quotient map $f(x) = \{\pm x\}$. As $f(x) = f(-x)$ the composition $s(f(x)) = s(f(-x))$.

We can also write $s(f(x)) = (\{\pm x\}, g(x)x)$ where $g(x) \in \mathbb{R}$ since the fiber vectors over x are multiples of x. So,

$$s(f(x)) = s(f(-x))$$

$$\implies (\{\pm x\}, g(x)x) = (\{\pm x\}, g(-x)(-x))$$

Therefore $g(x) = -g(-x)$, and by the intermediate value theorem, given the continuity of all of our maps and connectedness of S^n, there must be some point x^* on the sphere such that $g(x^*) = 0$. Applying the quotient map we get $s(f(x^*)) = (\{\pm x^*\}, g(x^*)x^*) = (\{\pm x^*\}, 0)$ contradicting our assumption that the section was nowhere zero. □

Definition 1.21. Cross-sections $s_1, s_2, ..., s_j$ of a bundle ζ over a manifold B are nowhere dependent if for every $b \in B$ the vectors $s_i(b)$ for $1 \leq i \leq j$ are linearly independent with respect to the vector space structure on the fiber over b. We can easily construct n nowhere dependent sections of the trivial n-plane bundle by having the i^{th} section take each element of the base space to the positive unit vector in the i^{th} coordinate.

Cross-sections will serve as a useful way to quantify the degrees of triviality of a bundle in light of the next few theorems.

Definition 1.22. Our next important notion is that of a *bundle map*, which is a morphism in the category of bundles. Given two n-plane bundles ζ and γ with base spaces B and M respectively, a bundle map is a continuous map f of total spaces $f : E(\zeta) \to E(\gamma)$ that is an isomorphism of vector spaces on each when restricted to each fiber, and satisfies the following commutative diagram:

$$\begin{array}{ccc} E(\zeta) & \xrightarrow{f} & E(\gamma) \\ \downarrow{\scriptstyle \pi_\zeta} & & \downarrow{\scriptstyle \pi_\gamma} \\ B & \xrightarrow[\hat{f}]{} & M \end{array}$$

where $\hat{f}$ is the map of base spaces induced by f.

Example 1.23. The map $f : M \times \mathbb{R}^n \to \mathbb{R}^n$ which takes $(x, v) \to (v)$ is a bundle map and takes the trivial bundle over M to the trivial bundle over a single point, the origin in R^n.

Example 1.24. The inclusion $i : \gamma_n^1 \to \gamma_{n+k}^1$ is a bundle map.

Definition 1.25. A *Whitney Sum* of two bundles ζ and γ over the same base, written $\zeta \oplus \gamma$, is a new vector bundle with fibers consisting of all elements of the form $(b, \alpha v_\zeta, \beta v_\gamma)$ where α and β are both scalars and v_ζ and v_γ are any elements of the fiber over b from ζ and γ respectively.

Definition 1.26. If a bundle ζ over base space B can be written as a subset of a larger bundle ν over the same base space such that the inclusion of ζ in ν respects both projection and the vector space structure on the fibers, then ζ is said to be a *sub-bundle* of ν. If we impose the requirement that ν have a continuous function which restricts to a Euclidean metric on each fiber, then we can take the orthogonal complement of any sub-bundle ζ. This is denoted $\zeta^\perp$ and satisfies:

$$\nu = \zeta \oplus \zeta^\perp$$

Theorem 1.27. *An R^n bundle ζ over M is trivial if and only if there exist n nowhere dependent cross sections.*

Proof. The forward direction is easier. If $E(\zeta)$ is trivial, it is homeomorphic to $M \times \mathbb{R}^n$. We can then construct n sections of $M \times \mathbb{R}^n$ as follows. Take $s_i(b) = (b, (0, 0..., 1, 0..., 0))$ where the 1 is in the i_{th} vector coordinate. These sections are linearly independent at each point and passing the sections back through the homemorphism to $E(\zeta)$ will preserve this since it carries each fiber isomorphically. So, there exist n nowhere dependent cross sections.

For the reverse direction, we can use the sections $s_i(b)$ to construct a basis for $\mathbb{R}^n$. First choose n nowhere dependent cross-sections: s_i for $1 \leq i \leq n$. For each b in the base space, notice that every vector v in fiber over b can be written as a unique linear combination $v = \sum_{i=1}^n \alpha_i s_i(b)$. It follows that a function that takes each $(b, v) \in E(\zeta)$ to $(b, (\alpha_1, \alpha_2..., \alpha_n)) \in \mathbb{R}^n$ will be an isomorphism of vector spaces on each fiber, and therefore defines the desired homeomorphism from $E(\zeta)$ to $B \times \mathbb{R}^n$. We conclude that ζ must be trivial. □

Theorem 1.28. *If an n-plane bundle ζ with a Euclidean metric admits k nowhere dependent nonzero sections, then ζ splits as Whitney sum into a trivial sub-bundle of degree k and its orthogonal complement.*

Proof. Let s_i for $1 \leq i \leq k$ be nowhere dependent nonzero sections of ζ. Taking the linear span of $s_i(b)_{1 \leq i \leq k}$ at each point will generate the needed k dimensional sub-bundle, and the sub-bundle is certainly trivial since it admits the same k sections from which it is spanned. It now follows that we can take its orthogonal complement. □

This is what the previous comment about sections quantifying degrees of non-triviality was referring to. Since trivial bundles are fairly simple, sections allow us to zoom in to the interesting parts of a bundle by ignoring the trivial parts as indicated by sections.

2. Stiefel Whitney Classes

We will now turn our attention to Stiefel-Whitney classes. Stiefel-Whitney classes, roughly speaking, can help identify which cohomology classes of a manifold obstruct the existence of linearly independent cross sections of a vector bundle. While the Stiefel-Whitney classes have historically been defined in various ways, the axiomatic definition below is fairly standard and all definitions should yield the same results.

Until stated otherwise, it will be assumed that all cohomology groups will be calculated with $\mathbb{Z}/2$ coefficients.

Definition 2.1. The following axioms define the Stiefel-Whitney (SW) classes:

Axiom 1: For each vector bundle ζ over a base space B there exists a corresponding sequence of cohomology classes:

$$w_i(\zeta) \in H^i(B, \mathbb{Z}/2), \text{ for all } i \in \mathbb{N} \tag{2.2}$$

called the SW classes of ζ. Further, $w_0(\zeta) = 1$ where 1 here denotes the unit element in the zeroth cohomology group and $w_i(\zeta) = 0$ whenever i is greater than the bundle dimension.

Axiom 2: *(Naturality)* If $f : B(\zeta) \to B(\nu)$ is covered by a bundle map from ζ to ν, then $w_i(\zeta) = f^* w_i(\nu)$. In short, the SW classes play well with bundle maps.

Axiom 3: (*The Whitney Product Theorem*) If ζ, ν are vector bundles over the same base space then:

$$w_k(\zeta \oplus \nu) = \sum_{i=0}^{k} w_i(\zeta) \smile w_{k-i}(\nu) \tag{2.3}$$

Here $\smile$ denotes the cup product operation. From this point forward, whenever two cohomology classes are written consecutively, for example $w_i(\zeta)w_{k-i}(\nu)$, the cup product operation should be assumed.

Axiom 4: For the twisted line bundle γ_1^1 over the circle P^1, the SW class $w_1(\gamma_1^1)$ is not zero. (Since we are working with mod 2 coefficients, we can now describe all the SW classes of γ_1^1 since they all follow immediately from the axioms). The zeroth class dictated by axiom 1 is $w_0(\gamma_1^1) = 1$. The 1st class is given by the 4th axiom $w_1(\gamma_1^1) = a$, where a is the unique nonzero element in the first cohomology group

We will black box the existence and uniqueness of these classes until we have developed a bit more of the theory. For now we will present some immediate results from the axioms.

Remark 2.4. If ζ is isomorphic to ν, then from axiom 2 $w_i(\zeta) = w_i(\nu)$.

Remark 2.5. If ζ is a trivial bundle then all of its SW classes will be zero, (aside from $w_0(\zeta)$). This follows from axiom 2 as well, because we can easily construct a bundle map from a trivial bundle to a bundle over a point.

Remark 2.6. If ϵ is a trivial bundle then $w_i(\zeta \oplus \epsilon) = w_i(\zeta)$. This follows from applying the previous remark using the Whitney Product axiom.

Remark 2.7. If ζ is a bundle of dimension n that admits a Euclidean metric and k nowhere dependent sections, then $w_i(\zeta) = 0$ if $i \geq n - k + 1$.

Proof. The proof of this remark relies on the fact that the Euclidean metric allows us to split ζ into the trivial sub-bundle spanned by the sections and its orthogonal complement. We then apply the above theorem and axiom 1 to derive the result. ☐

Definition 2.8. For any topological space B, we will let $H^*(B; \mathbb{Z}/2)$ be the graded cohomology ring $\bigoplus_{i \in \mathbb{N}} H^i(B; \mathbb{Z}/2)$ with elements:

$$a = a_0 + a_1 + a_2 + a_3 ..., \text{ for } a_i \in H^i(B; \mathbb{Z}/2) \tag{2.9}$$

equipped with the sum operation inherited from the sum operation in each cohomology group

$$a + b = (a_0 + b_0) + (a_1 + b_1)... \tag{2.10}$$

and the commutative and associative product operation derived from the cup product

$$ab = \sum_{i=0}^{\infty} \sum_{k+j=i} a_j b_k = (a_0 b_0) + (a_1 b_0 + a_0 b_1) + (a_2 b_0 + a_1 b_1 + a_0 b_2)... \tag{2.11}$$

Definition 2.12. This leads to the natural definition of the total SW class

$$w(\zeta) \in H^*(B; \mathbb{Z}/2) = \sum_{i=0}^{\infty} w_i(\zeta)$$

and the observation (using axiom 3) that $w(\zeta)w(\nu) = w(\zeta \oplus \nu)$.

The following application of the Whitney Product axiom will be a particular powerful tool for computation in some of the most interesting bundles. If the Whitney sum $\zeta \oplus \nu$ is trivial (tangent and normal bundles can form such a pair), then $w(\zeta \oplus \nu) = 1$. Therefore, for all $i \neq 0$:

$$\sum_{j+k=i} w_j(\zeta) w_k(\nu) = 0$$

$$\implies w_1(\zeta) + w_1(\nu) = 0, w_2(\zeta) + w_1(\zeta)w_1(\nu) + w_2(\nu) = 0...$$

This allows us to inductively solve for the classes of ζ as polynomials in the classes of ν.

The set of all $a \in H^*(B; \mathbb{Z}/2)$ such that $a_0 = 1$ forms a commutative ring under multiplication. The inverse elements $\bar{a}$ can be computed inductively as above with the first few terms being $\bar{a_1} = a_1$, $\bar{a_2} = a_1^2 + a_2$, $\bar{a_3} = a_1^3 + a_3$ and so on. We immediately get that if $\zeta \oplus \nu$ is trivial then $w(\nu) = \overline{w}(\zeta)$

Theorem 2.13. *Whitney Duality Theorem: If τ and ν are the tangent and normal bundles respectively of a manifold in Euclidean space then*

$$w(\tau) = \overline{w}(\nu) \tag{2.14}$$

Theorem 2.15. *The cohomology group $H^i(\mathbb{P}^n; Z/2)$ is isomorphic to $Z/2$ for $0 \leq i \leq n$ and is zero otherwise. Let a be the nonzero element $a \in H^1(\mathbb{P}^n; Z/2)$, then $H^i(\mathbb{P}^n; Z/2)$ is generated by a^i, where the exponent denotes iterated cup product.*

Example 2.16. Applying Axiom 2, the bundle map from γ_1^1 to γ_n^1 that covers inclusion $i : \mathbb{P}^1 \to \mathbb{P}^n$ allows us to determine that

$$w_1(\gamma_1^1) = i^*(w_1(\gamma_n^1)) \neq 0 \implies w(\gamma_n^1) = 1 + a$$

Taking γ_n^1 as a sub-bundle of ϵ^{n+1} let $\gamma^\perp$ be its orthogonal complement. We know the total class $w(\epsilon^{n+1}) = 1$ so we can compute $1 = w(\gamma^\perp)w(\gamma_n^1) = (w(\gamma^\perp))(1 + a)$.

Solving inductively, knowing that $w_0(\gamma^\perp) = 1$, then $a1 + 1w_1(\gamma^\perp) = w_1(\epsilon^{n+1}) \implies a = w_1(\gamma^\perp)$ (given mod2 coefficients).

Similarly $a^2 + 1w_2(\gamma^\perp) = 0 \implies a^2 = w_2(\gamma^\perp)$ and so on such that the total class $w(\gamma^\perp) = \sum_{i=0}^{n} a^i$

Theorem 2.17. *Letting τ be the tangent bundle of $\mathbb{P}^n$, the total Stiefel Whitney class is $w(\tau) = (1 + a)^{n+1}$.*

Proof. In order to prove this theorem we require the following lemma:

Lemma 2.18. *The canonical bundle γ_n^1 satisfies $\tau \approx Hom(\gamma_n^1, \gamma^\perp)$.*

A sketch of the proof follows: Consider the quotient map f from S^n to $\mathbb{P}^n$. The induced map on tangent manifolds identifies all pairs $(x, v), (-x, -v)$, where $x \in S^n, v \in R^{n+1}$ such that $x \cdot x = 1$ and $x \cdot v = 0$.

The space of all such pairs for a given $\pm x$ is isomorphic to the set of linear transformations from span(x) to span$(x)^\perp$ under the identification $((x, v), (-x, -v)) \to l_{x,v}$ where $l_{x,v}$ is the unique linear function for which $l(x) = v$. But the set of linear functions from span(x) to span$(x)^\perp$ for each $\pm x \in \mathbb{P}^n$ is exactly $Hom(\gamma_n^1, \gamma^\perp)$. □

Proof, continued. $Hom(y_n^1, y_n^1) \approx \epsilon^1$ since it has a nowhere zero section given by the identity homomorphism. From linear algebra, we have the identity

$$\tau \oplus \epsilon^1 \approx Hom(\gamma_n^1, \gamma^\perp) \oplus Hom(\gamma_n^1, \gamma_n^1) \approx Hom(\gamma_n^1, \gamma^\perp \oplus \gamma_n^1) \approx Hom(y_n^1, \epsilon^{n+1})$$

We can then split ϵ^{n+1} as an $n+1$ fold direct sum $\epsilon^1 \oplus \epsilon^1 \oplus ... \oplus \epsilon^1$. So:

$$Hom(\gamma_n^1, \epsilon^{n+1}) \approx Hom(\gamma_n^1, \epsilon^1 \oplus \epsilon^1 ... \oplus \epsilon^1) \approx Hom(\gamma_n^1, \epsilon^1) \oplus Hom(\gamma_n^1, \epsilon^1) \oplus ... \oplus Hom(\gamma_n^1, \epsilon^1)$$

Seeing as $Hom(y_n^1, \epsilon^1) \approx y_n^1$, we now have that $\tau \oplus \epsilon^1$ is isomorphic to an $n+1$ fold whitney sum of y_n^1. It follows from axiom 2 that

$$w(\gamma_n^1 \oplus \gamma_n^1 \oplus ... \oplus \gamma_n^1) = w(\tau \oplus \epsilon^1) = w(\tau)$$

Finally, an application of the Whitney Product formula gives

$$w(\tau) = w(\gamma_n^1 \oplus \gamma_n^1 \oplus ... \oplus \gamma_n^1) = (w(y_n^1))^{n+1} = (1 + a)^{n+1}$$

□

Keeping in mind that all coefficients here are mod 2, this formula allows us to easily calculate all SW classes of $\mathbb{P}^n$.

Remark 2.19. In fact, the following result will hold true for immersions as well, an immersion being a function from an n dimensional manifold M, into another manifold, N

$$f : M \to N$$

with weaker requirements than an embedding, allowing for self intersection of the image so long as the derivative has rank n everywhere.

Theorem 2.20. *If M is immersed in N, then the Whitney sum of the tangent bundle τ of M, and the normal bundle ν of M in N is isomorphic to the tangent space of N. In the case of an immersion*

$$f : \mathbb{P}^n \to \mathbb{R}^{n+k}$$

the necessary implication is that $\tau \oplus \nu \approx \epsilon^{n+k}$.

Example 2.21. Continuing with example of the canonical bundle γ_n^1 in Lemma 2.19, assume additionally that $\mathbb{P}^n$ can be embedded in Euclidean space $\mathbb{R}^{n+k}$.

By Theorem 2.21, $0 = w(\epsilon^{n+k}) = w(\tau)w(\nu) = w(\nu)(1 + a)^{n+1}$. Say $n = 8$, $w(\tau) = 1 + a + a^8$, therefore $w(\nu) = \overline{w(\tau)} = 1 + a + a^2 + a^3 + a^4 + a^5 + a^6 + a^7$. Since $w_7(\nu) = a^7 \neq 0$, it follows that ν must be a k-plane bundle for $k \geq 7$. This puts a lower bound $n + 7 = 15$ on the dimension of Euclidean space into which you can immerse P^8.

We have now seen that Stiefel-Whitney classes allow us to prove the impossibility of sections of a vector bundle, and to prove the non-immersibility of manifolds into Euclidean space.

Another set of interesting applications of SW classes is SW numbers, and Bordism and Cobordism groups which can help determine whether a manifold can be expressed as the boundary of a higher dimensional manifold.

3. The Grassmanian and Classifying Spaces

The length of this paper precludes the rigorous treatment that the Grassmanian deserves, however there is room to address construction and some key features.

Definition 3.1. The *Grassman Manifold* $G_n(\mathbb{R}^{n+k})$ is the set of all n planes passing through the origin (or the set of all n-dimensional subspaces) of R^{n+k} topologized as a quotient space of the Stiefel manifold.

The *Stiefel manifold* $V_n(\mathbb{R}^{n+k})$ is an open subset of $\mathbb{R}^{n+k} \times \mathbb{R}^{n+k} \times ... \times \mathbb{R}^{n+k}$ (where there are n copies of $\mathbb{R}^{n+k}$) consisting of all n-tuples of linearly independent real $n+k$ dimensional vectors.

The *quotient map* $q : V_n(\mathbb{R}^{n+k}) \to G_n(\mathbb{R}^{n+k})$ takes each n-tuple to the n-plane which it spans. A set in the Grassman is open if and only if its inverse image under the quotient map is open.

Remark 3.2. The Grassman manifold is a compact manifold of dimension nk, and the function $X \to X^{\perp}$ is a homeomorphism between $G_n(\mathbb{R}^{n+k})$ and $G_k(\mathbb{R}^{n+k})$.

Remark 3.3. $G_1(\mathbb{R}^{n+1})$ is $\mathbb{P}^n$.

Definition 3.4. We define a canonical n-plane bundle $\gamma^n(\mathbb{R}^{n+k})$ over $G_n(\mathbb{R}^{n+k})$ that is similar to our canonical line bundle for $n = 1$. The fiber at each point is simply the set of all vectors in the subspace identified with that point equipped with the usual vector space structure.

Definition 3.5. As k goes to infinity, we obtain the Grassmanian $G_n(\mathbb{R}^{\infty}) = G_n$. This is known as the infinite Grassmanian, topologized as the direct limit over all k of $G_n(R^{n+k})$, with respect to the natural maps coming from the inclusions $R^{n+k_1} \to R^{n+k}$, for $k_1 \leq k$. It has canonical bundle γ^n.

The infinite Grassman Manifold is known as the classifying space for vector bundles. because of the next few theorems.

Theorem 3.6. *For any n-plane bundle ζ over a compact base space B, there exists a bundle map from ζ to $\gamma^n(\mathbb{R}^{n+k})$ for k sufficiently large. If B is paracompact there exists a bundle map from ζ to γ^n. Further, any two bundle maps f, g from an n-plane bundle ζ to γ^n are homotopic to one another.*

Proof. In the compact case the proof is fairly straight forward. Find an open cover of B such that the restriction of ζ to each open set in the cover is trivial. Then, for each of the m open sets U_i in the finite subcover, since the bundle is trivial on each open set in the cover, there exists a bundle map h_i from $\pi^{-1}(U_i)$ to $\mathbb{R}^n$ which is linear on each fiber. If we assume our space is hausdorff then, by compactness, it is also normal. We can then make two refinements to our cover, $U_i \supset \bar{V}_i$, $V_i \supset \bar{W}_i$ and construct smooth maps $\lambda_i : B \to [0, 1]$ which take value 1 on the closure of W_i and are 0 on $(V_i)^c$.

Now, we can construct the map on the fibers $\hat{f} : E(\zeta) \to \gamma^n(\mathbb{R}^{n+k})$ as

$$\hat{f}(e) = (\lambda_1(\pi(e))h_1(e), \lambda_2(\pi(e))h_2(e)), ..., \lambda_m(\pi(e))h_m(e))$$

and let the map of base spaces take each $b \in B$ to the subspace $\hat{f}(fiber(b)) \in G_n(\mathbb{R}^{mn})$. In order to make the above expression well-defined outside U_i, when h_i is not defined, we have λ_i equals zero, and hence we take $\lambda_i(\pi(e))h_i(e)$ to mean the zero vector in R^n.

The paracompact case is slightly different. Again take two refinements to end up with a locally finite covering of potentially countably many open sets. Therefore the mapping $\hat{f}$ is from $E(\zeta)$ into the bundle γ^n:

$$\hat{f}(e) = (\lambda_1(\pi(e))h_1(e), \lambda_2(\pi(e))h_2(e)), ..., \lambda_m(\pi(e))h_m(e), ...)$$

recalling that γ^n is the canonical n-plane bundle over the infinite Grassman manifold. Since the coverings are locally finite, the map as constructed is still continuous.

This is an extremely important result and is the reason why the Grassmanians and their canonical bundles earn the titles of classifying spaces and universal bundles respectively. Axiom 2 of SW classes allow us to classify n-plane bundles by mapping them into γ^n.

Now we must show that any two bundle maps f, g from an n-plane bundle ζ to γ^n are homotopic to one another.

Let d_1 be the induced map on γ^n obtained from the linear map from $\mathbb{R}^\infty$ to $\mathbb{R}^\infty$ that takes the i^{th} coordinate to the $2i^{th}$ coordinate.

Similarly, let d_2 take the i^{th} coordinate to the $2i-1$ coordinate. We obtain that the following are homotopic:

$$f \sim d_1 \circ f \sim d_2 \circ g \sim g$$

with the one parameter families being

$$\alpha f + (1-\alpha)d_1 \circ f, \quad \alpha d_1 \circ f + (1-\alpha)d_2 \circ g, \quad \alpha g + (1-\alpha)d_2 \circ g \tag{3.7}$$

for $\alpha \in [0,1]$. Though I will not address the proof in any detail, d_1 and d_2 have been constructed to ensure linear independence vectors, such that for every α all these maps are still bundle maps.

□

Therefore, there is at most a single homotopy class of maps ζ into γ^n. Combined with the previous result, we have that every n-plane bundle ζ over B has a unique corresponding homotopy class of maps $\bar{f}$ into G^n generated by the induced map of base spaces from any bundle map.

Definition 3.8. Let Λ be any coefficient group or ring, let $c \in H^i(G_n, \Lambda)$, and let $\bar{f}_\zeta$ be the unique homotopy class of maps of base spaces into G_n determined by ζ. The unique cohomology class $\bar{f}^*_\zeta(c) \in H^i(B; \Lambda)$ is the characteristic class of ζ determined by c, denoted $c(\zeta)$.

Remark 3.9. The ring of all characterisitic classes of real vector bundles over paracompact base spaces with coefficient ring Λ is canonically isomorphic to $H^*(G_n; \Lambda)$. This illustrates the importance of the Grassmanian as a classifying space.

4. Existence of Stiefel-Whitney Classes

We now have enough machinery in place to prove the existence of Stiefel-Whitney Classes. The symbols for our total space, base space, fiber and bundle will be as usual and we will further introduce some new terminology. We will denote $F_0 = F \setminus 0$, where F is the fiber at b, the set of nonzero elements of the fiber, and $E_0 = E/B$, will be the set of all non zero elements of the total space, in other words the complement of the zero section in E. Assuming that our fiber dimension is n, we can immediately remark that the cohomology of F_0 and E_0 satisfies:

$$H^n(F, F_0; Z/2) = Z/2 \tag{4.1}$$

and is zero for all other dimensions.

Theorem 4.2. *The group $H^n(E, E_0)$ contains a unique non-zero element u, henceforth known as the fundamental cohomology class, such that the restriction of u to each fiber is the unique non-zero element of $H^n(F, F_0)$. Further, for every k, $x \mapsto x \cup u$ is an isomorphism from $H^k(E)$ to $H^{k+n}(E, E_0)$*

We also note that the everywhere zero section of E is a deformation retract of E onto B, providing us with another isomorphism from $H^k(B)$ to $H^k(E)$

Definition 4.3. We define the Thom Isomorphism $\phi : H^k(B) \to H^{k+n}(E, E_0)$ using the composition of our two isomorphisms above:

$$H^k(B) \to H^k(E) \to H^{k+n}(E, E_0) \tag{4.4}$$

Shortly, we will use the Steenrod Square cohomology operations in order to prove the Thom Isomorphism's existence.

Definition 4.5. The Steenrod Square operations are operations that satisfy the four following properties (still assuming Z/2 coefficients):

(1) For each pair $X \subset Y$ and all nonnegative integers n, i, there exists an additive homomorphism
$$Sq^i : H^n(X, Y) \to H^{n+i}(X, Y)$$
(2) The operations are natural with respect to continuous maps on pairs. For any $f : (X, Y) \to (X', Y')$ the identity $\bar{f} \circ Sq^i = Sq^i \circ \bar{f}$ holds.
(3) If $a \in H^n(X, Y)$ then $Sq^0(a) = a$, $Sq^n(a) = a \smile a$ (this is the cup product again) and $Sq^i(a) = 0$ for $i > n$.
(4) The Cartan formula: $Sq^k(a \cup b) = \sum_{i+j=k} Sq^i(a) \smile Sq^j(b)$

We will again define a total square $Sq(a) = \sum_{i=1}^{n} Sq^i(a)$, analogous to our total Stiefel-Whitney class, in that it allows us to simplify the product rule given by the Cartan formula to $Sq(ab) = Sq(a)Sq(b)$. Again, just as we had for Stiefel-Whitney classes, we also have the cartesian product relation $Sq(a \times b) = Sq(a) \times Sq(b)$

By this point, the similarities between the definitions of the Steenrod squares and the Stiefel-Whitney classes themselves are clear, so it would appear fruitful to use the assumed existence of the former to prove the existence of the latter. However, the existence of the Steenrod operations is completely divorced from vector-bundles and therein lies their utility. The Thom isomorphism, on the other hand, takes into account the topology of the total space of a fiber. Therefore, by combining these two tools, we can identify characteristic classes that satisfy the desired axioms while varying appropriately for different bundles over the same space.

Theorem 4.6. *The unique classes that are defined by the relations*

$$w_i(\zeta) = \phi^{-1} Sq^i(\phi(1)) \text{ and } w(\zeta) = \phi^{-1} Sq(\phi(1))$$

satisfy all the axioms of the Stiefel Whitney classes and total Stiefel Whitney classes, respectively, and therefore are the (total) Stiefel Whitney classes.

Proof. According to the given relations, for any bundle ζ,

$$w_0(\zeta) = \phi^{-1} Sq^0 \phi(1) = \phi^{-1}(id(\phi(1))) = 1$$

The naturality condition for SW classes comes from the the naturality condition on the Steenrod operations along with the fact that induced maps on cohomology preserve the fundamental class of a pair.

The Whitney product axiom requires more thorough work to verify rigorously but relies generally on arguing that the fundamental class of the Cartesian product of pairs of spaces is equal to the cross product of the fundamental class of the pairs.

Finally, the fourth axiom, that $w(\gamma_1^1) = 1 + a$ can simply be confirmed via direct computation. □

5. Orientation and the Euler Class

Just as Stiefel-Whitney classes helped quantify obstructions to the existence of cross sections of a vector bundle, the Euler class, a closely related characteristic cohomology class, will help measure obstruction to orientability. For this reason, we will move from coefficients in $Z/2$ to coefficients in Z.

Definition 5.1. An orientation of a real vector space is an equivalence class of bases in which two bases are equivalent if and only if the change of basis matrix from one basis to the other has positive determinant. It follows immediately that every real vector space has two orientations.

Definition 5.2. An orientation for the n-plane bundle ζ is a choice of preferred orientation for each fiber of ζ subject to a local compatibility restriction. Namely, for every point in the base space, there must exist an open set U and a homeomorphism $h : U \times \mathbb{R}^n \to \pi^{-1}(U)$ that preserves orientation of each fiber.

This leads to the important notion of a preferred cohomology generator for each fiber,

$$u_F \in H^n(F, F_0; Z)$$

and a generator,

$$u \in H^n(\pi^{-1}(U), \pi^{-1}(U)_0; Z)$$

whose restriction to each (F, F_0) over points in U is the preferred generator of $H^n(F, F_0; Z)$.

Theorem 5.3. *For any oriented n-plane bundle ζ, there exists a unique $u \in H^n(E, E_0; Z)$ whose restriction to each (F, F_0) is the preferred generator of $H^n(F, F_0; Z)$. Further, the function $a \mapsto a \smile u$ is an isomorphism from $H^k(E; Z)$ to $H^{n+k}(E, E_0; Z)$.*

Passing this preferred cohomology class, u, through the natural restriction homomorphism $j : H^n(E, E_0; Z) \to H^n(E; Z)$ which takes a to $a|E$, and then passing this element through the canonical isomorphism from $H^n(E; Z)$ to $H^n(B; Z)$ gives rise to our next characteristic class:

Definition 5.4. The Euler class of an oriented n-plane bundle is

$$(\pi^*)^{-1}(u|E) = e(\zeta) \in H^n(B; Z)$$

This class happens to be closely related to the Euler characteristic of oriented manifolds and allows for their computation, hence the name.

The Euler class satisfies many properties similar to those of Stiefel Whitney classes. Notably, it is natural with respect to orientation preserving bundle maps. If a bundle map $f : \zeta \to \zeta'$ preserves orientation then $e(\zeta) = f^*(e(\zeta'))$. Naturality is in fact a condition for all characteristic classes. If f reverses orientation then $e(\zeta) = -f^*(e(\zeta'))$.

This immediately gives us the result that, for trivial bundles $e(\zeta) = 0$, and whenever n is odd $e(\zeta) = -e(\zeta)$. This second statement follows from the fact that, whenever the fiber dimension is odd, the automorphism $(b, v) \to (b, -v)$ is orientation reversing.

Next, we note that the natural homorphism on cohomology induced by coefficient surjection $Z \to Z/2$ takes $e(\zeta)$ to $w_n(\zeta)$.

Further properties include the familiar identities:

$$e(\zeta \oplus \zeta') = e(\zeta) \cup e(\zeta'), e(\zeta \times \zeta') = e(\zeta) \times e(\zeta')$$

with the caveat that the orientation for each $F \oplus F'$ is obtained by taking the oriented basis for F followed by the oriented basis for F'. Proofs are omitted but can be found in the sources below.

Theorem 5.5. *If a bundle ζ admits a nowhere zero section, then $e(\zeta) = 0$*

Proof. We can split ζ into a trivial bundle ϵ and its complement $\epsilon^{\perp}$. We then have $e(\zeta) = e(\epsilon) \cup e(\epsilon^{\perp}) = 0 \cup e(\epsilon^{\perp}) = 0$. □

Example 5.6. For line bundles, realize that an orientation consists in a compatible choice of a preferred unit vector at each point. Such a choice is simply a global section, and any global section of a line bundle can be used to define an orientation. Therefore, a line-bundle is orientable if and only if it is trivial. Returning to our twisted line bundle γ_1^1, we immediately see that it is not orientable.

On the other hand, the tangent bundle of S^{2n} is certainly orientable and yet contains no non zero sections.

Theorem 5.7. *A manifold is orientable if and only if its tangent bundle is orientable.*

In fact, any orientation of the tangent bundle induces an orientation on the manifold itself and vice-versa. This arises from the natural homeomorphism from a small open neighbourhood of each point and the tangent bundle at that point. Roughly speaking, an orientation of coordinates for the tangent space can be viewed as a local orientation on the manifold.

Example 5.8. An application of the above generalizes the hairy ball theorem for higher dimensional spheres. Let $A \subset S^n \times S^n$ be the set of antipodal unit vectors $(v, -v)$, and let $B \subset S^n \times S^n$ be the diagonal set of points (v, v).

For each point $b \in S^n$, take the stereographic projection from S^n to R^n, where b sits on the origin and the projection is taken from the antipode of each point, to create a homeomorphism from the tangent manifold $E(\tau_{S^n})$ to $S^n \times S^n - A$

Using excision

$H^i(S^n \times S^n, A) \approx H^i(S^n \times S^n - A, (S^n \times S^n - B) - A) \approx H^i(E, E_0) \subset H^i(S^n \times S^n)$

With some details omitted, if n is even then $e(\tau) = \phi^{-1}(u \cup u) = 2a \in H^n(S^n, Z)$ where a is a generator of $H^n(S^n, Z)$. Since $e(\tau)$ is not zero, there exists no non-zero section of the n sphere.

Acknowledgments

It is a pleasure to thank my mentor, Shiva Chidambaram, for his invaluable help and guidance in writing this paper. I would also like to thank Professor Peter May for organizing the excellent University of Chicago REU program.

References

[1] J. W. Milnor and James D. Stasheff . Characteristic Classes. Princeton University Press and University of Tokyo Press. 1974.
[2] Allen Hatcher. Algebraic Topology. https://pi.math.cornell.edu/ hatcher/AT/AT.pdf
[3] Robert E. Mosher and Martin C. Tangora. Cohomology Operations and Application in Homotopy Theory. https://www.maths.ed.ac.uk/ v1ranick/papers/moshtang.pdf