A THEOREM ON THE CLASSIFYING SPACE OF A GROUP WITH TORSION

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Abstract. Let G be a discrete group and consider the Eilenberg-MacLane space $K(G,1)$. We prove that if there is a finite dimensional CW structure on this space then G cannot have any torsion.

Contents

1. Introduction

Henceforth any group mentioned in this paper is a discrete group. Our goal is to show that for a group G with torsion, there does not exist a finite dimensional CW-complex structure on $K(G,1)$. We first define the Eilenberg-MacLane $K(G,1)$ spaces, and construct a particular example, the classifying space BG. We then define the group homology $H_*(G;\mathbb{Z})$ of G and show that this is isomorphic to the singular homology $H_*(K(G,1);\mathbb{Z})$). We find that the homology of a cyclic group G is nonzero in infinitely many degrees, and thus that any CW structure on $K(G,1)$ must be infinite dimensional. Finally, a covering space argument allows us to generalize this to any group G with nontrivial torsion

2. $K(G,1)$ Spaces

Definition 2.1. Let G be a (discrete) group. A $K(G,1)$ space is a path-connected space with contractible universal cover and with fundamental group isomorphic to G.

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Example 2.2. The circle S^1 is a $K(\mathbb{Z}, 1)$ space with universal cover $\mathbb{R}$. Likewise, the torus T is a $K(\mathbb{Z} \times \mathbb{Z}, 1)$ space with universal cover $\mathbb{R}^2$. More generally, the n-dimensional torus

$$T^n = \underbrace{S^1 \times S^1 \times \cdots \times S^1}_{n \text{ times}},$$

can be obtained from the quotient of $\mathbb{R}^n$ by the action of $\mathbb{Z}^n$ translating each coordinate by integer amounts, and thus T^n is a $K(\mathbb{Z}^n, 1)$ space.

Example 2.3. Analogously to the real projective space $\mathbb{RP}^n$, we can define the infinite real projective space $\mathbb{RP}^\infty$. We define the infinite sphere S^∞ as the set of points $(x_i)_{i=1}^\infty$ in $\mathbb{R}^\infty$ (i.e. the $x_i \in \mathbb{R}$ are 0 for almost all i) such that $\sum x_i^2 = 1$, and $\mathbb{RP}^\infty$ is defined as the quotient of S^∞ by identifying antipodal points.

We can see that S^∞ is contractible, either by using an explicit deformation retract to a point, or by appealing to Whitehead's Theorem: every map $f : S^n \to S^\infty$ has a compact image, which must be contained in some $S^k \subseteq S^\infty$ (equatorial inclusion). Since the inclusion $S^k \subseteq S^{k+1}$ is nullhomotopic, the composite $S^k \subseteq S^{k+1} \subseteq S^\infty$ is nullhomotopic and thus so is f. Therefore $\pi_n(S^\infty, *) = 0$ for all $n \geq 0$ which implies that S^∞ is contractible.

$\mathbb{Z}/2$ acts on S^∞ by a covering space action and thus $\mathbb{RP}^\infty = S^\infty/(\mathbb{Z}/2)$ is a $K(\mathbb{Z}/2, 1)$ space.

Considering S^∞ to be the unit sphere in $\mathbb{C}^\infty$ rather than in $\mathbb{R}^\infty$, we can define a covering space action of $\mathbb{Z}/p$ on S^∞ for arbitrary p by multiplication by roots of unity, i.e.

$$[n] * (z_1, z_2, \ldots) = e^{\frac{2\pi i n}{p}}(z_1, z_2, \ldots).$$

Then, we define the infinite dimensional lens space as $S^\infty/(\mathbb{Z}/p)$, which is a $K(\mathbb{Z}/p, 1)$.

Here, we see that it is not too difficult to come up with $K(G, 1)$ spaces for certain groups G. However, there would not be much of a point in defining something as general as $K(G, 1)$ spaces if they only existed for G cyclic or free abelian. Certainly, if we want to use such spaces to study anything meaningful about groups, we need them to exist for arbitrary groups. In fact, we *can* construct a $K(G, 1)$ space for any group G, and the construction gives a rather strong space with important pointwise properties. Given a group G, this space is referred to as the classifying space BG of G.

3. Construction of the Classifying Space

To construct BG, we begin by defining a larger space EG, which will be the universal cover for BG.

Definition 3.1. Let G be a group. The space EG is the simplicial complex with one n-simplex for each $n + 1$-tuple $[g_0, g_1, \ldots, g_n]$ of elements of G, with a face attached to each $n - 1$-simplex of the form $[g_0, \ldots, \hat{g}_i, \ldots, g_n]$.

Notice that EG is contractible by a homotopy continuously moving any point x along the segment connecting x to $[e]$

We can consider the group action of G on EG given by $g[g_0, \ldots, g_n] = [gg_0, \ldots, gg_n]$. Clearly, the only element of g which sends any simplex to itself is the identity, and the image of every n-simplex in EG is an n-simplex. For any point x, we may find an open neighborhood of x which intersects only those simplices containing x, and so this is a covering space action.

We define the classifying space BG to be the quotient EG/G. As G acts by a covering space action and EG is contractible, the quotient map $EG \to BG$ is a universal covering. Thus, BG is a $K(G,1)$ space. Moreover, we can obtain a simplicial complex structure on BG from that on EG. We may write every n-simplex in EG uniquely as $[g_0, g_0g_1, \ldots, g_0g_1 \ldots g_n]$. Each simplex is sent by the quotient map to the equivalence class of $[e, g_1, g_1g_2, \ldots, g_1g_2 \ldots g_n]$, which we denote using "bar notation" as $[g_1|g_2| \ldots |g_n]$. Each equivalence class $[g_1| \ldots |g_n]$ defines a simplex in BG with faces $[g_2| \ldots |g_n]$, $[g_1| \ldots |g_ig_{i+1}| \ldots |g_n]$, and $[g_1| \ldots |g_{n-1}]$.

At first glance, BG may seem like too large of an object to work with, as it is infinite dimensional and has $|G|^n$ many simplices per dimension, but it has some important properties. For the purposes of this paper, BG simply serves as a proof of existence of a $K(G,1)$ space given an arbitrary group. However, in general, one advantage of BG over arbitrary $K(G,1)$ is that it defines a functor from the category of groups to the category of spaces, inducing for each homomorphism f a simplicial map $[g_1| \ldots |g_n] \mapsto [f(g_1)| \ldots |f(g_n)]$.

4. Uniqueness of $K(G,1)$ homotopy type

Having shown the existence of $K(G,1)$ spaces, we now state an important theorem which allows us to consider arbitrary $K(G,1)$s up to equivalence.

Proposition 4.1. *Let X be a connected CW complex and Y a $K(G,1)$ space. For every homomorphism $\varphi : \pi_1(X, x_0) \to \pi_1(Y, y_0)$, there exists a map f, unique up to homotopy fixing x_0, which induces φ.*

The proof of this proposition can be found in [Hatcher 2001]. From this, we get the following theorem:

Theorem 4.2. *Given a group G, let X and Y be CW-complexes and $K(G,1)$'s. Then, X and Y are homotopy equivalent.*

We could further require (by CW approximation) that any $K(G,1)$ space be homotopy equivalent to a CW-complex, in which case the group G totally determines the homotopy type of $K(G,1)$. From now on, we will restrict any discussion of $K(G,1)$ spaces to CW-complexes.

5. The Group Ring $\mathbb{Z}G$

We now take a somewhat lengthy detour into the algebraic realm. Our goal is to define the homology of a group, which will give us a relationship between $K(G,1)$ complexes and their associated group G. To do this, we need to examine the group ring $\mathbb{Z}G$.

Definition 5.1. Let G be a group. The group ring $\mathbb{Z}G$ is the free $\mathbb{Z}$-module generated by G.

Elements of $\mathbb{Z}G$ are sums of the form $\sum_i n_ig_i$, with $n_i \in \mathbb{Z}$ and $g_i \in G$. Addition of elements is defined as one would expect, by combining two sums, and multiplication is defined by the distributive property and group multiplication in G, giving

$$\left(\sum_i n_ig_i\right)\left(\sum_j n_jg_j\right) = \left(\sum_{i,j}(n_in_j)(g_ig_j)\right). \tag{5.2}$$

This gives us a ring structure on $\mathbb{Z}G$. In particular, $\mathbb{Z}G$ is associative and unital with unit $1 \cdot e_G$ (which we typically write as just 1), and is commutative if and only if G is abelian.

Having defined the ring structure on $\mathbb{Z}G$, we can consider modules over $\mathbb{Z}G$. By writing down the definitions we see that the structure of a $\mathbb{Z}G$ module (or G-module) on a set X is equivalent to X being a an abelian group with an additive group action from G.

In particular, for any group G, $\mathbb{Z}$ is a G-module equipped with trivial G-action. Another simple example which we can consider is the circle S^1 as a $\mathbb{Z}/p$-module, where multiplication in S^1 is defined by complex multiplication and $\mathbb{Z}/p$ acts by rotation (i.e. multiplication by roots of unity).

6. Chain Complexes

In order to define the homology of a group, we need to be able to talk generally about chain complexes, an immediate generalization of the chain complexes which appear in singular homology.

Definition 6.1. A *chain complex* is a sequence $(C_n)_{n\in\mathbb{Z}}$ of abelian groups or modules equipped with a sequence of homomorphisms $\partial_n : C_n \to C_{n+1}$ called *differentials* such that $\operatorname{im} \partial_{n+1} \subseteq \ker \partial_n$ for all n. We may refer to a chain complex (C_n) simply as C if it is unambiguous.

We can consider maps between chain complexes which preserve their structure.

Definition 6.2. Let C and C' be chain complexes with differentials ∂ and ∂'. A chain map $f : C \to C'$ is a sequence of homomorphisms $f_n : C_n \to C'_n$ such that $f_n\partial_{n+1} = \partial'_{n+1} f_{n+1}$ for all n.

So we have a category of chain complexes and chain maps.

Similar to maps between topological spaces, we can define an equivalence between chain maps which we refer to as homotopy.

Definition 6.3. Two chain maps f and g between C and C' are *homotopic* if there exists a sequence of homomorphisms $h_n : C_n \to C'_{n+1}$ such that $\partial' h + h\partial = g - f$. We then call h a *chain homotopy* and write $h : f \simeq g$. We define homotopy equivalence of chain complexes in the analogous way to homotopy equivalence of spaces.

Lastly, we can define homology groups for chain complexes:

Definition 6.4. The *homology* $H_*(C)$ of a chain complex C is the sequence of abelian groups or modules given by $\ker \partial_n / \operatorname{im} \partial_{n+1}$. Note that homology defines a functor.

Definition 6.5. A *quasi-isomorphism* is a chain map $f : C \to C'$ which induces isomorphisms on the homology groups.

Proposition 6.6. *A chain homotopy equivalence f is a quasi-isomorphism.*

Proof. To prove this, it is sufficient to show that homotopic chain maps induce the same map between homology groups. Let $f, g : C \to C'$ be homotopic chain maps. For a cycle x in C_n, we have $\partial' h(x) = f(x) - g(x)$. Since $f(x)$ and $g(x)$ differ by a boundary, they are sent to the same homology class. $\square$

7. Free Resolutions

A particular class of chain complexes which will allow us to construct the homology of a group are free resolutions of a module, which we define as a particular exact sequence of modules.

Definition 7.1. A sequence of homomorphisms

$$\dots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \dots$$

is called *exact* if $\ker \partial_n = \operatorname{im} \partial_{n+1}$ for all n, or equivalently if all homology groups are trivial.

Definition 7.2. Let M be a module over a ring R. A *free resolution* of M over R is an exact sequence

$$\dots \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\varepsilon} M \longrightarrow 0,$$

where C_* is a free R-module.

Free resolutions can be constructed for arbitrary modules M by choosing a surjection $\varepsilon : C_0 \to M$ for a free module C_0, then choosing a surjection $C_1 \to \ker \varepsilon$ and proceeding inductively.

We are particularly interested in free resolutions of $\mathbb{Z}$ over $\mathbb{Z}G$, where $\mathbb{Z}$ is treated as a G-module with trivial action. To be able to meaningfully construct the homology of groups, we need to be able to talk about free resolutions as if they are unique. The comparison theorem [Weibel 1994] tells us that any two free resolutions of a module M over a ring R are chain homotopy equivalent.

8. The Homology of a Group

The homology of a group is defined in terms of a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$, using *co-invariants*.

Definition 8.1. Let G be a discrete group and M be a G-module. The abelian group M/G of co-invariants is the quotient $M/gm \sim m$.

This defines a functor from G-modules to $\mathbb{Z}$-modules (i.e. abelian groups). For a map $f : M \to M'$ of G-modules, the induced map f/G of coinvariants sends $[m]$ to $[f(m)]$. Given a chain complex of G-modules C_* with differential ∂, we have induced maps $[\partial_n(-)] : C_n/G \to C_{n-1}/G$, and so C_* is associated with a chain complex of co-invariants C_*/G. Then, because the co-invariants define a functor, and additionally because $[m + m'] = [m] + [m']$ for elements m, m' in a G-module M, we can see from the definition of chain homotopy that for homotopic chain maps $f, g : C \to C'$ between chain complexes of G-modules, the chain maps induced on the chain complex of co-invariants are also homotopic.

Definition 8.2. Let G be group and C an augmented chain complex defining a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. The *homology groups* of G are given by $H_*(G; \mathbb{Z}) = H_*(C/G)$.

Since free resolutions are homotopy equivalent and homotopy equivalence is preserved by co-invariants, $H_*(G; \mathbb{Z})$ is unique up to isomorphism.

9. Relationship to $K(G,1)$ Complexes

For an arbitrary group G we can use the universal space EG to obtain a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.

Proposition 9.1. *The augmented simplicial chain complex of EG gives a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$, the "bar" resolution*

Proof. Since EG is contractible, its reduced simplicial homology groups are trivial, so the associated chain complex must be exact. □

Proposition 9.2. *The coinvariants $C_*(EG)/G$ of the bar resolution $C_*(EG)$ form the simplicial chain complex $C_*(BG)$ of BG.*

Proof. The simplicial chain group $C_n(EG)$ has a $\mathbb{Z}G$-basis defined by the simplices $[g_1|g_2|\dots|g_n]$ of BG since G freely permutes the simplices of EG giving one $\mathbb{Z}G$-basis element per G-orbit. Therefore $C_n(EG)/G$ has a $\mathbb{Z}$-basis given by the $[g_1|\dots|g_n]$, and so is canonically isomorphic to $C_n(BG)$. Using the formula for the differential in simplicial chains, one finds that the differential ∂_n is given on the basis of $C_n(EG)/G$ by

$$\partial_n[g_1|\dots|g_n] =$$
$$g_1[g_2|\dots|g_n] + \sum_{i=1}^{n-1}(-1)^i[g_1|\dots|g_{i-1}|g_ig_{i+1}|g_{i+2}|\dots|g_n] + (-1)^n[g_1|\dots|g_{n-1}].$$

This is precisely the differential for the chain group $C_n(BG)$. Therefore the chain complexes are isomorphic. □

From this we conclude that $H_*(G;\mathbb{Z}) \approx H_*(BG;\mathbb{Z})$ where on the left we have group homology and on the right (simplicial) homology of spaces. Then, since any $K(G,1)$ CW complex is homotopy equivalent to BG, we have:

Corollary 9.3. *Let Y be a $K(G,1)$ CW complex. Then, $H_*(G;\mathbb{Z}) \approx H_*(Y;\mathbb{Z})$.*

10. An Example: Calculating $H_*(\mathbb{Z}/p;\mathbb{Z})$

Suppose G is the cyclic group of order p, with generator g. We construct a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ of the form

$$\dots \xrightarrow{\partial_2} \mathbb{Z}G \xrightarrow{\partial_1} \mathbb{Z}G \xrightarrow{\partial_0} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0.$$

Let $N = 1+g+g^2+\dots+g^{p-1}$ be the norm element of G. We have a free resolution by defining $\varepsilon(g) = 1$ and

$$\partial_i(m) = \begin{cases} (1-g)m, & i \text{ is even} \\ Nm, & i \text{ is odd.} \end{cases}$$

Dividing out by the G-action, the chain complex of co-invariants is given by

$$\dots \xrightarrow{p} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{0} \mathbb{Z}.$$

Thus, we find that the homology groups are given by

$$H_i(G;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & i = 0 \\ \mathbb{Z}/p, & i \text{ is positive, even} \\ 0, & i \text{ is odd.} \end{cases}$$

As a corollary, we have the following statement regarding $K(G, 1)$ complexes:

Corollary 10.1. *Let G be a discrete group. Suppose there exists a finite-dimensional $K(G, 1)$ CW complex X. Then, G is torsionfree.*

Proof. Suppose that G is *not* torsionfree. Then, G contains some cyclic subgroup $\mathbb{Z}/p$ and so there exists a covering space $p : \tilde{X} \to X$ such that $\pi_1(\tilde{X}) = \mathbb{Z}/p$. Since X and $\tilde{X}$ have the same universal cover, $\tilde{X}$ is a $K(\mathbb{Z}/p, 1)$ complex, and so $H_i(\tilde{X}; \mathbb{Z})$ is nonzero for infinitely many i. However, this cannot be the case, since $\tilde{X}$ is the covering space of a finite-dimensional CW-complex, and so must itself be a finite-dimensional CW-complex. □

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