INTRODUCTION TO BROWNIAN MOTION AND ITS FUNDAMENTAL PROPERTIES

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Abstract. This is an expository paper that gives an introduction to Brownian motion and its fundamental properties.

Contents

1. Introduction

Imagine a glass of water with a single particle of pollen mixed in. As it floats in the water it will frequently collide with nearby water molecules, and each of these collisions will leave the pollen moving in a new direction. Moreover, since the collisions of the water molecules with the pollen will be random we can see that the motion of the pollen will also be random. Naturally the desire to model this type of motion arises and with this desire comes Brownian motion. Informally, Brownian motion can be thought of as random, continuous motion. Modelling random, continuous motion might seem like a tedious task, but this paper will serve to explore some of the surprisingly beautiful properties of Brownian motion. We will prove its nowhere differentiability and its strong Markov property. We will also explore the maxima and the zeroes of Brownian motion as well as briefly discuss fractional Brownian motion.

2. Relevant Definitions

Definitions 2.5, 2.15, 2.16, and 2.21 are from Introduction to Stochastic Processes [1]. Definitions 2.1-2.4 and 2.8-2.10 are from Stochastic Processes [2]. Definitions 2.17 and 2.18 are from Measure, Integral, and Probability [3]. Definition 2.19 is from Measure Theory [4]. Definition 2.13 is from Gaussian Processes [7]. Definitions 2.6 and 2.12 are from the e-Handbook of Statistical Methods [10]. Definitions 2.7 and 2.11 are from Introduction to Probability, Statistics, and Random Processes [11]. Definition 2.20 is from Brownian Motion [12]. Finally, definition 2.14 is from Introduction to Gaussian Processes [13].

In this section we will provide definitions from probability and measure theory for later use in our proofs. Some examples will be given to help the reader get acquainted to the new ideas and objects being defined.

Definition 2.1. Sample Space: A sample space of an experiment is the set of all possible outcomes of an experiment and is often denoted by S.

Remark 2.1. The probability of S, denoted by $P(S)$, is always equal to 1.

Definition 2.2. Event: An event of an experiment is a single possible outcome in the sample space of the experiment.

Remark 2.2. For any event E in S we know that $0 \leq P(E) \leq 1$ where $P(E)$ is the probability of E.

Example 2.3. Consider a coin with two sides: heads and tails. The sample space of the experiment is heads and tails as those are the possible outcomes. An event would be heads if the coin was tossed and came up heads.

Definition 2.3. Independent Variables: Two variables are independent if for two events A and B

$$P(A \cap B) = P(A)P(B).$$

Example 2.4. The events of rolling a 3 on a die and pulling a jack from a deck of cards are independent. On the other hand, the events of getting heads and tails on a single coin flip are not independent.

Definition 2.4. Random Variables: A random variable X is a function that assigns a real value to each outcome in S. For any set of real numbers A, the probability that X will assume a value that is contained in the set A is equal to the probability that the outcome of the experiment is contained in $X^{-1}(A)$. That is

$$P(X \in A) = P(X^{-1}(A))$$

where $X^{-1}(A)$ is the event consisting of all points $s \in S$ such that $X(s) \in A$.

Definition 2.5. Stochastic Processes: A stochastic process is a collection of random variables, X_t, indexed by time. We will generally consider the range of t to be some subset of the nonnegative real numbers.

Definition 2.6. Mean: The mean is the sum of all the values of the data points divided by the number of data points. We can write this as

$$\bar{Y} = \sum_{i=1}^{N} \frac{Y_i}{N}$$

where $\bar{Y}$ is the mean of the data set $\{Y_1, Y_2, ...Y_N\}$. If we collect data on more than one variable at a time then we can arrange the means of these variables in a mean vector.

Definition 2.7. Cumulative Distribution Function: The cumulative distribution function, $F_X(x)$, of a random variable X is defined as

$$F_X(x) = P(X \leq x)$$

for all x in $\mathbb{R}$.

Definition 2.8. Expected Value: For a random variable X with distribution function $F_X(x)$ we define the expected value of X, denoted by $E[X]$, to be

$$E[X] = \int_{-\infty}^{\infty} x dF_X(x).$$

Definition 2.9. Variance: The variance of a random variable X is defined by

$$Var(X) = E\left[(X - E[X])^2\right] = E\left[X^2\right] - E^2[X].$$

Definition 2.10. Covariance: For two random variables X and Y, we define their covariance as

$$Cov(X,Y) = E[XY] - E[X]E[Y].$$

For multiple random variables in a vector we can arrange their covariances into a matrix called the covariance matrix. In these matrices the i,j position of the matrix is the covariance between the ith and jth elements of the vector.

Definition 2.11. Probability Density Function: Consider a continuous random variable X with an absolutely continuous cumulative distribution function $F_X(x)$. If $F_X(x)$ is differentiable at a point y then we define the probability distribution function f_X as

$$f_X(y) = F'_X(y).$$

Definition 2.12. Multivariate Normal Distribution: A p-dimensional vector of random variables, $X = \{X_1, X_2, ..., X_p\}$ where all X_i for i in $\{1, 2, ..., p\}$ are finite, is said to have a multivariate normal distribution if its density function, $f(X)$, is of the form

$$f(X) = \left(\frac{1}{2\pi}\right)^{p/2} |\Sigma|^{-\frac{1}{2}} \, exp\left[-\frac{1}{2}(X - m)'\Sigma^{-1}(X - m)\right].$$

Here m is the mean vector and Σ is the covariance matrix of $\{X_1, X_2, ...X_p\}$.

Definition 2.13. Gaussian Process: A real-valued stochastic process $\{X_t, t \in T\}$ where T is an index set is a Gaussian process if all the finite-dimensional distributions have a multivariate normal distribution. That is to say for any choice of distinct values $t_1, ..., t_k \in T$, the random vector $X = (X_{t_1}, ..., X_{t_k})'$ has a multivariate normal distribution with mean vector $\mu = EX$ and covariance matrix $\Sigma = Cov(X, X)$ which will be denoted by $X \sim N(\mu, \Sigma)$.

Definition 2.14. Centered Gaussian Process: A Gaussian process $\{X_t, t \in T\}$ where T is an index set is called centered if for every s and t we know that $E[X_t] = 0$ and $Cov(X_t, X_s) = E[(X_s - E[X_s])(X_t - E[X_t])]$.

Definition 2.15. State Space: The state space of a stochastic process is the set of all possible values, or states, X_t can take.

Definition 2.16. Brownian Motion: Brownian motion is a stochastic process X_t taking real number values such that

(1) $X_0 = 0$;
(2) For any $s_1 \le t_1 \le s_2 \le t_2 \le ... \le s_n \le t_n$, the random variables $X_{t_1} - X_{s_1}, ..., X_{t_n} - X_{s_n}$ are independent;
(3) For any $s < t$ the random variable $X_t - X_s$ has a normal distribution with mean 0 and variance $(t - s)\sigma^2$;
(4) The paths are continuous, i.e., the function $t \mapsto X_t$ is continuous.

Definition 2.17. σ-Algebra: A family of sets that contains a base set and is closed under complements and countable unions is a σ-algebra.

Definition 2.18. Measureable Spaces: Let Ω be a set and let $\mathcal{F}$ be a σ-algebra of its subsets. Then the pair $(\Omega, \mathcal{F})$ is a measureable space.

Definition 2.19. Measureable Function: Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces. A function $f : X \mapsto Y$ is measurable if $f^{-1}(B) \in \mathcal{A}$ for every $B \in \mathcal{B}$.

Definition 2.20. Filtration: A filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ of a probability space $(\Omega, \mathcal{F}, P)$ is a family of σ-algebras indexed by t in $[0, \infty)$, all contained in F, such that

(1) We have
$$\mathcal{F}_\infty = \sigma(\cup_{t\geq 0}\mathcal{F}_t);$$
(2) If $s \leq t$, then $\mathcal{F}_s \subset \mathcal{F}_t$.

In the context of Brownian motion we will refer to the filtration of Brownian motion at time t as all the information contained in the Brownian motion up until time t. This is to make the intuition of the proof more obvious to the reader.

Definition 2.21. Stopping Time: A random real, nonnegative variable T is a stopping time for a Brownian motion if for each t, which is the indicator function of the event $\{T \leq t\}$, it is measurable with respect to $\mathcal{F}_t$.

Using these definitions we can now explore the elegance of Brownian motion.

3. Properties of Brownian Motion

As mentioned previously, Brownian motion has several underlying and elegant properties. Not the least of which is as follows.

Theorem 3.1. Let $(B_t)_{t\geq 0}$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$. Then with probability 1 the function $t \mapsto B_t$ for $t \geq 0$ is nowhere differentiable.

Proof. Let us first explore the consequences of differentiability on a real valued function, $f(t)$ on $[0, \infty)$. Assume $f'(t)$ exists at some time t_o. Choose $a \in \mathbb{R}^+$ such that $|f'(a)| \leq a$. If $|t - t_o| < \frac{3}{n_o}$, then there exists $n_o \in \mathbb{Z}^+$ such that for $n \geq n_o$ and
$$|f(t) - f(t_o)| \leq 2a|t - t_o|.$$

For $n \geq n_o$, let k be the integer such that
$$\frac{k-1}{n} \leq t_o \leq \frac{k}{n}.$$
Thus, the points, $\frac{k-1}{n}$, $\frac{k}{n}$, $\frac{k+1}{n}$, and $\frac{k+2}{n}$ are all within $\frac{3}{n}$ of t_o. Combining this with the differentiability of f at t_o shown previously, we obtain
$$\left|f\left(\frac{k+2}{n}\right) - f\left(\frac{k+1}{n}\right)\right| \leq \left|f\left(\frac{k+2}{n}\right) - f(t_o)\right| + \left|f(t_o) - f\left(\frac{k+1}{n}\right)\right|.$$

This in turn gives us
$$\left|f\left(\frac{k+2}{n}\right) - f(t_o)\right| + \left|f(t_o) - f\left(\frac{k+1}{n}\right)\right| \leq 2a\left(\left|\frac{k+2}{n} - t_o\right| + \left|t_o - \frac{k+1}{n}\right|\right).$$

Since $|\frac{k+2}{n} - t_o| \leq \frac{3}{n}$ and $|\frac{k+1}{n} - t_o| \leq \frac{2}{n}$, we get
$$2a\left(\left|\frac{k+2}{n} - t_o\right| + \left|t_o - \frac{k+1}{n}\right|\right) \leq \frac{10a}{n}.$$

Repeating this method then gives us $|f(\frac{k+1}{n}) - f(\frac{k}{n})| \leq \frac{10a}{n}$ and $|f(\frac{k}{n}) - f(\frac{k-1}{n})| \leq \frac{10a}{n}$. As such we get that

$$\max\left\{\left|f\left(\frac{k+2}{n}\right)-f\left(\frac{k+1}{n}\right)\right|,\left|f\left(\frac{k+1}{n}\right)-f\left(\frac{k}{n}\right)\right|,\left|f\left(\frac{k}{n}\right)-f\left(\frac{k-1}{n}\right)\right|\right\}\leq\frac{10a}{n}.$$

Having found this, we can apply this same method to the Brownian motion. For the sake of simplicity let

$$X_k=\max\left\{\left|B\left(\frac{k+2}{n}\right)-B\left(\frac{k+1}{n}\right)\right|,\left|B\left(\frac{k+1}{n}\right)-B\left(\frac{k}{n}\right)\right|,\left|B\left(\frac{k}{n}\right)-B\left(\frac{k-1}{n}\right)\right|\right\}.$$

Now let A be a set of ω's such that somewhere on $[0,1)$, the function $t\mapsto B_t(\omega)$ has a derivative bounded in absolute value by a. (Note that 1 is being chosen as a boundary only for convenience. This method of proof holds for any n in $\mathbb{R}^+$.) Any ω in A is, provided that n is large enough, in A_n where A_n is the set of ω's such that for at least one k in $1,2,...,n$ we have $X_k(\omega)\leq\frac{10a}{n}$. So, if $P(\lim_{n\to\infty}\inf A_n)=0$ we will have proved that on $[0,1)$ the paths have no derivative bounded by a. Because our increments of Brownian motion, $[\frac{i}{n},\frac{i+1}{n}]$ are independent and identically distributed we have

$$P(A_n)\leq\sum_{k=1}^{\infty}P\left(X_k\leq\frac{10a}{n}\right).$$

The right side of this equation is equal to

$$nP\left(\max\left\{\left|B\left(\frac{3}{n}\right)-B\left(\frac{2}{n}\right)\right|,\left|B\left(\frac{2}{n}\right)-B\left(\frac{1}{n}\right)\right|,\left|B\left(\frac{1}{n}\right)\right|\right\}\leq\frac{10a}{n}\right).$$

This expression then can be expressed as

$$n\left[P\left(\left|B\left(\frac{1}{n}\right)\right|\leq\frac{10a}{n}\right)\right]^3=n\left[\sqrt{\frac{n}{2\pi}}\int_{\frac{-10a}{n}}^{\frac{10a}{n}}e^{\frac{-nx^2}{2}}dx\right]^3.$$

Finally, the resulting integral converges to 0 as n goes to infinity. The event that B_t is differentiable for some t is contained in the union of all A_n's for all n. Since $P(A_n)=0$ for all n, we know for all n that $P(\cup A_n)\leq\cup P(A_n)$. Thus, on $[0,1)$ with bound a we have proved Brownian motion is nowhere differentiable with probability 1. And, as stated previously, though we only handled a small case with 1 being our bound, this argument holds true for any positive, real n. □

Not only is Brownian motion nowhere differentiable, but it also has a local maximum on any set interval with probability 1.

Theorem 3.2. Density of Maxima: Almost surely, the set of times where a Brownian motion B_t will attain a local maximum is dense in $[0,\infty)$.

Proof. First off, we must show that B_t will almost surely have a local maximum on every fixed interval. B_t will only attain a local maximum at a time on an interval if it is preceded by an increasing interval and followed by a decreasing interval. So, if B_t does not have a local maximum on an interval (a,b), then it is either monotonically decreasing on the interval or it is monotonically decreasing to some point $y\in(a,b)$ and then monotonically increasing from there. The probability of the first case is 0. Similarly, since for all y either $(a,\frac{a+b}{2})\subseteq(a,y)$ or $(\frac{a+b}{2},b)\subseteq(y,b)$, we know that the second case would require B_t to be monotonic on $(a,\frac{a+b}{2})$ or $(\frac{a+b}{2},b)$. This is because

B_t must be monotonic on (a, y) and (y, b). And, just like in the first case, both of the events that B_t is monotonic on $(a, \frac{a+b}{2})$ or $(\frac{a+b}{2}, b)$ occur with probability 0.

Now by taking the countable union of intervals with rational endpoints we can see that B_t almost surely attains a local maximum in every such interval. Moreover, the density of $\mathbb{Q}$ in $\mathbb{R}$ allows us to construct an interval with rational endpoints in any real interval. Thus, we can always find a local maximum in any real interval. □

Not only does Brownian motion have a local maximum on every interval, but it also has uncountably many zeros.

Theorem 3.3. Zeros of Brownian Motion: Let B_t for $t \geq 0$ be a Brownian motion. Then, the zeros set of Brownian motion,

$$\text{Zeros} = \{t \geq 0 : X_t = 0\},$$

is a closed set with no isolated points.

Proof. Notice that $\{0\}$ is a closed set, which follows directly from the continuity of Brownian motion because the preimage of a closed set must be closed. Now allow us to construct $\tau_q = \inf\{t \geq q : X_t = 0\}$ for all positive rational q's. Since this is a closed set, τ_q is the minimum of the set. Also note that τ_q is almost surely a finite stopping time. Now using the strong Markov property (see theorem 3.5) at τ_q we get $X_{T+\tau_q} - X_{\tau_q}$ which is a new Brownian motion that begins at 0. Since any small interval to the right will contain 0, we know that almost surely τ_q is not isolated from the right for all positive rationals. Now pick a point called t in the zero set that does not correspond to some τ_q. We can pick a sequence of rational numbers, $q(n)$, that converges to t. But then the sequence $\tau_q(n)$ converges to t. Hence, t is not isolated from the left. □

This theorem alone is of course not enough to prove that Brownian motion has uncountably many zeros; the following theorem will fill in the missing pieces.

Theorem 3.4. A closed set, A, with no isolated points is uncountable.

Proof. The set A cannot be finite since it is a collection of limit points, so it is either countably or uncountably infinite. Suppose by contradiction that A is countably infinite. Then we can write $A = \{a_1, a_2, ...\}$. Now let us define U_n to be $(a_n - 1, a_n + 1)$. We can then construct sets where $\bar{U}_n$ is the closure of the set U_n such that for all n

(1) $\bar{U}_{n+1} \subseteq U_n$;
(2) U_n does not contain any points a_j for $j < n$;
(3) U_n contains a_n.

Then consider the set $V = \cap_{n \in \mathbb{N}} (\bar{U}_n \cap A)$. Notice that for all n the set $\bar{U}_n \cap A$ is compact. Now by Cantor's intersection theorem we know that V must contain a point. But by the construction of our sets that point must not be enumerated in $\{a_1, a_2, ...\}$. So, V must contain an element not in the list which means that A is uncountable. Thus, we have proved the theorem. □

So, we have proven one of the most beautiful properties of Brownian motion, but there is still more to explore. Another beautiful part of Brownian motion is the strong Markov property.

Theorem 3.5. Strong Markov Property: Let T be less than or equal to t. Now let Y_t denote the Brownian motion after time T. Then,

$$Y_t = X_{t+T} - X_T$$

and Y_t is an independent Brownian motion.

Proof. Due to definition 2.8 it immediately follows that

$$Y_t = X_{t+T} - X_T$$

and that Y_t is an independent Brownian motion. □

What makes the strong Markov property so beautiful despite its trivial proof is that it allows the observer to start their experiment at any time without them missing relevant information. Let's go back to the pollen-in-water example given earlier. Due to the strong Markov property the observer does not need to begin observing the pollen immediately after it is placed in the water. Instead, one can begin the experiment at any arbitrary point in time without missing necessary motion that occurred before. Using this property we can obtain the following interesting result.

Lemma 3.6. Given any $T \geq 0$, let $\mathcal{F}_T$ be all the information contained in the Brownian motion up through the stopping time T (see definition 2.19), and let $T \leq t$. Then, we know

$$E(X_t|\mathcal{F}_T) = E(X_t|X_T).$$

Proof. Allow X_t to be a standard Brownian motion. Let $\mathcal{F}_T$ represent all the information stored in X_T. Now let $T < t$ and consider the expected value of $E(X_t|\mathcal{F}_T)$. We can see that

$$E(X_t|\mathcal{F}_T) = E(X_T|\mathcal{F}_T) + E(X_t - X_T|\mathcal{F}_T).$$

Now since the change in Brownian motion is independent of the information that came previously, we can notice that

$$E(X_t - X_T|\mathcal{F}_T) = E(X_t - X_T) = 0.$$

Moreover, seeing as X_T is $\mathcal{F}_t$ measurable, we get that

$$E(X_T|\mathcal{F}_T) = X_T.$$

This in turn gives us the beautiful result that

$$E(X_t|\mathcal{F}_T) = X_T = E(X_t|X_T).$$

□

Another interesting result of the strong Markov property is the reflection principle.

Theorem 3.7. Let T be a stopping time and B_t a Brownian motion. Then,

$$B_t^* = \begin{cases} 2B_T - B_t & t \geq T \\ B_t & 0 \leq t \leq T \end{cases}$$

is a Brownian motion.

Proof. We know by the strong Markov property that $B_{t+T} - B_T$ is a Brownian motion independent of $\{B_t|0 \leq t \leq T\}$. That means that $B_T - B_{t+T}$ is as well. We can attach B_t for $0 \leq t \leq T$ to the beginning of $B_{T+t} - B_T$ for $0 \leq t$. The resulting object is B_t^*. Now we can check that this fits the definition of a Brownian motion. First of all, all the properties of a Brownian motion hold within the individual segments that we connected together. Second, the strong Markov property ensures the two segments are independent of each other. And last, continuity holds as the two processes are equal at time T when we connect them. Thus, we are done. □

Now not only can we shift a Brownian motion using the strong Markov property, but we can also scale a Brownian motion.

Lemma 3.8. Let X_t be a standard Brownian motion. If $a > 0$ and $Y_t = a^{-1/2} X_{at}$, then Y_t is a standard Brownian motion.

Proof. The continuity of paths and independence of increments remain unchanged after the scaling. Since $X_0 = 0$ we know that $Y_0 = 0$. Now we must observe that $Y_{t+\epsilon} - Y_t = a^{-1/2}(X_{a(t+\epsilon)} - X_{at})$ which gives us the necessary variance and distribution. Thus, Y_t is a standard Brownian motion. □

Lemma 3.9. (Time Inversion). Suppose Y_t is a standard Brownian motion. Then, the process defined by X_t where

$$X_t = \begin{cases} 0 & t = 0 \\ tY_{1/t} & 0 < t \end{cases}$$

is a standard Brownian motion.

Proof. We immediately see that the expected value of the increments of the process is 0. As for the other properties note that

$$Cov[Y_t, Y_{t+s}] = E[Y_t Y_{t+s}] = E[Y_t(Y_{t+s} - Y_t + Y_t)] = E[Y_t^2] + E[Y_{t+s}]E[Y_t - Y_s] = t.$$

Now applying that to X_t we get

$$Cov[X_t, X_{t+s}] = Cov[tY_{1/t}, (t+s)Y_{1/t+s}] = t(t+s)Cov[Y_{1/t}, Y_{1/t+s}] = (t+s)\frac{t}{t+s} = t.$$

That in turn gives us

$$Cov[X_t, X_{t+s} - X_t] = Cov[X_t, X_{t+s}] - Var[X_t] = t - t = 0.$$

Moreover we know that

$$Var[X_{t+s} - X_t] = Var[X_{t+s}] + Var[X_t] - 2Cov[X_{t+s}, X_t] = (t+s) + t - 2t = s.$$

So our increments have the correct variance. We can also notice that the independence of the increments holds because X_t and X_{s+t} have 0 covariance and are normal variables. Lastly, we must prove continuity. For $t > 0$ this is trivial, so we must only prove it for $t = 0$. Since the distribution of X_t over the rationals is the same for a Brownian motion we have that for $t \in \mathbb{Q}$

$$\lim_{x \to 0} X_t = 0.$$

And with that we have proved the theorem.

□

4. Fractional Brownian Motion

As useful as Brownian motion can be for modeling systems, there are many kinds of systems that act in ways that are not quite random enough to be considered Brownian motions. Specifically, many systems do not have the independence of increments despite fitting in with the other specifications of Brownian motion. We call this kind of motion fractional Brownian motion. In this section we will formally define fractional Brownian motion and discuss some of its most basic properties.

Definition 4.1. Fractional Brownian Motion: A fractional Brownian motion is a centered Gaussian process (see definitions 2.13 and 2.14) $\{B_t^H, t \geq 0\}$ with a covariance function

$$E\left[B_t^H B_s^H\right] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)$$

where $H \in (0,1)$. We call H the Hurst parameter.

Remark 4.1. Note that if $H = \frac{1}{2}$ then the fractional Brownian motion, $B^{\frac{1}{2}}$, is really just a standard Brownian motion. From this we can see that fractional Brownian motion is the more general case of Brownian motion when the Hurst parameter differs from $\frac{1}{2}$.

Lemma 4.1. A fractional Brownian motion B_t^H has stationary increments.

Proof. Fix $t \geq 0$ and consider the process $Y_t = B_{t+s}^H - B_t^H$. From the definition of fractional Brownian motion we can see that

$$E\left[\left(B_{t_1}^H - B_{s_1}^H\right)\left(B_{t_2}^H - B_{s_2}^H\right)\right]$$
$$= \frac{1}{2}\left(|t_1 - s_2|^{2H} + |t_2 - s_1|^{2H} - |t_1 - t_2|^{2H} + |s_1 - s_2|^{2H}\right).$$

From this we can see that the covariance of Y_t is the same as B_t^H. Both Y_t and B_t^H are centered Gaussian processes so the equality of their covariance means that they have the same distribution. Thus, the distribution of B_t^H will be the same at any time in the future. As such, it has stationary increments. $\square$

Definition 4.2. H-Self-Similar: A stochastic process X_t for $t \geq 0$ is said to be H-self-similar if for any $a > 0$, there exists $H > 0$ such that

$$\{X_{at}\} \stackrel{d}{=} \left\{a^H X_t^H\right\}.$$

Lemma 4.2. A fractional Brownian motion is H-self-similar.

Proof. Fix $a > 0$ and let B^H be a fractional Brownian motion with Hurst parameter H. Now consider the process $Z_t = B_{at}^H$ for $t \geq 0$. We can see from the definition of fractional Brownian motion that Z_t has the same covariance and consequently the same distribution as $a^H B^H$. Thus, we have finished the proof. $\square$

Lemma 4.3. A fractional Brownian motion has dependent increments.

Proof. Assume that $s_1 < t_1 < s_2 < t_2$ so that the intervals $[s_1, t_1]$ and $[s_2, t_2]$ do not intersect. Now let $a_1 = t_2 - s_1, a_2 = t_2 - t_1, b_1 = s_2 - s_1, b_2 = s_2 - t_1$ and $f(x) = x^{2H}$. From here we can write

$$E\left[\left(B_{t_1}^H - B_{s_1}^H\right)\left(B_{t_2}^H - B_{s_2}^H\right)\right] = \frac{1}{2}\left(\left(f(a_1) - f(a_2) - \left(f(b_1) - f(b_2)\right)\right)\right).$$

We also know that $a_1 - a_2 = b_2 - b_1 = t_1 - s_1$. Combining all this with the fact that f is convex in this case gives us

$$E\left[\left(B_{t_1}^H - B_{s_1}^H\right)\left(B_{t_2}^H - B_{s_2}^H\right)\right] < 0 \text{ for } H \in \left(0, \frac{1}{2}\right)$$

and

$$E\left[\left(B_{t_1}^H - B_{s_1}^H\right)\left(B_{t_2}^H - B_{s_2}^H\right)\right] > 0 \text{ for } H \in \left(\frac{1}{2}, 1\right).$$

Thus, we have proved the lemma. $\square$

Intuitively this means that for $H \in \left(0, \frac{1}{2}\right)$, if the fractional Brownian motion was decreasing in the past it is likely to increase in the future and vice versa. This is called counterpersistence. On the other hand, for $H \in \left(\frac{1}{2}, 1\right)$ then if the motion was increasing in the past it is likely to continue doing so. The same holds true if the motion was decreasing in the past. We call this persistence.

5. Acknowledgements

I would like to thank my advisor Nixia Chen for all her hard work helping me with this project. I'd also like to thank Daniil Rudenko and Peter May for making the University of Chicago Apprentice REU possible. It was incredibly fun and definitely convinced me to pursue math research. And finally I'd like to thank Gregory Lawler for the lectures and texts that were instrumental in creating this project.

References

[1] Lawler, Gregory F. An Introduction to Stochastic Processes. Chapman and Hall. 2006.
[2] Ross, Sheldon M. Stochastic Processes. John Wiley & Sons Inc. 1996.
[3] Capiński, Marek and Kopp, Ekkehard. Measure, Integral, and Probability. Springer. 2004.
[4] Hunter, John K. Measure Theory.
`https://www.math.ucdavis.edu/~hunter/m206/measure_notes.pdf`
[5] Dahl, Andy. A Rigourous Introduction to Brownian Motion. 2010.
`http://www.math.uchicago.edu/~may/VIGRE/VIGRE2010/REUPapers/Dahl.pdf`
[6] Leiner, James. Brownian Motion and the Strong Markov Property. 2012.
`http://math.uchicago.edu/~may/REU2012/REUPapers/Leiner.pdf`
[7] Davis, Richard A. Gaussian Processes. 2006.
`http://www.stat.columbia.edu/~rdavis/papers/VAG002.pdf`
[8] Shevchenko, Gregoriy. Fractal Brownian Motion in a Nutshell. 2014.
`https://arxiv.org/pdf/1406.1956.pdf`
[9] Van Ness, J.W. Fractional Brownian Motions, Fractal Noises, and Applications. 1968.
`https://pdfs.semanticscholar.org/fe5c/19465b9e0488c19cba8e42120af3b362a495.pdf`
[10] NIST/SEMATECH. e-Handbook of Statistical Methods. 2012.
`http://www.itl.nist.gov/div898/handbook/`
[11] Pishro-Nik, Hossein. Introduction to Probability, Statistics, and Random Processes. 2014.
`https://www.probabilitycourse.com/`
[12] Lalley, Stephen P. Brownian Motion.
`https://galton.uchicago.edu/~lalley/Courses/313/BrownianMotionCurrent.pdf`
[13] Lalley, Stephen P. Introduction to Gaussian Processes.
`https://galton.uchicago.edu/~lalley/Courses/386/GaussianProcesses.pdf`