THE LICKORISH-WALLACE THEOREM

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Abstract. This paper presents a proof of the Lickorish-Wallace Theorem, a theorem that establishes a relationship between any closed connected orientable 3-manifold and S^3, as well as establishing a method to simply descibe any closed connected orientable 3-manifold. It begins with a short presentaton of important definitions relevant to the theorem and its proof, after which the proof and the machinery necessary for it (in particular, the notions of handlebodies, twists, and twist equivalence) are presented, largely following the method of proof presented by W.B. Raymond Lickorish in his book *An Introduction to Knot Theory*.

Contents

Introduction

This paper explores the Lickorish-Wallace Theorem, stated below, and a proof of the theorem. This proof largely follows the proof presented by W.B. Raymond Lickorish in chapter 12 of his book *An Introduction to Knot Theory* [2]. Before stating this theorem, a defintion is needed:

Definition 0.1. An *elemental r-surgery* is an operation on an n-mainfold M that is accomplished by removing from M an embedded

$$S^r \times D^{n-r},$$

and replacing it with a copy of

$$D^{r+1} \times S^{n-r-1}.$$

The replacement is made along the standard homeomophism between the boundary of $S^r \times D^{n-r}$ and of $D^{r+1} \times S^{n-r-1}$, taking advantage of the simple relationship between S^k and D^{k+1}. A *surgery* in general is then a sequence of these elemental surgeries.

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Here we will consider only 1-surgeries on 3-manifolds. This amounts to removing a torus from the 3-manifold in question, and replacing with an alternative parameterization of that torus, along the previously described homeomorphism.

With this definition, we are able to present the theorem that is the focus of this paper:

Theorem 0.2 (Lickorish-Wallace Theorem)**.** *Any closed connected orientable 3-manifold M can be obtained from S^3 by a collection of 1-surgeries.*

In addition to the insight that this theorem gives into the relationship between closed connected orientable 3-manifolds and S^3, this theorem gives a simple way to describe any such manifold. The proof of this theorem, and the machinery necessary for it, comprise much of the remainder of this paper. However, first a number of definitions pertaining to topological equivalence will be given for the sake of the reader:

Definition 0.3. A function $f\colon X \to Y$ is a *homeomorphism* if f is bijective, continuous, and has a continuous inverse. Two spaces X and Y are *homeomorphic* if there exists a homeomorphism between them.

Homeomorphisms are the isomorphisms of topological spaces, in that they preserve all the topological properties of any given space.

Definition 0.4. A *homotopy* between two continuous functions f and g from topological space X to Y is a continuous function

$$H\colon X \times [0,1] \to Y$$

such that

$$H(x,0) = f(x)$$

and

$$H(x,1) = g(x)$$

for all $x \in X$. Two functions are *homotopic* if there exists a homotopy between them.

A homotopy can be thought of as a continuous deformation between the two functions (which can, for example, represent objects by way of functions that embed these objects into some space).

Definition 0.5. Two spaces X and Y are said to be *homotopy equivalent* or of the same *homotopy type* if there existcontinous maps in both directions between them such that the composition of these maps is homotopic to the identity map in each space.

We can see that every homeomorphism is a homotopy equivalence by the definition of a homeomorphism. However, the inverse of this is not the case, since the bijectivity required of a homeomorphism is not gauranteed by homotopy equivalence.

Definition 0.6. An *isotopy* is a homotopy H between two embeddings $f, g\colon X \to Y$ such that for any $t \in [0,1]$, the function $H(x,t)$ of x is also an embedding.

It should also be noted that in a number of places in what follows, smooth manifolds, or piecewise linear topology, will be assumed where relevant, and consquences of this taken as fact without proof. This is done because the proofs are somewhat tangential to the important concepts that serve as the core of the proof of the

Lickorish-Wallace Theorem, and are beyond the scope of this paper. Proofs for these topics can be found in John M. Lee's *Introduction to Smooth Manifolds* [1] and Colin P. Rourke and Brian Joseph Sanderson's *Introduction to Piecewise-linear Topology* [3].

1. Handlebodies

The first notion that will be introduced here to prove the Lickorish-Wallace Theorem is that of a handlebody, and subsequently that of a Heegaard splitting of a manifold into a pair of handlebodies. The Heegaard splitting of a 3-manifold, and in particular the shared boundary of the two handlebodies it creates, forms the central idea of the proof of the Lickorish-Wallace Theorem.

Definition 1.1. For an n-manifold M, the operation of attaching an *r-handle* is done by joining the manifold M with a copy of $D^r \times D^{n-r}$ along the embedding of the boundary of the "handle" into the boundary of M (or the boundary created by removing two copies of D^r from M),

$$e \colon \partial D^r \times D^{n-r} \to \partial M.$$

A *handlebody* is then defined to be a 3-ball with 1-handles attached. The number g of handles that are attached determines the genus of the handlebody.

This parallels the notion of genus of a surface, as a handlebody aligns with the intuitive definition of a "filled surface", so for example, a filled torus can equivalently be referred to as a genus 1 handlebody.

This definition is largely in line with what one might have as intuition for a "handle"; attaching a 1-handle in particular results in a handle in the same sense as a handle on a mug.

Definition 1.2. A *Heegaard splitting* of a closed connected orientable 3-manifold M is a set of two handlebodies $U, V \subset M$ such that

$$U \cup V = M,$$

but also

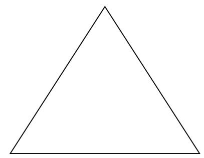
$$U \cap V = \partial U = \partial V.$$

In other words, these two handlebodies together make up M, but are disjoint with the exception of their shared boundary. As these two handlebodies share a boundary surface of some genus g, it then follows that both U and V are handlebodies of genus g.

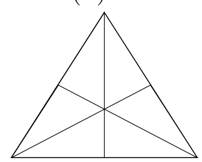
We will now show that M (or, any closed connected orientable 3-manifold) has a Heegaard splitting. We will then show that regardless of the nature of the handlebodies of this splitting, there is a homeomorphism between the boundaries of these handlebodies such that joining them along that homeomorphism results in a copy of S^3. This will serve as the basis of the relationship between M and S^3 used in the remainder of the proof.

Theorem 1.3. *Any closed connected orientable 3-manifold M has a Heegaard splitting.*

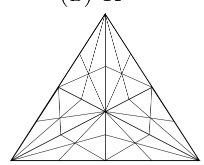
Proof. To do this, take a triangulation of M to a simplicial complex K. Take the second derived subdivision of this complex, $K^{(2)}$, where the derived subdivision of



(A) K



(B) $K^{(1)}$



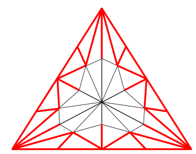
(C) $K^{(2)}$

FIGURE 1. An example of a simplicial complex K, and its first and second derived subdivisions.

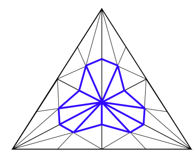
K is a subdivision of K that has vertices at the barycenters of the simplices of K (hence dividing 1-simplicies into two 1-simplicies, and so on), as shown in Figure 1.

Now, consider the 1-skeleton of K, the set of 1-simplcies and 0-simplices of K. If we consider this as a graph, with 0-simplices as vertices and 1-simplicies as edges, then we have the notion of a tree that spans this graph, since the 1-skeleton of K must be a connected graph if M is connected. Consider the neighborhood of this tree in $K^{(2)}$, in the sense of a neigborhood of a subgraph (as in Figure 2a), where K here is considered as a subgraph of $K^{(2)}$. This neighborhood is a simplcial complex and a subset of $K^{(2)}$. However, because we are considering the neighborhood of a tree, which must by definition have no cycles, this simplicial complex must therefore be genus 0, and therefore can then be taken to be S^3. The neighborhood of the remaining 1-simplices of K not in the tree can be taken to then be handles, which when joined with S^3, form a handlebody that is the neighborhood of K in $K^{(2)}$. An example of this for a 2-simplex can been seen in Figure 2a.

The closure of the complement of this handlebody in $K^{(2)}$ is, by nature of the second derived subdivision, also the neighborhood of a graph. Specifically, this graph is the union, over all 3-simplexes A of K, of cones C_A with vertex at the barycenter of A, over the barycenters of the 2-simplex faces of A. Then, since K is closed, it follows that the union of 2-simplices of K is connected. Because K is a triangulation of the 3-manifold M, each of these 2-simplices are faces of 3-simplices of K, and so this second graph is also connected. Additionally, by the same argument as above, the neighborhood of this graph is also a handlebody. Because of how the second handlebody was constructed in relation to the first, these two handlebodies have a common boundary, and therefore make up a Heegaard splitting of M. □



(A) The neighborhood of K in $K^{(2)}$



(B) The closure of the complement of (a).

FIGURE 2

2. Twists and Twist Equivalence

We now have that a closed orientable 3-manifold M has a Heegaard splitting into handlebodies U, V with the associated homeomorphism $h\colon \partial U \to \partial V$. The next step is to show the previously mentioned connection between these handlebodies (U and V) and S^3.

Let $p'_1, p'_2, \ldots, p'_g$ be disjoint simple closed curves on ∂U, each bounding a disc in U and "passing through" one of the holes of ∂U, analogous to a meridian of a torus, as seen in Figure 3a. Also, let $q_1, q_2, \ldots, q_g$ be disjoint simple closed curves on ∂V, each "around" a different hole of ∂V, analogous to a longitude curve of a torus, as seen in Figure 3b.

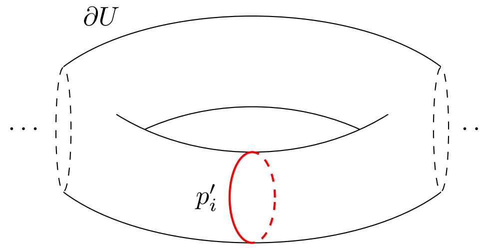
The manner in which we have chosen these curves then ensures that for a homeomorphism $h'\colon \partial U \to \partial V$, if $h'(p'_i) = q_i$ for $1 \leq i \leq g$, then

$$U \cup_{h'} V = S^3.$$

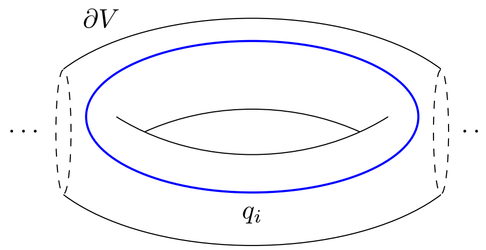
That this is the case is clear when $g = 1$, as this is the normal splitting of S^3 into two solid tori associated along their boundary. For other g, ∂U and ∂V are genus g surfaces, or the connected sum of g tori, T_{q_i} and T_{p_i}. If we consider each of these separately, we can choose h' such that the boundaries of the excluded discs for each connected sum are identified between T_{q_i} and T_{p_i}. Then the union of these tori, each with a disc excluded, gives S^3 with a copy of D^3 excluded. Then the connected sum of these gives S^3, as desired.

Now let $h(p'_i) = p_i$, a simple closed curve on ∂V, recalling that h is the homeomorphism associating ∂U and ∂V for the Heegaard splitting of M. If there is a homeomorphism on V that sends p_i to q_i, then h satisfies the requirements for h', meaning that M is S^3, and we are done. Otherwise, we need to introduce the notion of a twist.

Definition 2.1. For a simple closed curve C embedded in a connected compact oriented surface F, a *Dehn twist* (which from this point on will be referred to simply as a twist) about C is a homeomorphism that is isotopic to the homeomorphism



(A) An example of p_i' on ∂U.



(B) An example of q_i on ∂V.

FIGURE 3

$\tau\colon F \to F$ that is the identity outside of an annulus A on F, and on A,

$$\tau(e^{i\theta}, t) = (e^{i(\theta - 2\pi t)}, t),$$

where the coordinates are with respect to A when parameterized as $S^1 \times [0, 1]$. The effect of this is what one would expect from the name: the boundary of the annulus is unaffected, while the interior of the annulus is swept (or twisted) around the extent of the S^1 component of the annulus.

For example, twist about a meridian of a torus takes a longitude curve of the torus to a curve that goes around the torus, but also passes once through the hole, as show in Figure 4.

Additionally, we say that two oriented simple closed curves p, q on the interior of the surface of F are *twist equivalent*, written $p \sim_\tau q$, if $g(p) = q$ for a homeomorphism g of F generated by twists of F.

The next step will be to show that given two sets of n disjoint simple closed curves in the interior of a surface, not only is each curve twist equivalent to its partner in the other set, but in fact there is some single homeomorphism of the surface generated by twists that takes each curve to its partner in the other set. This will be shown in a number of steps.

Theorem 2.2. *Given simple oriented closed curves p and q on the interior of a surface F that intersect transverely at a single point, $p \sim_\tau q$.*

Proof. To show this, consider two additional curves, C_1 and C_2, that are slightly shifted copies of q and p on F, respectively (in particular, they should be chosen

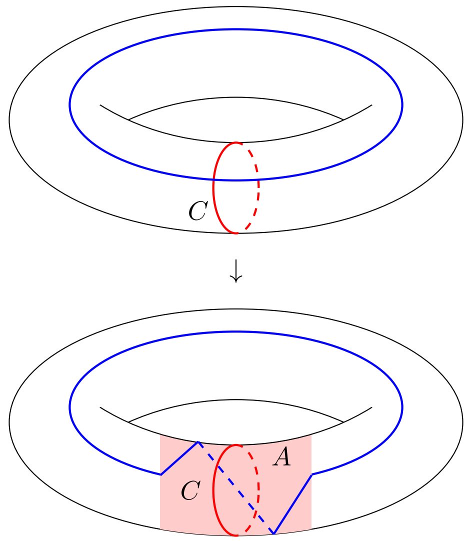


FIGURE 4. A twist about the curve C on a torus.

such that C_1 intersects p transversely exactly once, and vice versa). An example of such curves, as well as images depicting the rest of the proof for such an example, can be seen in Figure 5. Then twist p about C_1. This twist of p, $\tau_1 p$, closely follows the entirety of C_1 within the annulus of the twist, and is otherwise identical to p. Thus, within the annulus, $\tau_1 p$ is a slightly shifted copy of q. Now twist this resulting curve about C_2, resulting in $\tau_2\tau_1 p$. Since C_2 is a shifted copy of p, this twist can be performed such that it only impacts the potion of $\tau_1 p$ that closely follows q as a result of τ_1. Thus, a portion of this section is twisted to run parallel to p (and thus parallel to the remaining, unaffected portion of $\tau_2\tau_1 p$). However, this portion will have the opposite orientation, as both twists have the same "handedness". Thus, these portions of $\tau_2\tau_1 p$ make up a "kink" that, because the twist preserves the closedness of the curves, must double back upon itself, and so can be eliminated by a homeomorphism isotopic to the identity, an operation allowed in twist equivalence. The issue of matching the orientation of q rather than being opposite can be solved by using two twist inverses instead of twists, which does not impact the rest of the argument. Thus, for such p and q, $p \sim_\tau q$. $\square$

Theorem 2.3. *Given two oriented simple closed curves p, q on the interior of the surface of F that are disjoint (rather than intersecting transversely at a single point), where neither curve separates F, then $p \sim_\tau q$.*

Proof. Cut F along $p \cup q$. If this separates F (for example, if p and q are both meridians of a torus F), then let r be a simple oriented curve that begins on p and goes to q on one piece, and then continues on the other piece of F, returning to its original point on p. Then r is a simple oriented closed curve that intersects each of p and q transversely at a single point. Therefore, by the previous theorem, $r \sim_\tau p$, and also $r \sim_\tau q$, so then $p \sim_\tau q$. If $p \cup q$ does not separate F, then as before,

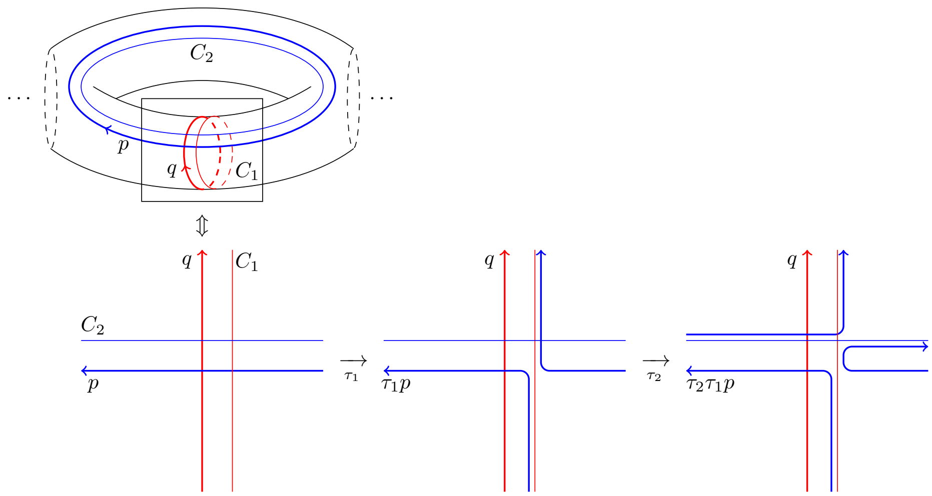


FIGURE 5. An example of the method used to prove Theorem 2.2.

let r be a simple oriented curve that begins on p and goes to q, but in this case simply intersects it transversely and then returns to its starting point on p such that it intersects p transversely as well, which will always be possible because F is not separated by $p \cup q$. Therefore, again using the previous theorem, $r \sim_\tau p$ and $r \sim_\tau q$, so $p \sim_\tau q$. □

Theorem 2.4. *Suppose that p and q are oriented simple closed curves on the interior of F that do not separate F (but now need not necessarily be disjoint). Then $p \sim_\tau q$.*

Proof. For any intersection that is not transverse, we can alter, by means of a homeomorphism isotopic to the identity, one of the curves to eliminate that intersection. Then we have that p and q intersect, always transversely, at n points. The cases of $n = 0$ and $n = 1$ have already been considered in the previous theorems. That $p \sim_\tau q$ for $n \geq 2$ will be shown by induction. As we are considering finitely many transverse intersections, and p is oriented, we can take two points p_0 and p_1 as consecutive points on p belonging to $p \cap q$. Now we will consider two cases.

In the first, p has the same orientation with respect to q at both p_0 and p_1 (that is, p crosses q in the "same direction" at both intersections). Then let r be a simple closed curve on the interior of F that starts near p_0, then follows close to p, approaching p_1, where it then returns to its starting point. Then r is a slightly shifted copy of p between the p_0 and p_1, except in a neighborhood of q. Because p is a simple curve and we have chosen r to follow it closely except in a neighborhood of q, r need not intersect p except in that neighborhood. Furthermore, r can be chosen such that it intersects p fewer than n times, since it need only cross p whenever there are intersections of p and q that lie between (in the context of q) p_0 and p_1, and need cross p near p_0 or near p_1, but not both, resulting in fewer than n intersections between r and p. Also important to note is that r does not separate F as a result of the fact that outside of the neighborhood of q, the portion of p

that it follows intersects q twice with the same relative orientation. Then $r \sim_\tau p$ by the induction hypothesis, and, as r intersects q transversely, exactly once, it also follows from Theorem 2.2 that $r \sim_\tau q$.

In the second case, the orientation of p at p_0 and p_1 is opposite, with respect to the orientation of q. In this case, let r_1 and r_2 be simple closed curves that both begin near p_0, follow p closely without intersecting it until approaching p_1, and then return to their starting point by following q. However, r_1 begins to the left of p, while r_2 begins to its right. Because p intersects q with opposite relative orientation at p_0 and p_1 and r_1, r_2 do not intersect p while following it, they remain on their respective sides of p. Suppose both of these curves separate F, and consider the portion of $r_1 \cup r_2$ that is near q. If we close this union by connecting the two curves with short segments across p near p_0 and p_1, then this new curve is isotopically homeomorphic to q. However, joining two curves that separate F in this manner must create a curve that also separates F, thus creating a contradiction, since we require that q not separate F. Therefore, at least one of these curves does not separate F; let this curve be r. Again by the inductive hypothesis (since, similarly to before, r must cross p fewer than n times) and a similar argument to before, $r \sim_\tau p$, and since r and q are disjoint, we have already shown as well that $r \sim_\tau q$.

Thus in either case, $p \sim_\tau q$. □

Theorem 2.5. *For a set of disjoint simple closed oriented curves $\{p_1, p_2, \ldots, p_n\}$ on the interior of F with $\cup p_i$ not seperating F, and another such set $\{q_1, q_2, \ldots, q_n\}$, there is some homeomorphism h generated by twists such that for each $1 \leq i \leq n$, $hp_i = q_i$.*

Proof. By the previous theorem, this holds for $n = 1$, and subsequent cases can be shown by an induction on n. Assume that such a homeomorphism h' exists for p_i, q_i, with $1 \leq i \leq n-1$ and cut F along $q_1 \cup \cdots \cup q_{n-1}$. Since p_n is disjoint from all other p_i, and $h'p_i = q_i$ for all other i, then it follows that $h'p_n$ does not intersect any of q_i for $1 \leq i \leq n-1$, and so $h'p_n$ is an oriented simple closed curve that does not separate F, as is q_n. We may therefore appy the previous theorem to these two curves, so they must be twist equivalent. An annulus in the interior of this cut copy of F must also be an annulus on F, and so this twist has no impact on $h'p_i$ and q_i for $1 \leq i \leq n-1$. Then $\tau \circ h'$, where τ here is used to mean the series of twists that equate $h'p_n$ and q_n, satisfies all the requirements for the desired homeomorphism h. □

3. The Lickorish-Wallace Theorem

Finally we can return to the initial theorem. At this point we have a splitting of M into two handlebodies U and V, with their boundaries associated by the homeomorphism h. We have also shown that there exists a homeomorphism h', such that $U \cup_{h'} V = S^3$. Now, recall the curves p_i' and q_i that were constructed in order to construct h' and the curves p_i' given by $h(p_i') = p_i$. As stated before, if there is a homeomorphism on V that sends p_i to q_i, then M must be homeomorphic to S^3 and we have nothing further to prove. Otherwise, we will make use of twists to make the connection between these two sets of curves.

By Theorem 2.5, $p_i \sim_\tau q_i$ for $1 \leq i \leq g$. Call the homeomorphism on ∂V associated with this equivalance ψ. Each of the twists τ that generates ψ is supported on an annulus A on ∂V. Now consider a collar neighborhood of ∂V, homeomorphic to

$$\partial V \times [0, 1],$$

where ∂V is identified with $\partial V \times 0$. We can extend each τ as $\tau \times 1$ on $A \times [0, 1/2]$, and as the identity on the closure of $V - (A \times [0, 1])$. The we have an extension of τ to all of V, with the exception of solid torus homeomorphic to $A \times [1/2, 1]$. Because then extension of τ need be $A \times [0, 1/2]$ only up to homeomorphism, we are free to vary how "deep" within V the tori are located, as this "depth" is just the component of the product space homeomorphic to the interval $[0, 1/2]$. This freedom allows us to ensure that the solid tori corresponding to the different twists that realize the twist equivalence are disjoint. Therefore, we are able to ensure that the extension of ψ to V (with the exception of these tori), the product of the extensions of the τ that generate it, is supported except on a collection of *disjoint* solid tori. Therefore, at the cost of these solid tori, we are able to map V to V by a homeomorphism (ψh) that sends p_i to q_i, and so joining U and V along this homeomorphism gives S^3, with the solid tori removed.

The last step in this proof is to note the behavior of the mapping τ, and in particular its effect on the solid torus $A \times [1/2, 1]$. Consider a meridian on the boundary of this solid torus (or equivalently, the boundary of a meridian disk of the torus), which, if we parameterize the solid torus as $S^1 \times D^2$, we can express as ∂D^2. Then τ maps this to a curve homologous to a longitude, plus some number of meridians, on the boundary of the solid torus. We can express this as S^1. Since this preserves the boundary of the solid torus, but changes the parameterization of the solid torus from $S^1 \times D^2$ to $D^2 \times S^1$, we can see that the boundary of the torus is affected by τ in exactly the same way as performing a 1-surgery on the manifold. Since we have been omitting the interior of the solid tori when considering the homeomophism between our manifold M and S^3, this means that a series of 1-surgeries, one on each of the solid tori constructed above in relation to each of the necessary twists, is sufficient to transform M into a manifold which, by the homeomorphism which has been otherwise outlined previously, is homeomorphic to S^3. This finally gives us the desired result:

Theorem 3.1 (Lickorish-Wallace Theorem)**.** *Any closed connected orientable 3-manifold M can be obtained from S^3 by a collection of 1-surgeries.*

Acknowledgements

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