ENRICHED 2-MONADS AND CODESCENT OBJECTS

SAMUEL HSU

Abstract. In this expository article we review some enriched 2-monad theory and homotopical motivations for studying it, primarily from Guillou–May–Merling–Osorno's works. After a very chatty introduction, we recall a definition of enriched 2-categories and some basic 2-categorical notions, along with their enriched counterparts. After that, we specialize to when the base $\mathscr{V}$ is a symmetric monoidal closed 2-category with (co)descent objects and coproducts. From there, we outline an enriched analogue of Lack's coherence theorem. Finally, we find that for the context we are interested in, the desired codescent objects exist and an enriched Lack's coherence theorem holds. **Note:** This is a revision, primarily to correct some errors and include a bit more motivation cf. Section 1.5.

Contents

1. Introduction

Section 1.1 gives an algebraic discussion of why we might care about enriched (lax) codescent, and Section 1.3 discusses motivations from the Barratt–Priddy–Quillen theorem. Although they are not exactly independent, the reader should of course feel free to skip to what interests them more.

1.1. **General motivation.** Monads, which made an early appearance via the Godement resolution in sheaf theory, have been found in a wide variety of applications: in general algebra as a type of algebraic theory, in computer science to model computational effects, in homological algebra in forms such as monadic homology, in topos theory with results deduced from the monadicity of certain functors or fibrational slices, in category theory itself,* and even in analysis where commutative monads give rise to notions of distributions and their theory. Here,

Date: 2019 REU, revised 2021.

*This includes definitions of generalized multicategories [CS10], and algebraic definitions of n-category via opetopic sets, globular operads, free objects in Penon's category $\mathcal{Q}$, and generalizations of the above to the cubical and multiple setting. Many of these use monads in a setting more general than the 2-category of categories.

we concern ourselves with extending the idea of monads as a way of presenting algebraic theories. We assume familiarity with basic category theory, including enriched 1-categories. Basic notions of category theory may be found in [ML98], and the standard reference for category theory enriched over a monoidal category is [Kel82]. Since we make use of 2-limits, a brief sketch is given in Section 3 and a more thorough explanation may be found in Chapter 6 of [Lac09].

Our algebraic theories are presented by some categorical structure, and therefore may be formulated in different variants of category theories. For example, PROs, sketches, and monads can be defined in the enriched setting. This is not generalization for the sake of abstraction, as the naturally occurring structures that a theory might try to describe tend to force us to consider enriched, internalized, fibered, or other cases. The fibered case appears rather frequently in computer science, often under the adjective "parametrized". For enriched structures, May's original operads [Kel05] were already enriched over symmetric monoidal categories, and the study of algebraic structures like abelian groups or commutative rings as algebras for a $\mathscr{V}$-adjoint commutative monad on a closed symmetric monoidal category $\mathscr{V}$ shows that there is a notion of tensor product provided that the base of enrichment admits coequalizers as shown in [Kei78]; Anders Kock has proved that if the base of enrichment admits equalizers then the category of algebras is closed, so combining both results we can understand where tensor products come from in less familiar settings. Lastly, internalized structures appear in a characterization of Shannon entropy using lax algebra morphisms in topological categories.[†]

Scholium 1.1. In addition to operads and monads, various gadgets like Lawvere theories, sketches, PROPS, caterads, algebrads, and others have been proposed over the years as some way of encoding or presenting a theory. The two works [Fuj18] by Fujii and [Ave17] by Avery have individually proposed unified frameworks. Fujii's framework is based on the observation that most of the listed notions are really monoids internal to some monoidal category, and in fact defines a *meta-theory* to be a large monoidal category, while Avery's thesis develops a construction which he calls a *proto-theory*. A proto-theory is defined as a 1-morphism $f : A \to B$ in a 2-category powered over $\underline{\underline{\mathrm{Cat}}}$ and equipped with a factorization system $(\mathcal{E}, \mathcal{M})$ on the underlying 1-category, so that $f \in \mathcal{E}$. The idea is that A represents the shapes or arities of a theory, and B represents the operations in that theory. Martin Hyland has also written about a comparison using Kleisli 2-categories in [Hyl14], as was pointed out to me by Eugenia Cheng. Berman appears to tie in PROPS, theories, and operads with categorified commutative algebra and connective spectra in [Ber18].

To put it in a way that emphasizes the formal category theory more, Avery's framework lends itself more easily to regular 2-categories, Hyland's theory fits into KZ-doctrines and Yoneda structures [Wal18], while Fujii already invokes proarrow equipments. The previous statement shouldn't be taken too seriously, and the notions interact e.g. in a setting like any existing notion of 2-topos where the factorization system induced by taking the codensity monad of a map within bimodules/(co)discrete cofibrations is compatible with the one from a regular 2-category. In such contexts, we can write down, for example, a monad-theory correspondence, in a vein similiar to [BMW12].

[†]The algebras here are algebras for the operad Δ, which has $\Delta(n) \coloneqq \Delta_n$, the standard n-simplex, and composition given by weighted sums.

Despite a rather well-developed theory of monads in algebra, there are convincing reasons to move up at least one dimension. We might be interested in algebras which live at category level 2 rather than category level 1, in the sense that the category of algebras should be something akin to categories with structure as opposed to sets with structures. For example, if we study programming language semantics, then we might be interested in collections of categories with structures. Or perhaps we care about the semantics of modal logic, which involves diagrams of categories, functors, and transformations. We might wish to explore categories in which the very useful statement that filtered colimits commute with finite limits holds as in [Fra11]. Similarly, we could be intent on studying *higher theories* either for reasons similar to the motivations for general algebra, or we might be attempting to chase down the "doctrine" in which we formulate ordinary structures. To illustrate, note that we can define monoids and bialgebras in a general monoidal category, but we cannot do so for a group because of the need for duplication of variables. A key difference in the structures that are put on a category here is that $\otimes$ in general might not come with a transformation with components $\Delta_A : A \to A \otimes A$. Another important difference is that in general, $\otimes$ is *additional stuff-like structure* on top of a category while the Cartesian product $\times$ on a category is *additional property-like structure*.[‡] Furthermore, when we specify something like associativity or commutativity for our internal algebras, we end up using an associator or a braid map, which can be viewed as a categorified version of that same property! A few examples of algebras as given in [BKP89] includes closed categories, topoi,[§] and categories with certain classes of colimits. In fact, even the Shannon entropy example is 2-categorical, since the laxness of the algebra morphisms uses the 2-cells in topological categories.

The corresponding models of our monads as theories would be algebras for the monads. We first sketch the non-enriched theory as worked out in [Lac02]. In the 2-dimensional setting there are several versions of 2-categories of T-algebras: (1) $\underline{\underline{T\text{-}\mathrm{Alg}_{\mathrm{s}}}}$ which consists of the strict algebras, strict morphisms, and transformations, (2) $\underline{\underline{\mathrm{Ps}\text{-}T\text{-}\mathrm{Alg}}}$ having pseudoalgebras, pseudomorphisms, and transformations, and (3) $\underline{\underline{\mathrm{Lax}\text{-}T\text{-}\mathrm{Alg}_{\ell}}}$ which is comprised of the lax algebras, lax morphisms, and transformations. Following the convention of [Lac02], strict means that the diagrams commute up to equality, pseudo means that the diagrams commute up to an isomorphism 2-cell, and (co)lax means that the diagrams commute only up to a 2-cell. For example, a lax morphism of algebras will consist of 1-cell $f : A \to B$ and a 2-cell ϕ drawn as

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d] \arrow[dr, phantom, "\swArrow_{\phi}"] & TB \arrow[d] \\ A \arrow[r, "f"'] & B \end{tikzcd} \tag{1.2}$$

[‡]One way to make this precise is to note that the homotopy fiber above the forgetful functor from categories with products to categories is almost a truth value, as once a product exists in a category the fiber is always contractible, while the homotopy fiber above the forgetful functor from categories with monoidal product to categories is in general a 1-category.

[§]This has to be with respect to the $(2, 1)$-category of categories, functors, and natural isomorphisms because Cartesian-closedness prevents us from allowing non-invertible natural transformations. Perhaps we could look at 2-monads acting on 2-categories equipped with an involution as studied by Shulman in [Shu18] or look at double monads on equipments.

satisfying two equalities of pasting diagrams. The additional layer of complexity contributed by the 2-cells gives rise to some interesting phenomena extending the 1-dimensional case. Some examples of these phenomena include the pseudodistributive laws of [CHP04] and [Gar08], and the pseudocommutative laws of [CG]. However, the coherence cells can become rather unwieldy if we always work in the absolute weakest environment possible. But similar to how a 2-category may be strictified into a strict 2-category,[¶] it is in this scenario that we wish for a *coherence result*, which allows us to work with a class W' of partially strictified widgets such that all results about W' apply to the class W of fully weak widgets. In our case, we might wonder when the inclusions $\underline{\underline{T\text{-Alg}_{\mathrm{s}}}} \hookrightarrow \underline{\underline{\text{Lax-}T\text{-Alg}_{\ell}}}$ or $\underline{\underline{T\text{-Alg}_{\mathrm{s}}}} \hookrightarrow \underline{\underline{\text{Ps-}T\text{-Alg}}}$ admit a left adjoint S, since when this is the case a lax morphism $f : A \to B$ of algebras corresponds to a strict morphism $\overline{f} : SA \to B$ which Lack calls the "strict morphism classifier". Our question now becomes

Question 1.3. *Under what hypotheses does this left (2-)adjoint exist? And when is its unit an equivalence?*

1.2. **(Lax) codescent objects.** Referring back to the one dimensional case, let T be a 1-monad with unit η and multiplication μ. Recall that a presentation for a T-algebra $\alpha : TA \to A$ always exists as exhibited by the split coequalizer pair

$$
\begin{tikzcd}
T^2A \arrow[r, shift left=2, "T\alpha"] \arrow[r, shift right=2, "\mu_A"'] & TA \arrow[l, "T\eta_A" description] \arrow[r, "\alpha"] & A
\end{tikzcd}
\tag{1.4}
$$

since η_A satisfies $\alpha \circ \eta_A = \mathrm{id}_A$. This gives a presentation of a T-algebra in terms of something analogous to a surjective morphism. Note that the split pair is actually a truncated simplicial object.

Back to when T is a 2-monad, to construct a left adjoint, Lack first examines when $\underline{\underline{T\text{-Alg}_{\mathrm{s}}}}$ admits 2-colimits. Next, he rewrites the morphisms of $\underline{\underline{\text{Lax-}T\text{-Alg}_{\ell}}}$ in terms of $\underline{\underline{T\text{-Alg}_{\mathrm{s}}}}$, so that the datum of 2-cells is made explicit. The truncated simplicial object which is analogous to the presentation in (1.4) is

$$
\begin{tikzcd}
T^3A \arrow[r, shift left=3, "T^2\alpha"] \arrow[r, "T\mu_A" description] \arrow[r, shift right=3, "\mu_{TA}"'] & T^2A \arrow[r, shift left=2, "T\alpha"] \arrow[r, shift right=2, "\mu_A"'] & TA \arrow[l, "T\eta_A" description]
\end{tikzcd}
\tag{1.5}
$$

along with five 2-cells

$$
\begin{aligned}
\delta &: T\alpha \circ T\eta_A \to \mathrm{id}_{TA} \\
\gamma &: \mathrm{id}_{TA} \to \mu_A \circ T\eta_A \\
\kappa &: T\alpha \circ T^2\alpha \to T\alpha \circ T\mu_A \\
\lambda &: \mu_A \circ \mu_{TA} \to \mu_A \circ T\mu_A \\
\rho &: \mu_A \circ T^2\alpha \to T\alpha \circ \mu_{TA}.
\end{aligned}
$$

These 2-cells are deduced from examining the definition of a lax algebra. This is called the *lax coherence data*. Given that the algebra following the split pair in (1.4) is a coequalizer, we might guess that some kind of 2-colimit of this diagram may be taken for the strict morphism classifier SA. In fact, a certain colimit called the

[¶]The same *cannot* be said about the inclusion of 3-categories **Str2Cat** $\hookrightarrow$ **2Cat**.

lax codescent object for the given lax coherence data is what is needed. Ensuring codescent objects exist in $\underline{\underline{T\text{-Alg}_{\mathrm{s}}}}$ is sufficient to guarantee the existence of a left adjoint $S : \underline{\underline{\text{Lax-}T\text{-Alg}_\ell}} \to \underline{\underline{T\text{-Alg}_{\mathrm{s}}}}$. A crucial step then would be to ensure that $\underline{\underline{T\text{-Alg}_{\mathrm{s}}}}$ has enough colimits to construct a codescent object. Lack has shown that it is sufficient for a 2-category $\mathscr{K}$ to possess coequifiers and coinserters to construct a codescent object of a truncated simplicial object. This is because we can first take the coinserter of μ_A and $T\alpha$, and then impose some equalities between 2-cells using two coequifiers. Indeed, it is not necessary that all colimits exist, just the above two to construct the codescent object for the truncated simplicial object (1.5). We will say more about the necessary 2-colimits in Section 3.

We can actually think of (1.4) as the two-dimensional version of a coequalizer, and furthermore a notion of surjection in a suitable 2-categorical context. This was also anticipated by Ross Street in [Str04], who showed that a functor is essentially surjective if and only if it exhibits its codomain as a codescent object. This is very closely related to the idea of regular 2-categories and reducible 2-categories by Bourn and Penon, and furthermore Lack has shown these in turn factor into Power's coherence result making use of an enhanced factorization system.

Another way to view codescent objects is to think of them as a lax simplicial bar construction $\mathsf{B}_{\bullet,\mathrm{lax}}(T, \mathscr{V})$ for a 2-monad T, albeit 2-truncated. In place of an (augmented) simplicial object we should use Δ_{LG}, the lax-Gray simplicial category as described by [MS21]. In principle the reason the 2-truncation suffices should be due to 2-finality[‖] for something like ∞-bicategories[**] applied to the inclusion $\Delta_{LG,\leq 2} \hookrightarrow \Delta_{LG}$. The (written) state of the art seems to be [Gar20], so we aren't quite there yet.

Scholium 1.6. As for a setting that will require the full non-truncated "lax simplicial bar construction", I believe David Kern, who I would like to thank for letting me know, is working on lax codescent objects and left adjoints to $\underline{\underline{\text{Ps-}T\text{-Alg}}} \hookrightarrow \underline{\underline{\text{Lax-}T\text{-Alg}_\ell}}$ in the $(\infty, 2)$-categorical setting.

Similar to the one dimensional case where it is natural to consider enriched contexts, applications for 2-monads and their algebras call for an enriched version to complement the $\underline{\underline{\mathrm{Cat}}}$ enriched case. As for real life examples in which the enriched case appears, we refer to [Mayb] or the next section, which is motivated by foundations for equivariant stable homotopy theory. The study of $\mathscr{V}$-enriched categories equipped with certain colimits is described in [Fra11], which also recognized the need for enriched 2-monads to discuss properties of the 2-theory of enriched 1-categories.[††]

Remark 1.7. There are other ways of examining issues of coherence depending on the situation. The 2-Yoneda lemma, clubs, fibrations, and variations on polynomials by Finster, Gambino, and others are invoked sometimes, but there does not seem to be a good general framework for coherence to this day even for two dimensional structures. These ideas are also closely related to the notion of representability and universal properties surviving truncations and thus can be used to describe high dimensional phenomena low dimensionally.

[‖] Using final-initial rather than cofinal-coinitial terminology.

[**] By ∞-bicategory we mean the scaled simplicial set model of $(\infty, 2)$-categories.

[††] A similar microcosm principle appears in [GS16] as explained in the introduction there.

1.3. **Motivation from stable homotopy theory.** There is no way we could do Guillou, May, Merling, and Osorno's works justice in this subsection, but we can at least try and discuss one tiny aspect which may be of interest. A summary with far more details is already given in [Maya], but we will still say a few words.

We start with the Barratt–Priddy–Quillen theorem, one basic form of which states

Theorem 1.8 (BPQ)**.** *The algebraic K-theory spectrum of the category of pointed finite sets and base-point preserving isomorphisms is the sphere spectrum :*

$$\mathbf{K}(\mathrm{Fin}_*^{\simeq}) \simeq \mathbf{S}.$$

This is perhaps not too surprising, since $\mathrm{Fin}_*^{\simeq}$ is the free symmetric monoidal category on one object, while the sphere spectrum can be thought of as the free spectrum on one object. There are a lot of interesting connections, ranging from the more geometric questions dealing with mapping spaces of spheres and the moduli space of compact 0-manifolds, to speculative ones dealing with "the K-theory of $\mathbf{F}_1$". But even just knowing that "the generator is preserved" is a very pleasing statement, we might say. Now there is more we could get out of $\mathrm{Fin}_*^{\simeq}$ by formally adjoining finite coproducts, upgrading $\mathrm{Fin}_*^{\simeq}$ into a 2-rig $\mathrm{Fin}_*^{\simeq,+}$. Does $\mathbf{K}$ see this extra monoidal structure i.e. does it output an (E_∞-)ring spectrum? The answer turns out to be yes, and we have

$$\mathbf{K}(\mathrm{Fin}_*^{\simeq,+}) \simeq \mathbf{S}[x]$$

where we define $\mathbf{S}[x]$ to be $\Sigma_+^\infty\left(\coprod_{n\in\mathbf{N}} \mathbf{B}\Sigma_n\right)$, to be thought of as the free E_∞-ring spectrum. This is one form of multiplicative BPQ, with a proof presented by Saul Glasman. We might then ask whether there is an equivariant version. Indeed, there is one due to Guillou–Merling–May–Osorno:

Theorem 1.9 (Equivariant Multiplicative BPQ, GMMO)**.** *Let G be a finite group. There is a component-wise weak equivalence lax monoidal natural map*

$$\alpha_+ : \Sigma_{G,+}^\infty \to \mathbf{K}_G \circ \mathbf{P}_{G,+} \circ \mathrm{codisc}.$$

The multiplicativity sits in the fact that α_+ is lax monoidal. We will need to unpack this statement, especially the right side. The right side is defined to take as input a G-space, where by topological space we mean a convenient category of topological spaces e.g. compactly generated (weakly Hausdorff spaces) or Δ-generated topological spaces, turn that into a G-category, apply a 2-monad $\mathbf{P}_{G,+}$ to that G-category, and finally feed that into the multiplicative algebraic K-theory $\mathbf{K}_G$, which will read in pseudoalgebras over a certain 2-operad (symmetric monoidal G-categories) and output a G-spectrum. Again, we refer the reader to [Maya] for more details. Some outstanding applications include a description of G-equivariant spectra as spectral Mackey functors, parallel to an ∞-categorical treatment given by Barwick et al., and an equivariant tom Dieck splitting.

Focusing on the right side, we need a few constructions before a quick summary. Let codiscr take as input a G-space X and output the codiscrete topological G-category, given by $X \times X \rightrightarrows X$ with trivial composition. To take care of basepoints we always freely adjoin on the trivial G-category $*$. Now define a 2-operad $\mathscr{P}_G$ defined on $k \in \mathbf{N}$ as $\mathscr{P}_G(k) \coloneqq \underline{\mathrm{Fun}}(\mathrm{codiscr}(G), \mathrm{codiscr}(\Sigma_k))$. The strict algebras of $\mathscr{P}_G$ are permutative G-categories and pseudoalgebras are symmetric monoidal

G-categories. To this 2-operad we can associate a 2-monad $\mathbf{P}_{G,+}$ defined by

$$\mathbf{P}_{G,+} \coloneqq \coprod_{j\geq 0} \mathscr{P}_G \times_{\Sigma_j} \mathscr{A}^j,$$

in the same way as how we associate a monad to an operad. Now what does $\mathbf{K}_G$ do? It is defined as the composite

$$\mathbf{K}_G \coloneqq \mathbf{S}_G \circ B^2 \circ \xi_\# \circ \mathbf{G}r \circ \mathbf{R}_G.$$

We don't have the space to get into how all of these are defined, but extremely roughly $\mathscr{D}_G$ is similar to a G-category of operators for $\mathscr{P}_G$, $\mathbf{R}_G$ is close to a pseudo left Kan extension and prolongation, $\mathbf{G}r$ is an internal discrete Grothendieck construction that reads in two enriched (co)presheaves and returns an internal category, $\xi_\#$ uses strictification via Lack and Power's coherence results, B^2 is a double bar construction, and $\mathbf{S}_G$ is an infinite loop space machine already defined by Guillou–Merling–May–Osorno. The most important piece for this document is $\xi_\#$, which involves the composition

$$\mathscr{D}_G\text{-}\mathbf{PsAlg} \xrightarrow{\mathbf{S}t} \mathscr{D}_G\text{-}\mathbf{AlgSt} \xrightarrow{\xi_*} \mathscr{F}_G\text{-}\mathbf{AlgSt}$$

where $\mathbf{S}t$ is a strictification invoking Lack and Power's coherence results, leading us back to Question 1.3. One big point to keep in mind is that the many steps of $\mathbf{K}_G$ are intended to refine the homotopical behavior of a somewhat cruder definition of $\mathbf{K}_G$, and allow pseudo rather than just strict algebras e.g. permutative G-categories versus symmetric monoidal G-categories.

Scholium 1.10. This still isn't quite the whole structure, and really a double multicategory i.e. a category that is (locally) internal to the (2-)category of multicategories, is sitting in the background. This is emphasized in [Mayb], and it is also mentioned there that this notion doesn't appear to be well-studied, and is distinct from albeit related to the cyclic double multicategories of Cheng–Gurski–Riehl. I believe there is at least a well-defined 2-category of these, so some formal category and monad theoretic constructions might be available.

Now where does enrichment come in? Note that the 2-category $\underline{\underline{\mathrm{Cat}(G\text{-}\mathrm{Top})}}$ of topological G-categories is enriched over G-categories, because G acts by conjugation on the hom-categories. But that's not all: we can upgrade this enrichment over G-categories to self-enrichment. This is because we are working with a convenient category of spaces, in particular it is (finitely) complete and Cartesian closed, and so by [BE72] we have Cartesian closure and (finite) completeness again.

Our aim for the rest of this article is to obtain Corollary 5.6 using enriched codescent.

1.4. **Notation and conventions.** By 2-category we mean a weak 2-category or a bicategory, and we will specifically say "strict" to emphasize that a 2-category is strict. A single underline such as $\underline{\mathrm{Set}}$ denotes a 1-category, or the underlying 1-category. Two underlines like in $\underline{\underline{\mathrm{Cat}}}$ denotes a 2-category, and a box such as with $\boxed{\mathrm{Bimod}}$ means a double category. When working in a 2-category, $\simeq$ refers to an equivalence while $\cong$ means "isomorphic to".

Whiskering a 2-cell $\alpha : f \to g : A \to B$ with a 1-cell $k : B \to C$ is denoted $k \circ_0 \alpha$ since we are composing along the matching zeroth dimensional component i.e. the object B as in

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\downarrow\alpha"] & B \arrow[r, "k"] & C \end{tikzcd} . \tag{1.11}$$

$\circ_0$ will also denote horizontal composition along 2-cells. For example, if $\beta : h \to k : B \to C$ then we can compose along the common object B for $\beta \circ_0 \alpha : h \circ f \to k \circ g : A \to C$ like in

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\downarrow\alpha"] & B \arrow[r, bend left, "h"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\downarrow\beta"] & C \end{tikzcd} . \tag{1.12}$$

Composition of 1-cells along their common object or vertical composition of 2-cells i.e. along their common 1-cell is denoted by $\circ$ with no indices, or ';' for compositional order occasionally, as in

$$\begin{tikzcd} A \arrow[r, bend left=60, "f"] \arrow[r, "g" description] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & B \end{tikzcd} . \tag{1.13}$$

We sometimes use generalized elements and morphisms, so in a 1-category when we refer to an X *shaped element of* C, we mean a morphism $X \to C$. In a 2-category, we write X *shaped objects of* C for morphisms $X \to C$. By a morphism between X shaped elements of C we mean a 2-arrow $\alpha : f \to g : X \to C$.

1.5. **About this revision.** There were some particularly concerning parts that appeared in the original article, including imposing a strange looking condition on the equivalences internal a certain $\mathscr{V}$-2-category $\mathscr{B}$ to make diagram chases easier, and whether the constructions were enriched enough. It took far too long to finally address these, given their relatively small size. For this I have no excuse, and I can only give the lame reason that there was a constant accumulation of ideas I wanted to talk about, before I finally decided to focus just on what I had originally set out to do back in 2019. There are still a few unanswered questions however, or at least items that deserve more elaboration on. Also, I did add a tiny number of retrospectives, which can largely be distinguished by the fact that they cite references more recent than 2019, or an earlier draft.

Acknowledgments

First and foremost, I wish to thank my mentor, Professor Peter May, for his guidance and also for his wisdom on other matters be it math or something else. My stay at the University of Chicago was greatly improved by my room mates and other peers at this REU, especially Jae, Vihn-Kha, and Matt Lipman, who patiently answered my questions about miscellaneous topics in our daily interactions and were good friends to me. I hope everyone is doing well, wherever they are now.

2. Enriched 2-categories and 2-functors

To save on space, we merely give an idea for the definitions of enriched 2-categories and 2-functors and leave the details to [GS16], especially for the coherence conditions that are easier to draw out using string diagrams.

Definition 2.1. A 2-category $\mathscr{B}$ enriched over a monoidal 2-category $\mathscr{V}$ consists of

- a set of objects $\mathscr{B}_0$ with a distinguished object $I \in \mathscr{B}_0$
- for each pair of objects $A, B \in \mathscr{B}_0$ a hom object $\mathscr{B}(A,B) \in \mathscr{V}$
- for each $B \in \mathscr{B}_0$ an I-shaped object of $\mathscr{B}(B,B)$ meaning a morphism $I_B : I \to \mathscr{B}(B,B)$
- for each triple of objects $A, B, C \in \mathscr{B}_0$ a morphism $c_{ABC} : \mathscr{B}(B,C) \otimes \mathscr{B}(A,B) \to \mathscr{B}(A,C)$
- invertible 2-morphisms

$$\begin{tikzcd}
I \otimes \mathscr{B}(A,B) \arrow[r] \arrow[dr, swap, "\lambda_{\mathscr{B}(A,B)}"] & \mathscr{B}(B,B) \otimes \mathscr{B}(A,B) \arrow[d, "c_{ABB}"] \arrow[dl, phantom, "\Leftarrow \sigma_{AB}"] \\
& \mathscr{B}(A,B)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{B}(A,B) \otimes I \arrow[r] \arrow[dr, swap, "\rho_{\mathscr{B}(A,B)}"] & \mathscr{B}(A,B) \otimes \mathscr{B}(A,A) \arrow[d, "c_{AAB}"] \arrow[dl, phantom, "\Leftarrow \tau_{AB}"] \\
& \mathscr{B}(A,B)
\end{tikzcd}$$

- an invertible 2-morphism

$$\begin{tikzcd}
(\mathscr{B}(C,D) \otimes \mathscr{B}(B,C)) \otimes \mathscr{B}(A,B) \arrow[rr] \arrow[d, swap, "c \otimes \mathrm{id}"] & & \mathscr{B}(C,D) \otimes (\mathscr{B}(B,C) \otimes \mathscr{B}(A,B)) \arrow[d, "\mathrm{id} \otimes c"] \\
\mathscr{B}(B,D) \otimes \mathscr{B}(A,B) \arrow[r, swap, "c"] & \mathscr{B}(A,D) \arrow[u, phantom, "\Leftarrow \alpha_{ABCD}"] & \mathscr{B}(C,D) \otimes \mathscr{B}(A,C) \arrow[l, "c"]
\end{tikzcd}$$

  where the top 1-arrow is the associator of $\mathscr{V}$

satisfying two equalities of string diagrams given in Definition 3.1 of [GS16].

Scholium 2.2. [GS16] ultimately wish to characterize 1-categories enriched over proarrow equipments[‡‡] $\mathcal{K} \to \mathcal{M}$, possibly with some cocompleteness properties. The underlying idea is that both of these contexts abstract out good properties of $\underline{\underline{\mathrm{Cat}}}$, specifically by enhancing the underlying 2-category with a notion of hom objects.[§§]

Scholium 2.3. In addition to accounts of enrichment over monoidal categories, we can define enrichment over $\mathscr{V}$ provided that there is at least some structure in the base of enrichment $\mathscr{V}$ that can accommodate composition. For example, we could have taken $\mathscr{V}$ to be a *multicategory* (that is, a category whose arrows have multiple inputs) so that we have an arrow $\mathscr{C}(B,C), \mathscr{C}(A,B) \to \mathscr{C}(A,C)$. Alternatively, we could have *delooped* $\mathscr{V}$ if $\mathscr{V}$ was a monoidal category. For us, this roughly means that we "push" the monoidal structure upwards so that $\otimes$ on the objects/0-morphisms is actually encoded by composition of the 1-morphisms; think about how some people say that "a monoid is a category with one object". In fact, there are some interesting examples of enrichment over 2-categories, one of which actually describes sheaves on a topological space! An even more general version of enrichment involves enriching over an even more general structure called a *virtual*

[‡‡]These may be thought of as double categories satisfying a universal filler condition on "niches". The point is that this "niche filling" which also like the universal property of composition characterizes profunctors, and a certain "corner turning" lemma involving companions and conjoints allows us to abstractly describe limits weighted by profunctors, which generalizes large swaths of category theory.

[§§]These and related structures have been studied in formal category theory. Our choice of base reflects the earliest works of formal category theory by Gray and Lawvere in which explicit notions of internal homs are present up in the 2-cells, while later attempts such as Street, Bénabou, Verity, and Riehl's notions of cosmoi, equipment, Yoneda structures, and yosegi boxes have a notion of hom implicit somewhere else such as fibration properties/structures or through some abstraction of the presheaf construction.

double category which combines both multicategories and 2-categories into a single structure. This fits into a rather general framework of enrichment for generalized multicategories. One example of the aforementioned enrichment is the description of what Borcherds has called a *relaxed multicategory*, which is used to define his notion of a vertex algebra. The interested reader may wish to read [Lei02] for details. There are sure to be interesting examples that may be found by enriching over 3-categories, 2-multicategories,¶¶ or 1-cubical 3-categories, but we do not pursue them here. There are arguments as to why we should use monoidal double categories or 2-categories internal to categories, including applications to field theories, which can be found in [Shu10] or references therein.

Definition 2.4. A $\mathscr{V}$-functor $F : \mathscr{A} \to \mathscr{B}$ between $\mathscr{V}$-enriched 2-categories $\mathscr{A}$ and $\mathscr{B}$ is comprised of

- a map $\mathscr{A}_0 \to \mathscr{B}_0$
- for each $A_1, A_2 \in \mathscr{A}_0$ a morphism of hom objects $\mathscr{A}(A_1, A_2) \to \mathscr{B}(FA_1, FA_2)$
- invertible 2-cells expressing the preservation of composition and unitality up to those 2-cells

satisfying three equalities of string diagrams given in Definition 3.5 of [GS16].

[GS16] also define (lax) natural transformations and modifications (along with icons), which as expected are appropriately coherent (or lax) versions of $\mathscr{V}$-natural transformations and modifications.

3. Some 2-categorical (co)limits

3.1. **Preliminaries.** We need to quickly summarize a few important (co)limits from 2-category theory. Arguably the most important class are the PIE (co)limits i.e products, inserters, and equifiers. The inserter of a diagram

$$\begin{tikzcd} B \arrow[r, bend left] \arrow[r, bend right] & C \end{tikzcd}$$

is the universal $A \to B$ such that whiskering yields

$$\begin{tikzcd} A \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & C \end{tikzcd}$$

and an equifier of two parallel 2-cells

$$\begin{tikzcd} A \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\alpha\Downarrow\ \Downarrow\beta"] & B \end{tikzcd}$$

is the universal 1-cell $f : A \to B$ such that whiskering yields $\alpha \circ_0 f = \beta \circ_0 f$. In effect we could think of these as "lax equalizers" and "equalizers for 2-arrows via whiskering" although the former term is often best reserved for lax descent objects instead. Generally speaking we can build a lot of 2-limits of interest using PIE limits, and in fact a lax descent object can be constructed by taking an inserter followed by two equifiers. An isoinserter is like an inserter but the 2-cell obtained is an isomorphism. As expected, a pseudo descent object may be constructed from taking an isoinserter and two equifiers. We can already obtain an isoinserter from inserters and equifiers, namely we first take the inserters $\alpha : f \Rightarrow g$ and $\beta : g \Rightarrow f$ in both directions, and equify the pairs $(\alpha \circ_1 \beta, \mathrm{id}_g)$ and $(\beta \circ_1 \alpha, \mathrm{id}_f)$ to force an isomorphism.

¶¶By this we mean a 1-multiple 2-category, so that the 2-morphisms have just one input rather than a list as an input.

Also, note that inserters and equifiers can be given as ($\underline{\underline{\mathrm{Cat}}}$-)weighted colimits. An inserter in a 2-category $\mathscr{C}$ can be given as the weighted limit $\{W, F\}$ where $F : \{\bullet \rightrightarrows \bullet\} \to \mathscr{C}$ and $W : \{\bullet \rightrightarrows \bullet\} \to \underline{\underline{\mathrm{Cat}}}$ has image $* \rightrightarrows [1]$ which are the maps of $*$ to the endpoints of the free living arrow. An equifier in $\mathscr{C}$ can be given as $\{W, F\}$ where

$$F : \left(\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\,\Downarrow"] & \bullet \end{tikzcd} \right) \longrightarrow \mathscr{C}$$

and W has image

$$\begin{tikzcd} * \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & {[1]} \end{tikzcd}$$

which is again the inclusion of $*$ as the endpoints of the free living arrow, and the two 2-cells are mapped to the single 2-cell encoding $\bullet \to \bullet$ in $[1]$.

We now extremely quickly summarize some language that is used by Garner and Shulman in order to set up the theory of weighted (co)limits in $\mathscr{V}$-2-categories. They define the notion of a bimodule W externally, with the hom-objects acting on the left or right of $W(x, y)$ as a morphism in $\mathscr{V}$, since they wish to avoid assuming (right) closure hence self-enrichment of the base 2-category $\mathscr{V}$. For our purposes, since we assume $\mathscr{V}$ is in particular symmetric monoidal closed, we will treat a bimodule $W : \mathscr{C} \nrightarrow \mathscr{B}$ as a functor $W : \mathscr{B}^{\mathrm{op}} \otimes \mathscr{C} \to \mathscr{V}$, where $\otimes$ is the monoidal product on $\underline{\underline{\mathscr{V}\text{-}\mathrm{Cat}}}$ analogous to enriched 1-categories. In particular, a right $\mathscr{V}$-module will essentially be an enriched presheaf $\mathscr{B}^{\mathrm{op}} \to \mathscr{V}$. They show that bimodules can be composed by taking their tensor product provided $\mathscr{V}$ has enough colimits, that there is an internal hom of bimodules $\langle V, W\rangle \in \mathscr{V}$ for two $V, W : \mathscr{B}^{\mathrm{op}} \otimes \mathscr{C} \to \mathscr{V}$ provided $\mathscr{V}$ is complete and (right) closed, and that a morphism $V \otimes W \to U$ of bimodules can be exhibited by a morphism $V \to \langle W, U\rangle$.

We will need to make use of the enriched Yoneda embedding. The Yoneda embedding よ : $\mathscr{B} \to \mathscr{M}\mathscr{B}$ to right $\mathscr{V}$-modules sends objects $b \in \mathscr{B}$ to $\mathscr{B}(-, b)$, and on hom-objects $\mathscr{B}(b, b') \to \langle$よ$b$, よ$b'\rangle$.

Theorem 3.1 (cf. [GS16, Section 9.5])**.** *The Yoneda embedding $\mathscr{B} \to \mathscr{M}\mathscr{B}$ is fully faithful i.e. is a local equivalence of $\mathscr{V}$-2-categories.*

3.2. **Weighted (co)limits.** Just as weighted (co)limits feature essentially in enriched 1-category theory and unenriched 2-category theory, we will need to talk about weighted (co)limits in enriched 2-category theory.

Definition 3.2 (cf. [GS16, Section 10])**.** Let $\mathscr{V}$ be complete and symmetric monoidal closed. Let W be a bimodule $\mathscr{D} \nrightarrow \mathscr{C}$ and $F : \mathscr{D} \to \mathscr{B}$ be a $\mathscr{V}$-functor. A W-weighted cylinder consists of an object $v \in \mathscr{B}$ and a morphism of bimodules $\phi : W \to \mathscr{B}(v, F)$. For each $c \in \mathscr{B}$, this gives rise to the bimodule morphism $\widetilde{\phi^{(-)}} : \mathscr{B}(v, -) \to \langle W, \mathscr{B}(F, \mathrm{id})\rangle$ representing

$$\phi^{(-)} : \mathscr{B}(v, -) \otimes W \xrightarrow{(\mathrm{id}, \phi)} \mathscr{B}(v, -) \otimes \mathscr{B}(F, v) \xrightarrow{c_{F(-),v,-}} \mathscr{B}(F, \mathrm{id}) .$$

We say v is the W-weighted limit of F and write it as $\{W, F\}$ if $\widetilde{\phi^{(-)}}$ is an equivalence. If we assumed $\mathscr{V}$ was cocomplete instead of complete, then this amounts to stating $\phi^{(-)}$ exhibits $\mathscr{B}(v, -)$ as $\langle W, \mathscr{B}(F, \mathrm{id})\rangle$. Dually we denote the W-weighted colimit of F as $W \star F$.

Actually, in this article we will only ever invoke the case when W is a right $\mathscr{V}$-module $G : \mathscr{X}^{\mathrm{op}} \to \mathscr{V}$ but Pseudoproposition 3.3 becomes conceptually easier to process if we let W be a bimodule.

We can also define lax (co)limits and ask whether the lax classifier can be weighted out by replacing the weight W with the lax classifier W'. One half of the answer can be answered in the positive once we can write down $\mathscr{V}$-codescent objects as we shall do soon. The other half turns out to be yes as well for $\mathscr{V}$ with enough colimits by applying Section 4 to the forgetful 2-functor $[\mathscr{C}, \mathscr{V}] \to [\mathrm{ob}\mathscr{C}, \mathscr{V}]$ and then unraveling the usual computation [BKP89], to obtain that pseudoalgebras and pseudomorphisms are $\mathscr{V}$-functors and natural transformations, while lax algebras and lax morphisms are lax $\mathscr{V}$-functors and natural transformations. An application we have in mind is the description of the lax 2-end formula for the lax natural transformations $\mathrm{LaxNat}(F, G)$ for $F, G : \mathscr{C} \to \mathscr{V}$ two $\mathscr{V}$-functors.

We will not make essential use of the following not-quite-proposition, but it is nice to have in mind when giving an alternate description of enriched codescent objects in terms of coequifiers and coinserters. Again, this is not logically required for our main focus at all, and an actual proof so far requires using a simplicial presheaf model of 3-categories, with [Hin20] invoked for 3-Yoneda. There is a tricategorical Yoneda lemma in the literature, but a direct proof using tri or Gray-categories would seem very daunting.

Pseudoproposition 3.3 (Not logically required)**.** *We have a "tensor-hom adjunction" for weighted limits:*

$$\{W \star X, Y\} \simeq \{W, \{X, Y\}\}.$$

Proof sketch. Schematically, if we know the proarrow 2-equipment of $\mathscr{V}$-categories, functors, and bimodules, this follows from the manipulations

$$\begin{aligned}
\mathscr{K}_m(D, \{W \star X, Y\}) &\simeq \mathscr{K}_b(W \star X, D \triangleright Y) \\
&\simeq \mathscr{K}_b(W, D \triangleright Y \triangleleft X) \\
&\simeq \mathscr{K}_b(W \otimes D, Y \triangleleft X) \\
&\simeq \mathscr{K}_b(X \otimes (W \otimes D), Y) \\
&\simeq \mathscr{K}_b(X, (W \otimes D) \triangleright Y) \\
&\simeq \mathscr{K}_b(W \otimes D, \{X, Y\}) \\
&\simeq \mathscr{K}_b(W, D \triangleright \{X, Y\}) \simeq \mathscr{K}_m(D, \{W, \{X, Y\}\})
\end{aligned}$$

where $\mathscr{K}_m$ and $\mathscr{K}_b$ refer to the 3-category of maps and the (flagged***) 3-category of bimodules in $\mathscr{K}$, and then invoking the 3-Yoneda lemma.

We assumed for convenience $\mathscr{K}$ is a closed equipment with composition of bimodules, but we could replace all expressions involving $\triangleleft, \triangleright$ with their represented hom 2-categories, and all compositions with their "universal niche-fillers"; for our specific bases $\mathscr{V}$ we actually have $\triangleleft, \triangleright$, but not necessarily $\otimes$ so we have a(n augmented) virtual closed 2-equipment, rather than a 2-equipment. □

Remark 3.4. If we had enough colimits and were only concerned with invertible phenomena e.g. pseudo codescent objects, meaning we didn't care about the non-invertible 2-cells, translating to non-invertible 3-squares in the 2-equipment

***By this we mean an essentially surjective 3-functor from a 3-groupoid to the underlying 3-category of bimodules, so that Cauchy equivalent $\mathscr{V}$-3-categories are not equivalent.

$\boxed{\mathscr{V}\text{-}\mathbf{Bimod}}$, then we could have used [Hau16] instead. Following Shulman, a higher equipment there is defined to be a double category so that the source and target category morphisms together $(d_0, d_1) : \mathscr{K}_1 \to \mathscr{K}_0 \times \mathscr{K}_0$ form a Cartesian fibration, or equivalently a coCartesian fibration. An enriched Yoneda lemma in that setting in turn is given in [Hin20].

We will briefly need copowers (tensors) by $\mathscr{V}$, which like their 1-categorical counterparts are special examples of weighted colimits by setting W to be $\mathscr{I} \to \mathscr{V}$ where $\mathscr{I}$ has one object $*$ with $\mathscr{I}(*, *) = I$, the monoidal unit of $\mathscr{V}$. The universal property then translates to

$$\mathscr{B}(V \odot B, C) \simeq \mathscr{V}(V, \mathscr{B}(B, C))$$

as usual. From this point on we follow Franco [Fra11] more than Garner and Shulman. In fact we mostly follow Franco for the remainder of this article, with only a few exceptions like the additional coherence aspect of codescent objects for pseudoalgebras.

3.3. **Free $\mathscr{V}$-2-categories.** Following Franco [Fra11], we will make some more assumptions which make defining weighted colimits easier. We want to make use of a left adjoint to the forgetful functor $\mathscr{V}\text{-}\mathbf{2Cat} \to \mathbf{2Cat}$. 1-categorically this is called the free $\mathscr{V}$-category, and the free $\mathscr{V}$-category on a category $\mathscr{C}$ has the same objects and $\mathcal{C}_{\mathcal{V}}(X, Y) := \coprod_{\mathcal{C}(X,Y)} I$. If we allow some assumption on the monoidal product $\otimes$ in $\mathscr{V}$ e.g. closedness or semi-Cartesianness then we can get composition using

$$\coprod_{\mathscr{C}(x,y)} I \otimes \coprod_{\mathscr{C}(y,z)} I \to \coprod_{\mathscr{C}(x,y)\times\mathscr{C}(y,z)} I \to \coprod_{\mathscr{C}(x,z)} I$$

where in the above cases the left morphism is actually an isomorphism. We are using the former. The fact that $(-)_{\mathscr{V}}$ is left adjoint to the underlying 1-category functor means weights $W : \mathscr{C}_{\mathscr{V}} \to \mathscr{V}$ are conjugate to weights $W_0 : \mathscr{C} \to \underline{\mathrm{Set}} \to \mathscr{V}_0$ where the right functor sends a set S to $S \odot I \simeq \coprod_S I$. To replicate this in the 2-categorical setting, we need to replace the coproduct $\coprod_{\mathscr{B}(X,Y)} I$ with the coinserter of

$$\coprod_{\mathscr{B}(f,g)(X,Y)} I \underset{t}{\overset{s}{\rightrightarrows}} \coprod_{\mathscr{B}(X,Y)} I, \tag{3.5}$$

which gives us the universal object $\mathscr{H}$ that encodes two objects X, Y and the morphisms $f, g : X \to Y$ between them, where

$$\begin{tikzcd} \coprod_{\mathscr{B}(f,g)(X,Y)} I \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \mathscr{H} \end{tikzcd}$$

encodes a generalized morphism. This is 1-truncated, which suffices because we care about the *free* $\mathscr{V}$-2-category. We would be working with 2-truncated codescent objects in a setting where $\mathscr{V}$ allows us to compare $\mathscr{V}$-2-categories and 2-categories internal to $\mathscr{V}$, since there we will have a lax codescent object encoding objects, morphisms, and 2-morphisms. In turn, a 2-truncated diagram would be enough there because we would be interested in a free internal 2-category to $\mathscr{V}$.

To get composition in this setting we apply composition of 1-arrows and 2-arrows to obtain morphisms $\coprod_{\mathscr{B}(f,g)(X,Y)} I \otimes \coprod_{\mathscr{B}(g,h)(X,Y)} I \to \coprod_{\mathscr{B}(f,h)(X,Y)} I$ and

$\coprod_{\mathscr{B}(X,Y)} I \otimes \coprod_{\mathscr{B}(Y,Z)} I \to \coprod_{\mathscr{B}(X,Z)} I$, and then invoke functoriality of 2-colimits with (3.5).

We can now define enriched notions of PIE (co)limits and (lax) (co)descent objects, recalling their unenriched definition via weighted $\underline{\underline{\mathrm{Cat}}}$-enriched (co)limits, by modifying their functors and weights under $(-)_{\mathscr{V}}$.

3.4. **Lax (co)descent objects.** Referring back to Section 1.2, we want to construct an enriched variant. For this the weighted definition is easier to get a grip on.

Proposition 3.6. *Let $\Delta^{\mathrm{op}}_{\mathrm{lax},\le 2}$ and $\Delta^{\mathrm{op}}_{\mathrm{ps},\le 2}$ be the diagrams for the lax coherence datum and pseudo coherence datum, respectively. The* op *has to do with Δ being cosimplicial rather than simplicial. Let $よ : \Delta^{\mathrm{op}}_{*,\le 2} \to \left[\Delta_{*,\le 2}, \underline{\underline{\mathrm{Cat}}}\right]$ where $* =$* lax *or* ps *be the Yoneda embedding, and postcompose by* colim $: \left[\Delta_{*,\le 2}, \underline{\underline{\mathrm{Cat}}}\right] \to \underline{\underline{\mathrm{Cat}}}$. *Given a (lax) coherence datum $F : \Delta_{*,\le 2} \to \mathscr{C}$ in an ordinary 2-category $\mathscr{C}$ and letting $W = \mathrm{colim} \circ よ$, the weighted colimit $W \star F$ is the codescent object corresponding to the (lax) coherence datum F.*

The point is that this definition is not very explicit at all, and shoves the details under taking a colimit following the Yoneda embedding. We now use Section 3.3 to modify Proposition 3.6 and take this as the definition of an enriched codescent object. At this point we will also need codescent objects to exist in $\mathscr{V}$.

Definition 3.7. Letting W be as before, under taking free $\mathscr{V}$-2-categories $(-)_{\mathscr{V}}$ this gives a weight $よ_{\mathscr{V}} : \left(\Delta^{\mathrm{op}}_{*,\le 2}\right)_{\mathscr{V}} \to \mathscr{V}$ and $\mathscr{V}$-functor $F_{\mathscr{V}} : \left(\Delta^{\mathrm{op}}_{*,\le 2}\right)_{\mathscr{V}} \to \mathscr{B}$ where $\mathscr{B}$ is a $\mathscr{V}$-2-category. The $\mathscr{V}$-codescent object associated to the $*$ coherence datum $F : \Delta^{\mathrm{op}}_{*,\le 2} \to \mathscr{B}_0$ is defined to be the weighted $\mathscr{V}$-colimit $W_{\mathscr{V}} \star F_{\mathscr{V}}$.

Recalling that codescent objects can be constructed from taking a co(iso)inserter, followed by two coequifiers, we might ask the same about $\mathscr{V}$-codescent objects. Keeping in mind we can also dualize with $\mathscr{V}$ symmetric monoidal, we can then deduce from Pseudoproposition 3.3 that (lax) codescent objects can still be constructed by taking a co(iso)inserter and then taking two coequifiers, by modifying their weighted colimit description with $(-)_{\mathscr{V}}$.

4. Enriched 2-monads and their algebras

Definition 4.1. A $\mathscr{V}$-2-monad acting on a $\mathscr{V}$-2-category $\mathscr{B}$ is a 2-monad in the strict 3-category $\mathscr{V}$**-2Cat**. Following Kelly, these 2-monads are strict, so they consist of (strict) transformations $\mu : T^2 \to T$ and $\eta : \mathrm{id} \to T$ which satisfy the diagrams

$$\begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "\eta T"] \arrow[dr, equal] & T^2 \arrow[d, "\mu"] & T \arrow[l, "T\eta"'] \arrow[dl, equal] \\ & T & \end{tikzcd}$$

up to equality.

Definition 4.2. An object of $\underline{\underline{T\text{-}\mathrm{Alg}_{\mathrm{s}}}}$, $\underline{\underline{\mathrm{Ps}\text{-}T\text{-}\mathrm{Alg}}}$, and $\underline{\underline{\mathrm{Lax}\text{-}T\text{-}\mathrm{Alg}_{\ell}}}$ in $\mathscr{B}$ is a strict, pseudo, and lax T-algebra over T on the underlying 2-category $\mathscr{B}_0$. For all the relevant diagrams we refer the reader to [Lac02, Section 1].

After this, we can define $\underline{\underline{T\text{-Alg}_{\mathrm{s}}}}$, $\underline{\underline{\text{Ps-}T\text{-Alg}}}$, and $\underline{\underline{\text{Lax-}T\text{-Alg}_\ell}}$ as $\mathscr{V}$-2-categories.

Definition 4.3. The strict hom-object between two objects $\alpha : TX \to X$ and $\beta : TY \to Y$ is the 2-equalizer of

$$\mathscr{B}(X,Y) \underset{\mathscr{B}(\alpha,Y)}{\overset{T;\mathscr{B}(TX,\beta)}{\rightrightarrows}} \mathscr{B}(TX,Y)$$

in $\mathscr{V}$.

This is a reasonable definition because we are asking for the universal object $\mathscr{H} \to \mathscr{B}(X,Y)$ encoding $(TX \to X \to Y) = (TX \to TY \to Y)$, with no further coherence data hence a 2-equalizer.

Definition 4.4. The hom-object between two objects $\alpha : TX \to X$ and $\beta : TY \to Y$ in $\underline{\underline{\text{Lax-}T\text{-Alg}_\ell}}$ is the lax descent object of

$$\begin{tikzcd} \mathscr{B}(X,Y) \arrow[r, shift left=2, "T;\mathscr{B}(TX,\beta)"] \arrow[r, shift right=2, "\mathscr{B}(\alpha,Y)"'] & \mathscr{B}(TX,Y) \arrow[l, "\mathscr{B}(\eta,Y)" description] \arrow[r, shift left=2, "\mathscr{B}(\mu_X,Y)"] \arrow[r, "\mathscr{B}(T\alpha,Y)" description] \arrow[r, shift right=2, "T;\mathscr{B}(T^2X,\beta)"'] & \mathscr{B}(T^2X,Y) \end{tikzcd}$$

in $\mathscr{V}$. The hom-object in $\underline{\underline{\text{Ps-}T\text{-Alg}}}$ in turn is defined to be the pseudo descent object of the above diagram. Note that for $\underline{\underline{\text{Lax-}T\text{-Alg}_\ell}}$ we have lax coherence data, while for $\underline{\underline{\text{Ps-}T\text{-Alg}}}$ we have pseudo coherence data.

These are also reasonable definitions, since this time we are either taking an inserter or isoinserter, followed by two equifiers, which is what we expect in place of a 2-equalizer in the lax and pseudo cases. Furthermore, what allows us to define ι : $\underline{\underline{T\text{-Alg}_{\mathrm{s}}}} \to \underline{\underline{\text{Ps-}T\text{-Alg}}}$ and $\underline{\underline{T\text{-Alg}_{\mathrm{s}}}} \to \underline{\underline{\text{Lax-}T\text{-Alg}_\ell}}$ is the universal morphism from the 2-equalizer in Definition 4.3 to the pseudo or lax codescent objects in Definition 4.4. This sets us up for the next section.

5. Lack's coherence theorem in the $\mathscr{V}$-setting

Proposition 5.1. *Suppose T is a $\mathscr{V}$-2-monad acting on a $\mathscr{V}$-2-category $\mathscr{B}$, which admits $\mathscr{V}$-codescent objects and T preserves those $\mathscr{V}$-codescent objects. Then the inclusions $\iota : \underline{\underline{T\text{-Alg}_{\mathrm{s}}}} \hookrightarrow \underline{\underline{\text{Ps-}T\text{-Alg}}}$ and $\underline{\underline{T\text{-Alg}_{\mathrm{s}}}} \hookrightarrow \underline{\underline{\text{Lax-}T\text{-Alg}_\ell}}$ admit left adjoints $L_{\mathrm{ps}} : \underline{\underline{\text{Ps-}T\text{-Alg}}} \to \underline{\underline{T\text{-Alg}_{\mathrm{s}}}}$ and $L_{\mathrm{lax}} : \underline{\underline{\text{Lax-}T\text{-Alg}_\ell}} \to \underline{\underline{T\text{-Alg}_{\mathrm{s}}}}$.*

Note that the left adjoint usually gets called $(-)'$. Lack had originally set up lax descent objects specifically so that the unenriched inclusions $\iota : \underline{\underline{T\text{-Alg}_{\mathrm{s}}}} \to \underline{\underline{\text{Ps-}T\text{-Alg}}}$ and $\underline{\underline{T\text{-Alg}_{\mathrm{s}}}} \to \underline{\underline{\text{Lax-}T\text{-Alg}_\ell}}$ would admit left adjoints, and here we have made this true nearly by definition, since hom-objects are computed as lax descent objects.

For the next proposition I need to make an assumption that I still haven't figured out how to remove yet:

Proposition 5.2. *In the case of $\underline{\underline{\text{Ps-}T\text{-Alg}}}$, when $\underline{\underline{T\text{-Alg}_{\mathrm{s}}}}$ and $\underline{\underline{\text{Ps-}T\text{-Alg}}}$ have $\mathscr{V}$-copowers and the inclusion $\iota : \underline{\underline{T\text{-Alg}_{\mathrm{s}}}} \hookrightarrow \underline{\underline{\text{Ps-}T\text{-Alg}}}$ preserves them, we find that the left adjoint L_{ps} is actually a local equivalence i.e.*

$$\underline{\underline{\text{Ps-}T\text{-Alg}}}(X,Y) \simeq \underline{\underline{\text{Ps-}T\text{-Alg}}}(X,\iota L_{\mathrm{ps}}Y) \simeq \underline{\underline{T\text{-Alg}_{\mathrm{s}}}}(L_{\mathrm{ps}}X, L_{\mathrm{ps}}Y)$$

as objects of $\mathscr{V}$.

Right now the assumption is to ensure the 2-adjunction is actually enriched, but it does sound a bit unsatisfying. This should be doable in principle assuming $\mathscr{V}$ is semi-Cartesian, which is good enough for our motivations, and if T preserves $\mathscr{V}$-copowers then the fact ι does will follow without much work due to the way we defined internal homs.

Proof. With the assumptions that we have, the proof reduces to checking the underlying 2-category. This is [Lac02, Theorem 3.1], but just to give the gist: given a 2-algebra $\alpha : TA \to A$, we can take the codescent object $L_{\mathrm{ps}}A$, which comes with a 1-cell $e : TA \to L_{\mathrm{ps}}A$, a canonical 2-cell $\xi : e \circ \mu_A \to e \circ T\alpha$ and a canonical morphism $q : L_{\mathrm{ps}}A \to A$. Using the unit $\eta_A : A \to TA$ at A, we will be done if we can show

$$\begin{tikzcd} L_{\mathrm{ps}}A \arrow[r, "q"] \arrow[rrr, equals, bend right, "\wr|"'] & A \arrow[r, "\eta_A"] & TA \arrow[r, "e"] & L_{\mathrm{ps}}A \end{tikzcd}$$

then we will be done, because $e \circ \eta_A$ is equivalent to the unit of the adjunction. A universal property of the codescent objects stipulates that using $\xi : e \circ \mu \simeq e \circ T\alpha$ from the coherence data, if we can construct an invertible 2-cell $\zeta_0 : e \simeq e \circ \eta_A \circ q \circ e$ such that

$$\begin{tikzcd} & TA \arrow[dr, "e"'] \arrow[dr, bend left=50, "e\circ\eta_A\circ q\circ e"] \arrow[dr, phantom, bend left=20, "\Leftarrow\zeta_0"] \arrow[dd, phantom, "\xi\Downarrow"] & \\ T^2A \arrow[ur, "\mu_A"] \arrow[dr, "T\alpha"'] & & L_{\mathrm{ps}}A \\ & TA \arrow[ur, "e"'] & \end{tikzcd} = \begin{tikzcd} & TA \arrow[dr, "e\circ\eta_A\circ q\circ e"] \arrow[dd, phantom, "\Downarrow e\circ\eta_A\circ q\circ_0\xi"] & \\ T^2A \arrow[ur, "\mu_A"] \arrow[dr, "T\alpha"'] & & L_{\mathrm{ps}}A \\ & TA \arrow[ur, "e\circ\eta_A\circ q\circ e"] \arrow[ur, bend right=50, "e"'] \arrow[ur, phantom, bend right=20, "\zeta_0\Downarrow"] & \end{tikzcd}$$

then there is a unique invertible 2-cell $\zeta : e \circ \eta_A \circ q \simeq \mathrm{id}_{L_{\mathrm{ps}}A}$ such that $\zeta \circ_0 e = \zeta_0$. We can directly give an invertible 2-cell which precisely satisfies that:

$$e = e \circ \mu_A \circ \eta_{TA} \xrightarrow{\xi\circ_0\eta_{TA}} e \circ T\alpha \circ \eta_{TA} = e \circ \eta_A \circ \alpha = e \circ \eta_A \circ q \circ e.$$

And so we are done. $\square$

Remark 5.3. Here, it's natural to ask whether an enriched Power's coherence theorem may be obtained by examining an enhanced $\mathscr{V}$-factorization system whose left class of arrows $\mathscr{E}$ can be detected by $\mathscr{V}$-codescent objects. On a $\mathscr{V}$-2-category $\mathscr{B}$, at least assuming the left class of arrows $\mathscr{E}$ is preserved under powering by enough objects of $\mathscr{V}$ this notion does seem to work. In particular when comma objects and lax codescent objects exist we can give a construction analogous to the congruence of a 1-cell in a 2-category, and for $\underline{\underline{\mathscr{W}\text{-}\mathrm{Cat}}}$ enriched over itself when $\mathscr{W}$ is nice, this appears to detect bijective-on-object $\mathscr{W}$-functors.

5.1. **Back to our motivation.** We return to Section 1.3 in this last section.

Proposition 5.4. *When $\mathscr{V} = \underline{\underline{\mathrm{Cat}(\mathrm{G}\text{-}\mathrm{Top})}}$, the $\mathscr{V}$-2-category $\mathscr{B} = [\Psi, \mathscr{V}]$ admits $\mathscr{V}$-codescent objects. Furthermore, the 2-monad $\mathbf{D}$ we are interested in preserves these, thus $\underline{\underline{\mathrm{Ps}\text{-}\mathbf{D}\text{-}\mathrm{Alg}}}$ and $\underline{\underline{\mathbf{D}\text{-}\mathrm{Alg}_{\mathrm{s}}}}$ do as well.*

Proof. Since we are working in functor categories with target $\mathscr{V}$, it is enough to show $\mathscr{V}$-codescent objects exist in $\mathscr{V}$ itself. This will follow from the existence of codescent objects in the underlying 2-category $\mathscr{V}_0$ and $\mathscr{V}$-copowers. Copowers by

$\mathscr{V}$ immediately follows from the fact that the 2-category of categories internal to G-spaces is monoidal closed. On the underlying 2-category, we can look for the existence of (κ-)filtered colimits. Although general colimits e.g. anything resembling pushouts or coequalizers in $\underline{\underline{\mathrm{Cat}(\mathscr{E})}}$ are difficult to obtain,[†††] (κ-)filtered 2-colimits are known to exist in $\underline{\underline{\mathrm{Cat}(\mathscr{E})}}$ for $\mathscr{E}$ admitting enough colimits since $\underline{\underline{\mathrm{Cat}(\mathscr{E})}}$ is a localization of $[\Delta_{\leq 2}, \mathscr{E}]$, as a limit sketch. But $\mathscr{E} = \underline{G\text{ - }\mathrm{Top}}$ which is cocomplete.

As for $\mathbf{D}$ preserving $\mathscr{V}$-codescent objects, as a $\mathscr{V}$-2-monad $\mathbf{D}$ is the composite of a left adjoint $\underline{\underline{\mathrm{Cat}(\mathscr{E})}}^{\Psi} \to \underline{\underline{\mathrm{Cat}(\mathscr{E})}}^{\mathscr{D}}$ followed by the right adjoint $\underline{\underline{\mathrm{Cat}(\mathscr{E})}}^{\mathscr{D}} \to \underline{\underline{\mathrm{Cat}(\mathscr{E})}}^{\Psi}$ given by restriction. Therefore it preserves (κ-)filtered 2-colimits and cotensors by $\mathscr{V}$ and hence $\mathscr{V}$-codescent objects. □

Remark 5.5. If we our topological spaces were defined to be e.g. Δ-generated topological spaces then we could also use Section 2.5 of [Ver92], but the proof we gave above will also work if we take $\underline{\mathrm{Top}}$ to be compactly generated Hausdorff spaces. We have left unanswered a few questions about what kind of colimits in $\underline{\underline{\mathrm{Cat}(\mathscr{E})}}$ when $\mathscr{E}$ is locally bounded. In the internal setting, we might not have anything as nice as [KL01], but it may be possible to combine this with base-change in Verity's thesis.

We already saw in Proposition 5.4 that $\underline{\underline{\mathrm{Ps}\text{ - }\mathbf{D}\text{ - }\mathrm{Alg}}}$ and $\underline{\underline{\mathbf{D}\text{ - }\mathrm{Alg}_{\mathrm{s}}}}$ admit $\mathscr{V}$-copowers. Furthermore, since we know our $\mathscr{V}$-2-monad $\mathbf{D}$ preserves $\mathscr{V}$-copowers, we can use that and the descent object definition of internal homs of algebras to verify ι preserves $\mathscr{V}$-copowers. We now conclude:

Corollary 5.6. *The strictification of pseudo* $\mathbf{D}$*-algebras to strict* $\mathbf{D}$*-algebras we were asking for back in Section 1.3 exists.*

References

[Ave17] Tom Avery. *Structure and Semantics*. PhD thesis, University of Edinburgh, 2017. arXiv:1708.01050.

[BE72] Andrée Bastiani and Charles Ehresmann. Categories of sketched structures. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 13(2):104–214, 1972.

[Ber18] John D. Berman. On the commutative algebra of categories. *Algebraic and Geometric Topology*, 18(5):2963–3012, August 2018.

[BKP89] R. Blackwell, G. M. Kelly, and A. J. Power. Two-dimensional monad theory. *Journal of Pure and Applied Algebra*, 59(1):1–41, July 1989.

[BMW12] Clemens Berger, Paul-André Melliès, and Mark Weber. Monads with arities and their associated theories. *Journal of Pure and Applied Algebra*, 216(8):2029–2048, August 2012.

[CG] Alexander S. Corner and Nick Gurski. Operads with general groups of equivariance, and some 2-categorical aspects of operads in Cat. arXiv:1312.5910.

[CHP04] Eugenia Cheng, Martin Hyland, and John Power. Pseudo-distributive laws. *Electronic Notes in Theoretical Computer Science*, 83, 2004.

[CS10] G.S.H. Cruttwell and Michael Shulman. A unified framework for generalized multicategories. *Theory and Applications of Categories*, 24:580–655, 2010. arXiv:0907.2460.

[Fra11] Ignacio López Franco. Pseudo-commutativity of KZ 2-monads. *Advances in Mathematics*, 228(5):2557–2605, 2011.

[Fuj18] Soichiro Fujii. *Foundations of Algebraic Theories and Higher Dimensional Categories*. PhD thesis, University of Tokyo, 2018.

[Gar08] Richard Garner. Polycategories via pseudo-distributive laws. *Advances in Mathematics*, 218(3):781–827, 2008.

[†††]Relatedly, the notoriously difficult-to-read [Kel80] by Kelly comes to mind.

[Gar20] Fernando Abellán García. Marked colimits and higher cofinality. October 2020. arXiv:2006.12416.

[GS16] Richard Garner and Michael Shulman. Enriched categories as a free cocompletion. *Adv. Math*, 289:1–94, 2016. arXiv:1301.3191.

[Hau16] Rune Haugseng. Bimodules and natural transformations for enriched ∞-categories. *Homology, Homotopy and Applications*, 18(1):71–98, 2016.

[Hin20] Vladimir Hinich. Yoneda lemma for enriched ∞-categories. *Advances in Mathematics*, 367:107129, June 2020.

[Hyl14] J. M. E. Hyland. Elements of a theory of algebraic theories. *Theoretical Computer Science*, 546:132–144, August 2014.

[Kei78] William F. Keigher. Symmetric monoidal closed categories generated by commutative adjoint monads. *Cah. Topol. Géom. Différ. Catég.*, 1978.

[Kel80] G. M. Kelly. A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on. *Bulletin of the Australian Mathematical Society*, 22(1):1–83, 1980.

[Kel82] G. M. Kelly. *Basic Concepts of Enriched Category Theory*, volume 64 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 1982. Also available online in Reprints in Theory and Applications of Categories, No. 10 (2005) pp. 1-136.

[Kel05] G. M. Kelly. On the operads of J. P. May. *Reprints in Theory and Applications of Categories*, (13):1–13 (electronic), 2005.

[KL01] G. M. Kelly and Stephen Lack. $\mathscr{V}$-Cat is locally presentable or locally bounded if $\mathscr{V}$ is so. *Theory Appl. Categ.*, 8(23):555–575, 2001.

[Lac02] Stephen Lack. Codescent objects and coherence. *Journal of Pure and Applied Algebra*, 175(1-3):223–241, 2002. Special volume celebrating the 70th birthday of Professor Max Kelly.

[Lac09] Stephen Lack. A 2-categories companion. In John C. Baez and J. Peter May, editors, *Towards Higher Categories*, volume 152 of *The IMA Volumes in Mathematics and Its Applications*, pages 105–192. Springer, 2009. arXiv:math.CT/0702535.

[Lei02] Tom Leinster. Generalized enrichment of categories. *Journal of Pure and Applied Algebra*, 168(2-3):391–406, 2002. Category theory 1999 (Coimbra).

[Maya] J. Peter May. From Structured G-categories to Structured G-spectra.

[Mayb] J. Peter May. The monadic theory of operadic multicategories.

[ML98] Saunders Mac Lane. *Categories for the Working Mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer, second edition, 1998.

[MS21] John Lauchlin MacDonald and Laura Scull. General 2-Dimensional Adjunctions, Universal Monads and Simplicial Structures. March 2021. arXiv:2103.02001.

[Shu10] Michael Shulman. Constructing symmetric monoidal bicategories. arXiv:1004.0993, 2010.

[Shu18] Michael Shulman. Contravariance through enrichment. *Theory and Applications of Categories*, 33(5):95–130, 2018. arXiv:1606.05058.

[Str04] Ross Street. Categorical and combinatorial aspects of descent theory. *Applied Categorical Structures*, 12:537–576, October 2004.

[Ver92] Dominic Verity. *Enriched Categories, Internal Categories, and Change of Base*. PhD Thesis, University of Cambridge, April 1992. Macquarie Mathematics Report No. 93-123.

[Wal18] Charles Walker. Yoneda structures and KZ doctrines. *Journal of Pure and Applied Algebra*, 222(6):1375–1387, June 2018.