RIEMANN MAPPING THEOREM AND PLANAR MODELS OF HYPERBOLIC GEOMERTRY

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Abstract. Hyperbolic geometry examines spaces with constant negative curvature. This paper will present and discuss three planer models of this geometry: the Poincaré disk model, the upper half-plane model, and the Klein disk model. This paper will also provide a proof of the Riemann Mapping Theorem and discuss the role it plays with these models, as well as what it means for creating other planer models of hyperbolic geometry.

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1. Introduction/overview

Euclid's fifth postulate states there exists a unique line passing through a given point parallel to a given line. The debate over the necessity of this assertion sparked three major types of geometry: Euclidean, spherical, and hyperbolic. We write on flat paper and live on a sphere making it more common to experience both Euclidean and spherical geometries. Hyperbolic geometry, however, is less frequently encountered. As such, hyperbolic geometry often seems the most alien and difficult to picture of the three geometries. Yet, the Riemann Mapping Theorem shows that hyperbolic geometry may be the easiest geometry to model. This theorem can be used to show that all open simply connected subsets of the plane are locations for conformal hyperbolic models. Therefore, hyperbolic geometry can be displayed truthfully on a finite sheet of paper without distorting angles and small images.

In this paper, I introduce and connect three models of hyperbolic geometry and prove the Riemann Mapping Theorem. The second section presents the Poincaré, upper half-plane, and Klein models using cross-ratios based on Daniil Rudenko's

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REU 2019 apprentice lectures. The third and fourth sections provide necessary background for and a proof of the Riemann Mapping Theorem. In the fifth and final section, I will discuss consequences of the Riemann Mapping Theorem and the Schwarz Lemma. In particular, I will focus on the models displayed in section two and the implications for planar conformal models of hyperbolic geometry. Finally, I will mention a generalization of the Riemann Mapping Theorem, the Uniformization Theorem, and describe what it means for constructing planar models of constant curvature geometries.

2. Models of Hyperbolic Geometry

This section will exposit a construction of three models of hyperbolic geometry. These models depict the same space, yet some models are more conducive to certain types of proofs or calculations. Before we introduce the first model we will provide a definition for two fundamental concepts in studying geometry: geodesics and isometries.

Definition 2.1. A path between two points in a metric space is called a **geodesic** if and only of it is the shortest path between those points.

Definition 2.2. A transformation in a metric space is know as an **isometry** if and only if it is a bijection and the distance between any two points in the image is the same distance as the preimage of those two points.

2.1. Poincaré disk model.

One of the most common models is the Poincaré disk model. This model is restricted to the unit disk within the *Riemann sphere*, the complex plane with an infinity point. We denote the space of the model, the unit disk, with $\mathbb{D}$,

$$\mathbb{D} = \{z \in \bar{\mathbb{C}} : |z| < 1\}.$$

Notice this does not include the unit circle. The unit circle can be thought of as the boundary at infinity for this model. In this model, geodesics are Euclidean straight lines through the origin and arcs of Euclidean circles orthogonal to the unit circle. There is an example of both types of geodesics in Figure 2-1.

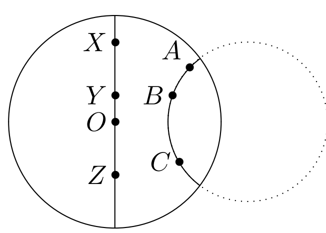


Figure 2-1

The points A,B,C lie on a hyperbolic line, which is the arc of a circle perpendicular to the unit circle. The points O,X,Y,Z also lie on a hyperbolic line which is a Euclidean line passing through the origin (labeled O).

To motivate this definition consider the following construction. Near the origin of the Poincaré model, distances are shorter. Thus, if we try to find a short path between two points, the ends of this path would first move toward the origin before connecting. It is not hard to see that curving toward the center will shorten the length of this path. We must move along a circle to do this consistently. Small steps along the unit circle are infinitely long; the path should first move directly to the center. Our short path described here is the arc of a circle orthogonal to the unit circle - a geodesic in the Poincaré model.

Negative curvature implies that different paths diverge as we move away from the origin. In Euclidean geometry, curvature is zero and geodesics are lines which separate at a constant rate. On a circle, geodesics curve back toward each other. Think of how different paths moving directly south from the north pole will curve back to each other and eventually intersect again at the south pole. Geodesics do the opposite in hyperbolic geometry. Instead of curving back together, curves separate. This separation of geodesics allows Euclid's fifth postulate to no longer hold.

This Poincaré model is important because it is conformal; meaning, it preserves angles. (The lack of formality in this meaning of conformal is discussed in section 3 with holomorphic maps.) Drawing two intersecting geodesics, the Euclidean angle they make in the model is the angle they make in hyperbolic space. This may seem like a rather simple addition, but this is the only model in the unit disk for which this is true, including for other geometries of constant curvature.

Proposition 2.3. *There exists a unique geodesic between every pair of points.*

This proposition is a necessity for all metrics. It can be shown geometrically, but the proof is not enlightening and conceptually much easier in other models and we shall prove it later. Now equipped with geodesics, we can begin to measure distances. The definition we will use relies on cross-ratios.

Definition 2.4. Take the geodesic between two points A and B, extend the geodesic to the boundary, and denote the boundary points as X and Y, where A is between X and B. Denote the **Poincaré hyperbolic distance** between A and B to be

$$d(A, B) = |log([X, A, Y, B])|,$$

where $[X, A, Y, B]$ is the cross-ratio of X, A, Y, B.

Remark 2.5. The readers who have seen other methods of measuring distance can easily check this is an equivalent definition. In particular, one method is to show the distance between the origin and a point z is $log(\frac{1+|z|}{1-|z|})$, as it is in other metrics. Then we use the fact this definition has the same group of isometries as other metrics. Because there exists an isometry sending any point to the origin, the distance from the origin generalizes to distances between arbitrary points.

With a metric and geodesics defined, it is now possible to find isometries. For distances to be well defined, isometries must do a few other things. They must preserve the space: map the unit disk and the unit circle to themselves separately. They must also send geodesics to geodesics (that is, circles orthogonal to the unit circle and lines through the origin must be sent to circles orthogonal to the unit circle and lines through the origin). We can show that functions with a few important properties are isometries in the Poincaré disk model.

Theorem 2.6. *Bijective maps that preserve the unit disk, unit circle, cross-ratios, angles, and the class of circles and lines are isometries of the Poincaré disk model.*

Proof. Let f be a bijective map that preserves the unit disk, cross-ratios, angles, and the class of circles and lines.

Fix A and B in $\mathbb{D}$. Then there exists a geodesic, g, between them that intersects the unit circle at X and Y. Thus, by definition $d(A, B) = |log([X, A, Y, B])|$. Now since f preserves the unit disk, $f(A)$ and $f(B)$ are still in the unit disk. As such there is a geodesic,g', that connects $f(A)$ and $f(B)$. This geodesic is the unique circle or line that passes through $f(A)$ and $f(B)$ while intersecting the unit disk at right angles.

Notice that, $f(g)$, the image of g under f is a circle or a line since the class of circles and lines is preserved by f. Likewise since g intersected the the unit circle at right angles, $f(g)$ intersects the image of the unit circle - the unit circle - at right angles. Furthermore, A and B are in g, thus $f(A)$ and $f(B)$ are in $f(g)$. As such $f(g)$ is by definition the geodesic connecting $f(A)$ and $f(B)$. Since these geodesics are unique, $f(g) = g'$. The curve g intersected the unit disk at X and Y and since the unit disk is preserved g' intersects the unit disk at $f(A)$ and $f(B)$. Thus, we can see the distance before and after the maps is conserved,

$$d(f(A), f(B)) = |log([f(X), f(A), f(Y), f(B)]) = |log([X, A, Y, B]) = d(A, B).$$

We have just shown that the distances between points in the unit disk are preserved under the function f and so f is an isometry.

□

The group of transformations that meets these criteria is known as the Möbius transformations preserving the unit disk. All Möbius transformations are maps from the Riemann sphere to itself which preserve angles (*conformal*), preserve cross-ratios, and send circles and lines to circles and lines (in this extended space of the Riemann sphere lines can be thought of as circles containing the infinity point in which case they send circles to circles). There is a special class of these transformations which are called orientation preserving (think angles are measured counterclockwise across transformations). Möbius transformations have a general form of a fraction of two polynomials:

$$\frac{az + b}{cz + d} : ad - bc \neq 0.$$

Where the a, b, c, d, z are all complex numbers. Möbius transformations, in general, are useful in the discussion of hyperbolic geometry due to the fact they are conformal and bijection functions. These two properties will be discussed in more detail in later sections with the concepts of biholomorphic maps. The group of Möbius transformations preserving the unit disk have an important subset of transformations which are reflections across lines through the origin and inversions about circles orthogonal to the unit circle.

These isometries are closely linked to reflections over geodesics of the space. Circles inversions are a kind of reflection over an arc instead of a straight line. In Euclidean geometry, every isometry can be written as the composition of one, two, or three reflections over geodesics (lines). This fact is also true in hyperbolic space. To understand it, consider an inversion that will fix the new center of the model. Then one reflection across a line through the center can fix a second point. Finally, there are only two locations for any third point given its angle and distance from the second point. The third transformation is either unnecessary or moves the third point to the proper location. Three points determine the plane, so everything else will be fixed.

Proposition 2.7. *The definition of distance in the Poincaré model is a metric.*

We will be taking the proposition as a fact. Being a *metric* requires a few properties. Distances are non-negative and only zero if the points are identical. The distance between two points does not depend on the order of the points. The triangle inequality holds; geodesics are indeed the shortest paths between points. The first two are easier to show, but triangle inequality is more difficult. A hint as to prove it is to send one vertex of the triangle to the origin and compute the cross-ratios. The proof will follow from the Euclidean triangle inequality.

At this point we have defined a notion of geodesics, distances, angles, and isometries. We effectively have a hyperbolic "straight edge" and "compass" as such we have defined all the foundational tools to geometrically study this space.

2.2. **Upper half-plane model.** A very similar model to the disk model is the upper half-plane model. Instead of the domain being within the unit disk, it is on the half-plane above the real number line. The domain of the space is denoted with $\mathbb{H}$.

$$\mathbb{H} = \{z \in \bar{\mathbb{C}} : Im(z) > 0\},$$

where $Im(z)$ is the imaginary part of z.

Pictorially, if the unit circle was a rubber band we could imagine cutting it at a point and stretching the ends off to infinity. More formally, the boundary is a stereographic projection of the unit circle onto a tangent line. All the points within the model can then be identified as the intersection points of geodesics. Since boundary points define geodesics, this gives a full map of the space. A more precise map between the two models is known as the *Cayley map* which is denoted as ϕ.

$$\phi : \mathbb{H} \mapsto \mathbb{D} : \phi(z) = \frac{z-i}{z+i}$$

This function has a few interesting properties. First, notice it is a Möbius transformation. The map is thus bijective so it has an inverse. Geodesics remain circles and lines perpendicular to the boundary (now the real number line) with some passing through the infinity point as one of the intersections with the boundary (these are vertical lines). In addition, the distance between two points has the same definition as with the Poincaré disk model since cross-ratios are preserved. Finally, since angles are preserved this means angles in the Upper-half plane model will be the same angles as those perceived in the space meaning the Upper-half plane model is a conformal model just like the Poincaré disk model.

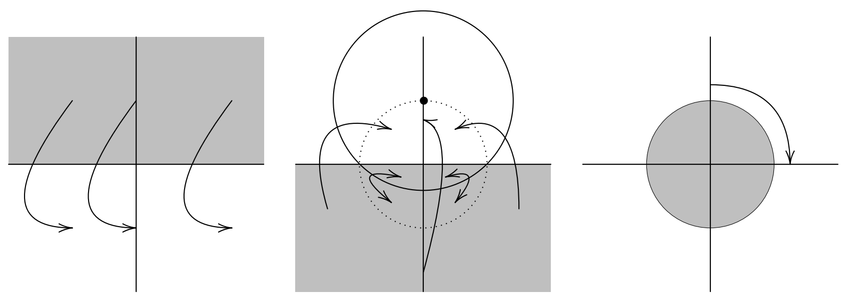
There is another way of understanding this map, and that is to break it down into the component transformations. It is possible to check that this map is equivalent to three separate transformations:

$$f(z) = \bar{z}$$

$$g(z) = i + \left(\frac{\sqrt{2}}{|z-1|}\right)^2 (z-i)$$

$$h(z) = -iz.$$

When composed in the correct order, we get the Cayley map: $\phi(z) = h(g(f(z)))$. We can examine each of these transformations individually to see how they change the model. The first function, f, is a reflection over the real number line making this the upper half-plane instead of the lower half-plane model. The last function, h, is a rotation, and is present because it is convenient to have 1 map to infinity and -1 map to 0 instead of i mapping to infinity and $-i$ mapping to 0. These two transformations will send circles to circles, send lines to lines, preserve angles, and preserve cross-ratios. The real change is in the middle function, g. This is an inversion with respect to a circle centered at i and with radius $\sqrt{2}$. This is where the unit circle is sent to the half-plane. Figure 2-2 shows the affect of these three transformations on the model represented by the shaded regions.



$f(z)$ $g(z)$ $h(z)$

Figure 2-2

There are a few facts about circle inversions which are another way to see the symmetries between the Poincaré disk and the half-plane models. First, inversions with respect to circles preserve angles, intersection points, and cross-ratios. They have the property that they send lines that do not pass through the center of the circle being inverted over to circles, and lines through the center are sent to themselves. Circles that do not pass through the origin of the circle of the inversion remain circles and circles that do pass through the center of the inversion become lines. Thus, all the circles that pass through the point $z = 1$ become vertical lines and all the lines not passing through the point $z = 1$ become circles.

This model can be useful for various calculations and proofs. As an example, recall Proposition 2.3 that claims geodesics are unique is simpler in this model since circles orthogonal to the boundary have centers that lie on the boundary. Thus, we will now provide a proof of it.

Proof. Proposition 2.3

It suffices to show that there exists a unique geodesic between every two points in $\mathbb{H}$ since the Cayley map is bijective and geodesics in one model are the images of geodesics in the other model. Let x and y be distinct points in $\mathbb{H}$. Either $Re(x) = Re(y)$ that is x and y lie on a vertical line or they don't and $Re(x) \neq Re(y)$.

Case 1 $Re(x) = Re(y)$.

No other vertical line can connect them and all circles containing x and y must have a center of the perpendicular bisector of the vertical line that connects them. This perpendicular bisector thus must be parallel to the real number line since they are both perpendicular to the vertical line connecting x and y. Thus, the centers all circles containing x and y do not lie of the real number line and so they do not intersect the real number line twice at right angles and so they are not geodesics. Therefore, this vertical line is the unique geodesic between x and y.

Case 2 $Re(x) \neq Re(y)$.

The perpendicular bisector between x and y is not parallel to the real number line and so it intersects the real number line exactly once. That means there is a unique circle containing x and y which intersects the real number line twice at right angles. Since there is no possible vertical line between x and y this circle creates the unique geodesic between x and y. □

Like the Poincaré disk, this model is the unique model in the upper half-plane where angles are as they appear. Moreover, notice how straight forward this construction was given we already knew the Poincaré disk model. We will see why this is the case while discussing the Riemann Mapping Theorem.

2.3. **Klein disk model.** One popular model of hyperbolic geometry that does not preserve angles is the Klein disk model. Like the Poincaré model, this model lives on the unit disk. Unlike the other models, all geodesics in the Klein model are Euclidean straight lines. Nonetheless, distances can still be measured with cross-ratios.

Definition 2.8. Take two points A and B in the unit disk and connect them with the Euclidean line l. This line will intersect the unit circle twice say at two points X and Y where A lies between X and B on l. The **Klein hyperbolic distance** between A and B is denoted $d(A, B)$.

$$d(A, B) = |\frac{1}{2}(log[X, A, Y, B])|$$

There is a map from the Poincaré model into the Klein model denoted ψ.

$$\psi(z) = \frac{2z}{|z|^2 + 1}$$

Intuitively understanding how this is a map between the models is more difficult than understanding the Cayley map. Most of the proofs lie on the computational side rather than the geometric. To understand the model, try this construction: there is no need to change the points on the boundary, so we keep them fixed. Geodesics are straight, no longer arcs of circles. Points have to stay on their geodesics as we move them and so they remain between their endpoints. As such, points move radially out from the origin. To flatten out these circles imagine, a "pushing out" of all points that is lessened the farther away they are from the origin, much like a magnet in the center repelling smaller magnets on the disk.

Moreover, to see this numerically, we plug in corresponding values to the $\psi(z)$ equation. A point p is on the unit circle if and only if $|p| = 1$. We can see that this map preserves the unit circle:

$$\psi(p) = \frac{2p}{|p|^2 + 1} = \frac{2p}{1^2 + 1} = \frac{2p}{2} = p.$$

In addition, the origin is sent to itself:

$$\psi(0) = \frac{2(0)}{|0|^2 + 1} = \frac{0}{1} = 0.$$

This map also preserves direction from the origin:

$$\frac{\psi(x)}{|\psi(x)|} = \frac{\frac{2x}{|x|^2+1}}{|\frac{2x}{|x|^2+1}|} = \frac{\frac{2x}{|x|^2+1}}{\frac{2|x|}{|x|^2+1}} = \frac{x}{|x|}.$$

Proposition 2.9. *The function ψ is a bijective map from the unit disk to itself.*

This is the first of the computation heavy relations between the Klein and Poincaré models. It is easy to see that the origin and the boundary are fixed. If a point A is closer to the origin than B, then the image of A will be closer to the origin than the image of B for all A and B on the unit disk or on the unit circle. Therefore, points on the unit disk (closer to the origin than the unit circle) will be sent to the unit disk. Bijectivity follows from the construction of an inverse function which is constructed by finding the inverse function for the absolute value of z, then specifying the direction from the origin.

Proposition 2.10. *Poincaré model geodesics are mapped by ψ to Klein geodesics.*

The key to this proof is showing that the change in distance from the origin will send a point with an arbitrary geodesic going through the origin to a point on the Klein geodesic with the same end points. To illustrate how this can be achieved consult Figure 2-3. The proof shows the point P is sent to the point Q. It relies on the fact that OXC, OBX and OPA, OQB are two pairs of similar triangles.

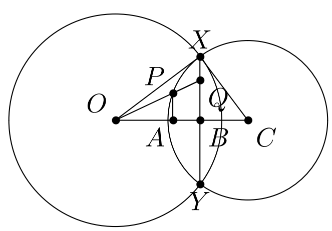


Figure 2-3

This construction is now more or less complete, yet at this point the relationship between these disk models is not fully clear. It is simple enough to understand how they are both models of hyperbolic geometry. Intersection points match so all the axioms concerning parallel lines match up. Like in the Poincaré model, we see distances shorten as we move away from the center, but no longer uniformly across all directions. This lack of uniformity makes it no longer effective to curve toward the center to shorten a path and enables straight lines to be our geodesics.

Part of the difficulty in translating between these two models is they are profoundly different types of models. The defining feature of the Poincaré model was angle preservation. The special feature of the Klein model is the preservation of "straightness." Formally, geodesics in the model are geodesics in the space. There is a connection between the Poincaré and upper half-plane models because they both preserve angles. There is no such connection between the Klein and Poincaré models.

3. Background for the Riemann Mapping Theorem

We have now seen a few models of hyperbolic geometry and can start asking questions about other possible models. How special is the connection between the half-plane model and the Poincaré disk model? What other models are possible and do any of them have a similar connection to the Poincaré disk model? This section gives background for the Riemann Mapping Theorem which can help answer these questions.

Theorem 3.1 (Riemann Mapping Theorem)**.** *Every simply connected proper open subset of the complex plane is conformally equivalent to the unit disk.*

For clarification we need to define some of the terms used above.

Definition 3.2. An open set X is called **simply connected** if and only if every continuous loop in X can be shrunk down to a point while remaining in the domain and every two points in X has a continuous path between them in X.

Simply connected has two requirements. The first is every closed loop bounds a subset of the set. This means that if a loop can be made around a point, then that point is in the set. There are no holes. The second requirement is that every two points can be connected by a continuous path; the set must all be one piece.

Definition 3.3. Two sets open sets are said to be **conformally equivalent** if there exists a bijective map between them that preserves angles between curves.

The definition of conformal equivalence relies on the idea of angle. What is the angle between two curves? The basic concept is the angle between two curves at an intersection point is the angle between their tangent lines at the intersection point. A fraction of complex numbers is only positive if the two points corresponding to the numerator and denominator lie on the same ray from the origin. Informally the numerator and denominator are pointing in the same direction. As such there is a way to provide a symbolic description of angle with a fraction of complex derivatives where the numerator is rotated so that the whole is positive. Consider two differentiable curves c_1 and c_2 which intersect at a point a in the complex plane at times t_1 and t_2 respectively such that,

$$e^{i\theta}\frac{c_1'(t_1)}{c_2'(t_2)} > 0 : 0 \leq \theta < 2\pi.$$

There is only one value of θ for which this is true. We can see that this produces an adequate definition of angle. With this definition we can see there exists an equivalent definition of a conformal bijection in complex analysis called a holomorphic bijection (or biholomorphic map). Holomorphic can be thought of as "complex differentiable" or differentiable in all complex directions. Most elementary calculus derivative rules apply to complex derivatives: product rule, power rule, quotient rule, and chain rule. Biholomorphic implies conformal by the chain rule:

$$\frac{[f(c_1(t_1))]'}{[f(c_2(t_2))]'} = \frac{f'(c_1(t_1))c_1'(t_1)}{f'(c_2(t_2))c_2'(t_2)} = \frac{f'(a)c_1'(t_1)}{f'(a)c_2'(t_2)} = \frac{c_1'(t_1)}{c_2'(t_2)}.$$

Bijective functions cannot have zero derivative over an interval and derivatives can be written as limits and so take place on intervals where the derivative is non-zero. Thus, we can avoid dividing by zero from the derivative if $f'(a)$ is zero. We have just shown that biholomorphic functions preserve angle. Unfortunately, showing the other direction is not as easy and will be taken as a fact. Recall that we took it as a fact that Möbius transformations are conformal bijections. We can now show this by showing that they are holomorphic. Recall they have a general form of a fraction of two polynomials:

$$\frac{az+b}{cz+d} : ad - bc \neq 0.$$

Using basic calculus rules we can see the derivative is $\frac{ad-bc}{(cz+d)^2}$. We can see the derivative is defined everywhere the function is and can be piecewise defined as infinite at $z = \frac{-d}{c}$ just like the Möbius transform can be defined as infinite at that point. The functions are holomorphic justifying our earlier assertion that they are conformal. We can use tools in complex analysis to show sets are conformally equivalent by showing the existence of a biholomorphic map between them and so we can make a more precise definition of conformally equivalent.

Definition 3.4. Two sets open sets are said to be **conformally equivalent** if there exists a biholomorphic map between them.

The upper half plane is a simply connected proper open subset of the complex plane, so the Riemann mapping theorem guarantees the existence of a conformal bijection from it to the unit disk. In fact, we have already exhibited an example of such a map: the Cayley map. However, there appears to be one problem. The Cayley map is a bijective function from the Riemann sphere to itself not the complex plane to itself. This is not a problem since we are only considering part of the map from the upper half-plane to the unit disk. What this map does to points outside the upper half-plane is of no consequence to this theorem.

The Schwarz Lemma states there are only two types of holomorphic functions preserving the unit disk and sending zero to zero. The first are rotations about the origin. The second are those that "pull points to the center." This can be described mathematically as decreasing the absolute value, but a visual picture is an attraction to the origin

Lemma 3.5 (Schwarz Lemma). *If f is a holomorphic function from the unit disk to itself such that $f(0) = 0$, then one of the following holds:*

(i) $|f'(0)| = 1$ *and* $f(z) = e^{i\theta} z$ *for some* θ *such that* $0 \leq \theta < 2\pi$
(ii) $|f'(0)| < 1$ *and* $|f(z)| < |z|$ *for all* z *not equal to zero*

Proof. Define the function $g(z)$ as follows:

$$g(z) = \begin{cases} \frac{f(z)}{z} & z \neq 0 \\ f'(0) & z = 0 \end{cases}$$

Everywhere except $x = 0$, g is a composition of two continuous functions (f is continuous since it is holomorphic, since differentiability implies continuity). The only thing to check in order to ensure g is continuous is the case at zero:

$$\lim_{z\to 0} g(z) = \lim_{z\to 0} \frac{f(z)}{z} = \lim_{z\to 0} \frac{f(z)-0}{z-0} = \lim_{z\to 0} \frac{f(z)-f(0)}{z-0}.$$

This last expression is precisely the definition of $f'(0)$, so the limit of g as z goes to zero equals g when z is zero. Thus, the function is continuous at zero, and so on the entire unit disk. Being a continuous composition of holomorphic functions, f is holomorphic. We will now invoke a fact from complex analysis known as the

Maximum Modulus Theorem. This theorem says a holomorphic function can only attain a maximum value can on an interior point if the function is constant. An intuitive way to realize this is to note that holomorphic functions have derivatives that are independent of direction; if a function is increasing in one direction, it will decrease by the same amount in the opposite direction. These functions take a kind of average of the values around it (formally they have Laplacian equal to zero). Maximums are greater than all the points around them so they can not be this kind of average. This principle can give an upper bound for g. Since f maps into the unit disk, we have that $|f(z)| < 1$ for all z.

$$|g(z)| = \frac{|f(z)|}{|z|} < \frac{1}{|z|}$$

Combining this with the maximum modulus principle, we can find a bound on g at zero and so f' at zero. In essence, g at zero has the same bound as the bounds of points a given distance away.

$$|f'(0)| = |g(0)| \leq \sup\{|g(z)| : |z| = r\} < \frac{1}{r} : 0 < r < 1$$

We can improve this by taking the limit as r approaches one from the left, where it is defined.

$$|g(0)| < \lim_{r \to^- 1} \frac{1}{r}$$

Taking the limit the inequality no longer remains strict. Think of this as every strict upper bound greater than one can be improved by taking r to be close to one.

$$|g(0)| \leq 1$$

This inequality is true for all points on the unit disk. As such, the maximum modulus principle says it is equal only if the function is constant. Otherwise, the function must be strictly less than one. The first case is if equality holds.

$$|f'(0)| = 1 = |g(z)| = \frac{|f(z)|}{|z|} \Rightarrow |f(z)| = |z|$$

The only holomorphic functions from the unit disk to itself that preserve distances from the center are rotations. Now consider the second case where equality does not hold and $|g(0)| < 1$. The Maximum Modulus Theorem requires the function $|g|$ to be one nowhere on the unit disk, because it would be a maximum. The function is non-constant, so maximums are not allowed. Consequently, this inequality holds for all z on the unit disk:

$$|g(z)| = \frac{|f(z)|}{|z|} < 1 \Rightarrow |f(z)| < |z| : z \neq 0.$$

It is clear these two cases are those described in the lemma statement. □

For the proof of Riemann Mapping Theorem, we use the idea that there are only two types of holomorphic functions preserving the unit disk: rotations and those that decrease the derivative of zero. We could classify them differently; for our purposes, it is helpful to think of options in this way. The proof of Riemann Mapping Theorem in this paper will use other theorems from complex analysis which are beyond the scope of this paper. The first of them is known as the Open Mapping Theorem.

Theorem 3.6 (Open Mapping Theorem)**.** *If X is an open connected subset of the complex plane and f is a holomorphic non-constant function from X to the complex plane, then $f(X)$ is open.*

In short, this theorem says holomorphic functions send open sets to open sets. Think of the holomorphic functions discussed so far. We have referred to those that operate on the unit disk and the upper half-plane. Both of these spaces are open. The Cayley map sends an open set (the upper half-plane) to an open set (the unit disk). A way of understanding this is to note that the derivative is approximating a point in all directions. The value approximates the change of the functions very well on a small disk around a point necessitating this small disk exists.

The next theorem we are going to discuss is known as Montel's Theorem. It relates qualities about families of functions. When we refer to families of functions, they are a set of functions all with like properties. An example of a family of functions is the holomorphic functions which preserve the unit disk and send zero to zero. So, the Schwarz Lemma is a lemma about a family of functions. Montel's Theorem says that if a family of functions is locally uniformly bounded, then the family of functions is normal.

Definition 3.7. A family of functions $\mathcal{F}$ is **locally uniformly bounded** on a set K if and only if for all compact subsets of K, C , there exists a constant M_C such that:

$$\sup\{|f(z)| : z \in C, f \in \mathcal{F}\} < M_C.$$

Definition 3.8. A family of functions $\mathcal{F}$ form a set X to Y is **normal** if and only if every sequence of functions, $\{f\}$, in $\mathcal{F}$ has a subsequence, $\{f_n\}$, such that there exists a function, F, so that for all compact subsets of X, C, the sequence $\{f_n\}$ converges uniformly to F on C.

Theorem 3.9 (Montel's Theorem)**.** *If a family of holomorphic functions, $\mathcal{F}$, from an open set O to the complex plane is locally uniformly bounded, then $\mathcal{F}$ is normal.*

This theorem tells us that a family of holomorphic functions that is locally bounded must have locally convergent subsequences. This should seem very similar to a Bolzano–Weierstrass type theorem that says bounded sequences have convergent subsequences. This theorem is more specific requiring holomorphic functions and implying uniform convergence. For a proper proof look at Datar's paper [3].

Various nice qualities carry over from normal families to the functions they converge to. The proof of the Riemann Mapping Theorem will use some of these qualities that pass through to limit. Notably, holomorphic normal families will converge to holomorphic functions. The derivative at a point will be the limit of the converging functions' derivatives. These are relatively basic properties of uniformly converging functions. *Hurwitz's Theorem* is about one property that only sometimes passes through the limit: injectivity.

Theorem 3.10 (Hurwitz's Theorem)**.** *Let X be an open connected subset of the complex plane and $\{f_n\}$ be a sequence of holomorphic injective functions from X to the complex plane which converges uniformly on compact sets to a function F on X. Then, F is either injective or constant.*

This theorem enforces that a normal family of holomorphic injective functions does not necessarily have subsequences that converge to injective functions. Instead, there are two options: either injectivity holds or the result is a constant function. This can be conceptualized by imagining a function that gets close to a value at two points. A path between these points cannot hit the same value twice so it has to stay between values on the path. Then, the whole path is close to the value and will become constant. If one path is constant, then all paths are constant. Datar's paper provides a formal proof[3].

4. Proof of Riemann Mapping Theorem

We now have all the tools required to prove the Riemann Mapping Theorem. Before the proof starts, it may be helpful to sketch out a brief road map. The main idea of the proof is that we can create a family of injective holomorphic functions that send a point to zero from an arbitrary open connected subset of the complex plane to the unit disk. This family has a subsequence that converges to our desired function. There are three major steps in the proof. The first is showing our family of functions is non-empty. Next, we will show there is a member of the family which maximizes the derivative at zero. Finally, we will show this function is also surjective and so has all of our desired properties.

Riemann Mapping Theorem. Let X be an open simply connected subset of the complex plane which is not the whole complex plane. Let a be some point in X. Let $\mathcal{F}$ denote the family of injective holomorphic functions from X to the unit disk that send the point a to the origin.

Finding an element of $\mathcal{F}$ is relatively simple, since it doesn't have to take up the entire disk, just part of it. Since X is not all of the complex plane, there is a point p not in X. We can then create $f(x)$, a square root distance function, which is holomorphic on X:

$$[f(x)]^2 = x - p.$$

Now we can easily check this is injective:

$$h(x) = h(y) \Rightarrow x - p = h(x)^2 = h(y)^2 = y - p \Rightarrow x = y.$$

Furthermore, this function does not send X to the entire plane; in particular, there is a small disk not in the image of X. Since X is open (and so is $f(X)$), there is a small disk around the image of a which its negative is not in the image due to the square. Let r be the radius of the small disk surrounding $f(a)$ that is contained in $f(X)$. This r exists because $f(X)$ is open by the Open Mapping Theorem. Now let c be in this small disk centered at $f(a)$. We know c is in $f(X)$ we want to show that $-c$ is not in $f(X)$.

Claim 1 The point $-c$ is not in $f(X)$.

We will show this by contradiction. Suppose that $-c$ is in $f(X)$. Let $f(x) = c$ and $f(y) = -c$.

$$x - p = f(x)^2 = c^2 = (-c)^2 = f(y)^2 = y - p$$

This implies that $x = y$ and so x has two images, which means that f is not a function. Thus, we see this is a contradiction, and so it is not possible for such a y to exist in X. We conclude that $-c$ is not in $f(X)$.

End of Claim 1

Now, we can translate this center of the hole to the origin. This function has a lower bound on the absolute value, so we can invert it. This inverted function is still holomorphic and injective, and now bounded function as well. All that is left is to scale the output to be inside of the unit disk. The resulting function is

$$g(x) = \frac{r}{f(x) + f(a)}.$$

Now take a look at what happens to a point x in X. Recall no points in X are sent to any point less than r away from $-f(a)$, so the image is scaled down to be within the unit disk.

$$|g(x)| = \frac{|r|}{|f(x) + f(a)|} \leq \frac{|r|}{|r|} = 1$$

Thus, we can see that g is an injective holomorphic function from X to the unit disk with the unit circle. Now since X is open, $f(X)$ must also be open by the Open Mapping Theorem and so it cannot contain the boundary (the unit circle). So, in reality, g is an injective holomorphic function from X to the unit disk. Now, we know there exists an orientation preserving Möbius transform, $t(x)$, preserving the unit disk sending $g(a)$ to zero. Let this function composed with $g(x)$ be $h(x)$:

$$h(x) = t(g(x)).$$

Now t is an orientation preserving Möbius transformation: it is injective and holomorphic. We can see that h is a function that sends X to the unit disk and a to zero and is both injective and holomorphic. We see $\mathcal{F}$ is not empty because h is in $\mathcal{F}$ by definition. Now there are two cases: $\mathcal{F}$ is finite or infinite. If $\mathcal{F}$ is finite, we can find F in $\mathcal{F}$ such that $|F'(x)| = S = max\{|f'(a)| : f \in \mathcal{F}\}$. When $\mathcal{F}$ is infinite we will require a bit more work to find F. We can no longer find S with a maximum and instead will need the supremum:

$$S = \sup\{|f'(a)| : f \in \mathcal{F}\}.$$

Since $\mathcal{F}$ is infinite, we can create an infinite sequence of functions in $\mathcal{F}$, $\{f\}$, such that the sequence of real numbers $\{|f(a)|\}$ converges to S. Now since all of the functions in $\mathcal{F}$ send X to the unit disk, they are all bounded. Being bounded on all of X means also being bounded on compact subsets, so we can see that $\mathcal{F}$ is locally bounded. We can see that $\mathcal{F}$ meets the hypothesis of Montel's Theorem and hence $\mathcal{F}$ is normal. Since $\mathcal{F}$ is normal for every sequence, there is a locally uniformly converging subsequence.

Find a locally uniformly convergent subsequence of $\{f\}$, $\{f_n\}$, which converges to F. To mirror the finite case, we wish to show F is in $\mathcal{F}$. Now we know, by being a uniformly converged to function, F takes on important qualities. Holomorphic functions uniformly converge to holomorphic functions, so F is holomorphic. The range of F could be defined on the closure of the union of the ranges of each f, so the closed unit disk. We can use our usual trick: since F is holomorphic then the image of X is open, and so the range of F is restricted to the unit disk. Due to the fact $f(a)$ is zero for all f in $\mathcal{F}$, $F(a)$ must also be zero.

Applying Hurwitz's Theorem, notice F is either injective or constant. Note that each f in the sequence is injective, and so not constant having non zero derivative. Thus, S is greater than zero. We know F is not constant - and thus injective - because $|F'(a)|$ is equal to S, which is not zero. To recap, F is an injective holomorphic function from X to the unit disk sending a to zero. Therefore, F is in $\mathcal{F}$.

It turns out that this F is the function we desire. All that is left to show is the surjectivity of F, and this comes from the fact F maximizes the derivative at zero.

Claim 2 F is surjective.

This is a proof by contradiction. Suppose F is not surjective. If F is not surjective, then there exists a point, q, in the unit disk that is not the image of any x in X. We can now define a clever function in $\mathcal{F}$ with a derivative at zero its absolute value is greater than S, a contradiction.

There is an orientation preserving Möbius transformation preserving the unit disk, m, that sends the point q to zero. Now choose the holomorphic branch of the square root function R such that:

$$e^{\frac{1}{2}log(m(F(x)))} = R(m(F(x))).$$

Finally, find $R(m(0))$ and there is an orientation preserving Möbius transformation, n, sending this point back to zero. Notice that orientation preserving Möbius transforms are bijective holomorphic functions. The square root function R is a holomorphic function injective function. All three of these transformations preserve the unit disk. The functions n and m have well-defined inverses by being bijective. The square root function, R, also has an inverse, call this the square function: $s(x) = x^2$. We can shorten these functions by calling them G and H:

$$G(x) = n(R(m(x)))$$

$$G^{-1}(x) = H(x) = m^{-1}(s(n^{-1}(x))).$$

Notice, since G is holomorphic and injective sending zero to itself, we can compose F with G and get an element of $\mathcal{F}$, $I(x)$:

$$I(x) = G(F(x))$$

It is, thus, clear to see that I is an element of $\mathcal{F}$. We can write this equivalently with H.

$$H(I(x)) = F(x)$$

Now H is a holomorphic function sending the unit disk to itself and zero to zero. The Schwarz Lemma thus demands that there are two options for H. The first is that it is a rotation. The function H contains the square function which naturally sends x and $-x$ to the same place so it cannot be a rotation. The other option is that the absolute value of the derivative of H at zero is less than one, meaning F does not attain the largest absolute value derivative at a, S.

$$S = |F'(a)| = |(H(I(a)))'| = |H'(I(a))I'(a)| = |H'(0)||I'(a)| < |I'(a)| \leq S$$

This contradicts the fact that F had the largest absolute value of the derivative at a. As such our assumption that F was not surjective is false. Therefore, F is surjective.

End of Claim 2

So we can see that F meets all of our requirements it is a bijective holomorphic function from X to the unit disk. □

We have thus found a bijective conformal function from an arbitrary simply connected open set to the unit disk. For a bit of notation, we will call these functions Riemann maps. One question is whether or not this map is unique. Uniqueness is pertinent to imposing hyperbolic metrics with these maps.

5. Significance of Riemann Mapping Theorem

Applications of the Riemann Mapping Theorem are vast. One particular application hinted at throughout this paper is making models of hyperbolic geometry in arbitrary simply connected open spaces.

Definition 5.1. Take a simply connected subset of the complex plane, O, which is not the whole complex plane. For any two points x and y in O we can say that the **Poincaré hyperbolic distance** between x and y in O is

$$d_{P_O}(x, y) = d_P(\zeta(x), \zeta(y)),$$

where d_P is the hyperbolic distance in the Poincaré disk model and ζ is a Riemann map from O to the unit disk.

At first glance, this definition may seem ill-defined because there could be multiple bijective conformal maps. It turns out these mappings will be unique up to orientation preserving isometries of the Poincaré disk model. To show this, we will first use the Schwarz Lemma to show bijective conformal functions from the unit disk to itself are orientation preserving isometries of the Poincaré disk model.

Lemma 5.2. *A bijective conformal function maps the unit disk to itself if and only if it is an orientation preserving Möbius transformation preserving the unit disk.*

Proof. Let ζ be a bijective conformal map from the unit disk to itself. Then there exists an orientation preserving Möbius transformation preserving the unit disk, M, that sends $\zeta(0)$ to zero. The composition of these two functions is a bijective conformal map form the unit disk to itself. In addition, $M(\zeta(0))$ is zero, so this function meets the criteria for the Schwarz Lemma.

We now have two cases: either $M(\zeta)$ is a rotation or it has derivative less than one at zero. Notice the inverse of $M(\zeta)$ is also holomorphic sending zero to itself. The composition of $M(\zeta)$ and its inverse is the identity map with derivative one everywhere.

$$1 = |[\zeta^{-1}(M^{-1}(M(\zeta(0))))]'| = |[\zeta^{-1}(M^{-1}(0))]'||[M(\zeta(0))]'| \leq |[M(\zeta(0))]'| \leq 1$$

Clearly equality holds and so $M(\zeta)$ is a rotation. Notice that a rotation is another orientation preserving Möbius transformation preserving the unit disk. Therefore we can see the composition of this rotation and the inverse of M, $M^{-1}[M(\zeta(x))]$, is as well. Since ζ is this composition, it is an orientation preserving Möbius transform preserving the unit disk.

The other direction is simple. Since all orientation preserving Möbius transforms preserving the unit disk are bijective and conformal maps that send the unit disk to itself. □

With this it is possible to show that the Poincaré distance between two points in O from the definition earlier is well defined and does not depend on the Riemann map.

Proof. Let ζ and θ be two different bijective holomorphic maps from an open simply connected subset of the complex plane that is not the whole complex plane, O, to the unit disk. Consider two points x and y in O. We want to show the Poincaré disk distance between the image of these points is the same for both maps.

There is a straightforward map between these two images being $\theta(\zeta^{-1})$.

$$\theta(\zeta^{-1}[\zeta(x)]) = \theta(x)$$

Now notice $\theta(\zeta^{-1})$ is a bijective holomorphic map sending the unit disk to itself. By our lemma, we know this function composition is an orientation preserving Möbius transform preserving the unit disk - an isometry of the Poincaré disk model.

$$d_P(\zeta(x), \zeta(y)) = d_P(\theta(\zeta^{-1}[\zeta(x)]), \theta(\zeta^{-1}[\zeta(y)])) = d_P(\theta(x), \theta(y))$$

Thus, it is clear to see Poincaré distance between the images of points is the same for any two arbitrary Riemann maps. □

What this culminates to say is that there is a conformal model of hyperbolic geometry in any arbitrary open simply connected subset of the complex plane, which is not the whole plane. This model would have geodesics, which are the images of geodesics in the Poincaré disk model, under some Riemann map between the two spaces. Distance between two points, likewise, is the distance according to the Poincaré disk model between the image of these two points under any Riemann map. We have already seen this in the upper half-plane model, where we defined distances and geodesics as they were in the image of the Poincaré disk. One special quality about this model is that these geodesics and distances have meaning in the upper half-plane in their own right.

This theorem seems also to neglect Klein's model, which it indeed does. Since Riemann maps are unique up to Poincaré isometries and not Klein isometries, we find it is not possible to define a Klein metric in arbitrary simply connected open sets in the way we did for Poincaré. It is natural to wonder what makes the Poincaré model so special. Intuitively, the Poincaré model is the only model of hyperbolic geometry in the unit disk that has angles that are the same as the respective Euclidean angles (this can be shown from Lemma 5.2). The Riemann Mapping Theorem works around conformal maps. Since the Klein disk is a non-conformal model, we cannot extend the Klein metric with the Riemann Mapping Theorem.

There is a much stronger version of the Riemann Mapping Theorem known as the *Uniformization Theorem*. This theorem considers the whole Riemann sphere instead of just the complex plane and classifies all simply connected subsets into three classes. Each class is conformally equivalent to precisely one of the following: the unit disk, the complex plane, or the Riemann sphere. The complex plane is a site of conformal Euclidean geometry. The Riemann sphere is a site of conformal spherical geometry. The unit disk has a conformal model of hyperbolic geometry. Thus, each class of simply connected subsets has exactly one conformal model of constant curvature geometry.

This theorem does not prohibit the models of hyperbolic geometry in subsets of the Riemann sphere not in the class of those conformally equivalent to the unit disk. For example, it is possible to create a model of hyperbolic geometry using the whole complex plane. Angles in such a model are not the Euclidean angles they appear to be. For every simply connected subset of the Riemann sphere, there is only one conformal model of constant curvature geometry. If this subset is open and missing at least two points, then this geometry is hyperbolic geometry by the Riemann Mapping Theorem. In the introduction, I made a subjective claim that hyperbolic geometry may be the easiest geometry to model. This claim is motivated by how often we work on finite pieces of paper and consider only the inside and not the boundary of the paper. Thus, for angles to be as they appear, we must use a model of hyperbolic geometry.

Regardless of how prevalent these models are, do they have applications outside of mathematics? The answer is an emphatic yes. Hyperbolic geometry's negative curvature enables higher surface area shapes. For example, circles in hyperbolic geometry have an area that grows exponentially with radius, much faster than Euclidean or spherical geometries. In nature, there are multiple examples of things taking on hyperbolic geometry: coral reefs, jellyfish, and some plant leaves (https://seagardens.wordpress.com/hyperbolic-crochet/hyperbolic-shapes-in-nature/). Surface area maximization can be quite useful. Hyperbolic geometry can be used in industry as well, through circle packing. The changing areas in hyperbolic geometry can help find optimal circle packing techniques. In general, hyperbolic geometry's surface area increases can be quite useful outside of mathematics.

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