AN INTRODUCTION TO CURVATURE IN $\mathbb{R}^3$

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ABSTRACT. In this paper we will discuss curvature in $\mathbb{R}^3$. We will start with a brief explanation of regular surfaces and parameterized curves. We will then use these to discuss the first fundamental form and the Gauss map. Finally, we will discuss three notions of curvature of regular surfaces: the second fundamental form, Gaussian curvature, and mean curvature.

CONTENTS

1. INTRODUCTION

Through the second fundamental form and the Gauss map, we can gain a more intimate understanding of the geometry of a surface. As with a curve, an important geometric property to be understood is curvature. Through the definition of the Gauss map we can define the principal curvatures on a surface, as well as the Gaussian curvature and mean curvature. The second fundamental form also gives a related geometric interpretation of curvature on a surface.

2. REGULAR SURFACE

To begin the study of surfaces it is imperative to define a Regular Surface.

Definition 2.1. A subset $S \subset \mathbb{R}^3$ is defined to be a regular surface if, for every point $p \in S$, there is a neighborhood $V \subset \mathbb{R}^3$ and a map $f : U \to V \cap S$ of an open set $U \subset \mathbb{R}^2$ onto $V \cap S \subset \mathbb{R}^3$ such that
(1) f is differentiable
(2) f is a homeomorphism
(3) For each $q \in U$, the differential $df_q : \mathbb{R}^2 \to \mathbb{R}^3$ is injective.

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Condition (1) specifies that for

$$f(u,v) = (x(u,v), y(u,v), z(u,v)), (u,v) \in U$$

the functions $x(u,v), y(u,v), z(u,v)$ have continuous partial derivatives of all orders in U. Throughout this paper we will refer to the partial derivative of f with respect to u as f_u.

Since f is continuous by condition (1), condition (2) means that f has an inverse.

From this point onwards the term surface will be used to mean a smooth and regular surface.

Definition 2.2. The map f as defined above is called a parameterization. The neighborhood $V \cap S$ of p in S is called a coordinate neighborhood.

In order to understand much of the geometry later in this paper, a quick aside on the geometry of parameterized curves is necessary.

Definition 2.3. A parameterized differentiable curve is a differentiable map $\alpha : I \to \mathbb{R}^3$ of an open interval $I = (a,b)$ of the real line $\mathbb{R}$ into $\mathbb{R}^3$.

Parameterized curves have two important geometric properties: tangent vectors and curvature. The tangent vector is denoted $\alpha'(t)$ for $t \in I$ and is defined as usual in calculus.

Definition 2.4. Let $\alpha : I \to \mathbb{R}^3$ be a parameterized curve. The curvature of α at s is defined to be $k(s) = |\alpha''(s)|$.

Remark 2.5. We can write $\alpha''(s) = k(s)n(s)$ where $n(s)$ is defined as the normal vector in the direction of $\alpha''(s)$. Moreover, $n(s)$ is normal to $\alpha'(s)$ and called the normal vector.

Returning to surfaces, the first application of these properties of curves is in defining a tangent vector to a surface.

Definition 2.6. A tangent vector to S at a point $p \in S$ is defined to be the tangent vector $\alpha'(0)$ of a differentiable parameterized curve $\alpha : (-\epsilon, \epsilon) \to S$ with $\alpha(0) = p$.

This definition is used for the notion of tangent space.

Definition 2.7. The tangent space T_pS is the set of all tangent vectors to S at p.

Proposition 2.8. *Let $f : U \subset \mathbb{R}^2 \to S$ be a parameterization of a regular surface S and let $p \in U$. The subspace $df_p(\mathbb{R}) \subset \mathbb{R}^3$, coincides with the set of tangent vectors to S at $f(p)$.*

Proof. Let w be a tangent vector at $f(p)$, $w = \alpha'(0)$, where $\alpha : (-\epsilon, \epsilon) \to f(U) \subset S$ is differentiable and $\alpha(0) = f(p)$. The curve $\beta = f^{-1} \circ \alpha : (-\epsilon, \epsilon) \to U$ is differentiable.

$$df_p(\beta'(0)) = df_p(df_p^{-1}(\alpha'(0)) = \alpha'(0) = w$$

This can best be seen by computing $\beta'(0)$ in the canonical basis for $\mathbb{R}^2$ and $\mathbb{R}^3$. Then we can write $\alpha(t) = (u(t), v(t)), \quad t \in (-\epsilon, \epsilon)$ and $f^{-1} = (x(u,v), y(u,v), z(u,v))$.Therefore,

$\beta = f^{-1} \circ \alpha = (x(u(t), v(t)), y(u(t), v(t)), z(u(t), v(t)))$. Using the chain rule and taking derivatives at t=0, we obtain

$$\beta'(0) = (\frac{\partial x}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial t})(1,0,0) + (\frac{\partial y}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial t})(0,1,0) + (\frac{\partial z}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial t})(0,0,1)$$
$$= \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial t} \end{pmatrix} = df_p^{-1}(\alpha'(0))$$

therefore, $w \in df_p(\mathbb{R}^2)$.

On the other hand, take $w = dx_p(v)$, where $v \in \mathbb{R}^3$. We can write a curve $\gamma(t) = tv + p, \quad t \in (-\epsilon, \epsilon)$ where v is the velocity vector of the curve. We can take $\alpha = x \circ \gamma$. By the definition of the differential, $w = \alpha'(0)$ and is therefore the tangent vector. $\square$

3. First Fundamental Form

At this point we are ready to define one of the most important geometric properties of surfaces.

Definition 3.1. By restricting the natural inner product of $\mathbb{R}^3$ to T_pS of a regular surface S we get an inner product on T_pS. This inner product is called the first fundamental form and is denoted I_p.

This inner product $I_p : T_pS \to \mathbb{R}$ is given by $I_p(v, w) = \langle v, w \rangle$ where $\langle v, w \rangle$ denotes the standard Euclidean dot product in $\mathbb{R}^3$.

We can express the first fundamental form in terms of a basis f_u, f_v associated to a parameterization $f(u, v)$ at a point p. Taking a tangent vector $w \in T_pS$, w is the tangent vector to a parameterized curve $\alpha(t) = f(u(t), v(t))$ for $t \in (-\epsilon, \epsilon)$ at a point $p = \alpha(0) = f(u_0, v_0)$.
We may then compute:

$$\begin{aligned} I\langle \alpha'(0), \alpha'(0) \rangle &= \langle f_u u' + f_v v', f_u u' + f_v v' \rangle \\ &= \langle f_u, f_u \rangle (u')^2 + 2\langle f_u, f_v \rangle u'v' + \langle f_v, f_v \rangle (v')^2 \\ &= E(u')^2 + 2Fu'v' + G(v')^2, \end{aligned}$$

where u' and v' are the respective derivatives of u and v. The functions E, F, G computed at $t = 0$ are

$$\begin{aligned} E(u_0, v_0) &= \langle f_u, f_u \rangle \\ F(u_0, v_0) &= \langle f_u, f_v \rangle \\ F(u_0, v_0) &= \langle f_v, f_v \rangle, \end{aligned}$$

the coefficients of the first fundamental form in the basis f_u, f_v of T_pS.

4. Orientability

Definition 4.1. A regular surface S is called orientable if it is possible to cover it will a family of coordinate neighborhoods in such a way that, if a point $p \in S$ lies in two neighborhoods of this family, then the change of coordinates has positive Jacobian at p. The choice of such a family is called an orientation of S and S is called oriented. If such a choice is not possible, the surface is called nonorientable.

On a regular surface in $\mathbb{R}^3$ given a system of coordinates $f(u, v)$, we can get a more geometric idea of orientability via a unit normal vector N at p by:

$$N = \frac{f_u \times f_v}{|f_u \times f_v|}(p) \tag{4.2}$$

In another system of local coordinates $\bar{f}(\bar{u}, \bar{v})$ at p

$$\bar{f}_{\bar{u}} \times \bar{f}_{\bar{v}} = (f_u \times f_v)\frac{\partial(u, v)}{\partial(\bar{u}, \bar{v})} \tag{4.3}$$

Where $\frac{\partial(u,v)}{\partial(\bar{u},\bar{v})}$ is the Jacobian of the coordinate change. The above equation tells us that the sign of N will be invariant to coordinate change if the Jacobian is positive and will change sign when the Jacobian is negative. From this it can be seen that a surface is orientable if N keeps its direction.

Now it may be helpful to give an example of a nonorientable surface.

Example 4.4. To give an example of a non-orientable surface take a Mobius strip, M. The Mobius strip is non-orientable because you cannot find a differentiable field of normal vectors defined on the entire surface. We can define a vector field $N : M \to \mathbb{R}^3$ of unit normal vectors. By following the middle circle of the Mobius strip we can see that the field returns as $-N$, which contradicts the continuity of N.

5. Gauss Map

Definition 5.1. Let $S \subset \mathbb{R}^3$ be a surface with an orientation N. The map $N : S \to \mathbb{R}^3$ takes values in the unit sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

The map $N : S \to S^2$ is called the Gauss map of S.

Remark 5.2. The sign of the Gauss map is dependent on the orientation of the surface.

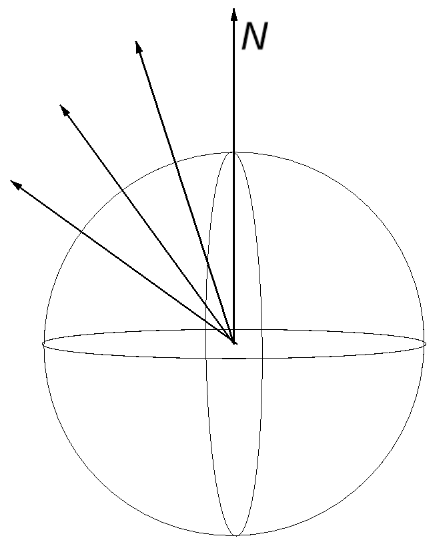
It can be seen that the Gauss map is differentiable. The differential dN_p of N at $p \in S$ is a linear map from T_pS to $T_{N(p)}S^2$.

Example 5.3. Consider the unit sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. Take the parameterized curve $\alpha(t) = (x(t), y(t), z(t))$ in S^2. Then

$$2xx' + 2yy' + 2zz' = 0$$

So the vector (x, y, z) is normal to the sphere at the point (x, y, z). Therefore, $N = (x, y, z)$ is a field of unit normal vectors in S^2 which points toward the center of the sphere. If the normal vector is restricted to the curve $\alpha(t)$ we obtain $N(t) = (x(t), y(t), z(t))$. Taking the derivative we obtain

$$dN(x'(t), y'(t), z'(t)) = (x'(t), y'(t), z'(t)).$$



Proposition 5.4. *The differential $dN_p : T_pS \to T_pS$ of the Gauss map is self-adjoint.*

Proof. Since dN_p is linear it is sufficient to show that $\langle dN_p(w_1), w_2\rangle = \langle w_1, dN_p(w_2)\rangle$ for a basis w_1, w_2 of T_pS. Take a parameterization $f(u,v)$ of S at p and f_u, f_v as the associated basis of T_pS. If $\alpha(t) = x(u(t), v(t))$ is a parameterized curve in S, with $\alpha(0) = p$ we obtain

$$\begin{aligned} dN_p(\alpha'(0)) &= dN_p(f_u u'(0) + f_v v'(0)) \\ &= \frac{d}{dt} N(u(t), v(t))|_{t=0} \\ &= N_u u'(0) + N_u v'(0) \end{aligned}$$

in particular, $dN_p(f_u) = N_u$ and $dN_p(f_v) = N_v$. Therefore, to prove that dN_p is self-adjoint, it suffices to show that $\langle N_u, f_v\rangle = \langle f_u, N_v\rangle$.

This can been seen by taking the derivative with respect to u of $\langle N, f_u\rangle = 0$ and the derivative with respect to v of $\langle N, f_v\rangle = 0$. We obtain

$$\langle N_v, f_u\rangle + \langle N, f_{uv}\rangle = 0$$
$$\langle N_u, f_v\rangle + \langle N, f_{vu}\rangle = 0$$

Therefore, $\langle N_u, f_v\rangle = -\langle N, f_{uv}\rangle = \langle N_v, f_u\rangle$. □

6. Second Fundamental Form

Using the Gauss map, multiple notions of curvature of a surface can be defined. The first of them is the second fundamental form.

Definition 6.1. The second fundamental form $II_p : T_pS \to \mathbb{R}$ given by:

$$II_p(v, w) = -\langle dN_p(v), w\rangle$$

A geometric interpretation of the second fundamental form can be understood using the fact that dN_p is self-adjoint. $N(s)$ denotes the restriction of the normal vector N to a parameterized curve $\alpha(s)$ where $\alpha(0) = p$.

$$\begin{aligned} II_p(\alpha'(0), \alpha'(0)) &= -\langle dN_p(\alpha'(0)), \alpha'(0)\rangle \\ &= -\langle N'(0), \alpha'(0)\rangle \\ &= \langle N(0), \alpha''(0)\rangle. \end{aligned}$$

Since $\alpha''(0) = k(s)n(s)$ from remark (2.5) we can write

$$II_p(\alpha'(0), \alpha'(0)) = \langle N, kn\rangle(p).$$

Definition 6.2. We can define the normal curvature of a surface along a parameterized curve as

$$k_n = kcos(\theta), \qquad cos(\theta) = \langle n, N\rangle.$$

Where θ is the angle between n, which is both the normal vector to the curve and tangent to the surface, and N the normal vector of the surface.

Using the above definition it can be seen that $II_p = \langle N, kn\rangle(p) = k\langle n, N\rangle = kcos\theta = k_n(p)$ giving a geometric understanding of the second fundamental form.

7. Gaussian Curvature

In addition to the second fundamental form, we can define two simple scalar quantities that measure the curvatures of a surface namely the Gaussian Curvature and Mean Curvature. All of these curvatures depend on the following definition:

Definition 7.1. The principal curvatures of a surface S at a point p are the maximum normal curvature k_1 and the minimum normal curvature k_2. The directions of k_1 and k_2 are called the principal directions at p.

Based on the definition of principal curvatures we can rewrite $II_p(v)$ in terms of k_1 and k_2. Given $v \in T_pS$ where $|v| = 1$ and take e_1 and e_2 which form an orthonormal basis of T_pS. We have $v = e_1 cos\theta + e_2 sin\theta$, where θ is the angle from e_1 to v in T_pS. We can write the normal curvature along v as

$$\begin{aligned} k_n = II_p(v) &= -\langle dN_p(v), v\rangle \\ &= \langle dN_p(e_1cos\theta + e_2sin\theta), e_1cos\theta + e_2sin\theta\rangle \\ &= \langle e_1k_1cos\theta + e_2k_2sin\theta, e_1cos\theta + e_2sin\theta\rangle \\ &= k_1cos^2\theta + k_2sin^2\theta. \end{aligned}$$

Returning to the differential of the Gauss map we can write dN_p in the basis e_1, e_2 as a matrix in terms of the principal curvatures. In fact,

$$dN_p = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$$

in the basis e_1, e_2.

Definition 7.2. Let $p \in S$ and $dN_p : T_pS \to T_pS$ be the differential of the Gauss map. The determinant of dN_p is the Gaussian curvature of K of S at p. The negative of half the trace of dN_p is called the mean curvature H of S at p.

Using the above matrix for dN_p we can write

$$K = k_1 k_2, \qquad H = \frac{k_1 + k_2}{2}$$

and we can also write K and H in local coordinates. Take a parameterization $x(u, v)$ at a point $p \in S$ and $\alpha(t) = x(u(t), v(t))$ a parameterized curve of S with $\alpha(0) = p$. In the following section all functions denote their value at the point p to simplify notation.

The tangent vector to $\alpha(t)$ at p is $\alpha' = x_u u' + x_v v'$ and $dN(\alpha') = N'(u(t), v(t)) = N_u u' + N_p v'$.

We can write $dN(\alpha') = (a_{11}u' + a_{12}v')x_u + (a_{21}u' + a_{22}v')x_v$ where $N_u = a_{11}x_u + a_{21}x_v, N_v = a_{12}x_u + a_{22}x_v$. So dN is given by the matrix $(a_{ij}), i, j = 1, 2$ in the basis x_u, x_v.

We can then write,

$$II_p(\alpha') = -\langle dN(\alpha'), \alpha' \rangle = -\langle N_u u' + N_v v', x_u u' + x_v v' \rangle.$$

We can then define e, f, g as

$$e = -\langle N_u, x_u \rangle = \langle N, x_{uu} \rangle$$

$$f = -\langle N_v, x_u \rangle = \langle N, x_{uv} \rangle = -\langle N_u, x_v \rangle$$

$$g = -\langle N_v, x_v \rangle = \langle N, x_{vv} \rangle$$

Using these we can write $II_p(\alpha') = e(u')^2 + 2fu'v' + g(v')^2$.

We can express e, f, g in matrix form as,

$$-\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}.$$

It is easy to check that

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} = \frac{1}{EG - F^2} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = \begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{1}$$

so, $K = det(a_{ij}) = \frac{eg - f^2}{EG - F^2}$.

In order to fully understand all of these curvatures take the example of the torus.

Example 7.3. Take a torus parameterized by

$x(u, v) = ((a + rcos(u))cos(v), (a + rcos(u))sin(v), rsin(u)), 0 < u < 2\pi, 0 < v < 2\pi.$

We need to compute the coefficients e, f, g so we need x_{uu}, x_{uv}, x_{vv}

$$x_u = (-rsin(u)cos(v), -rsin(u)sin(v), rcos(u))$$
$$x_v = (-(a + rcos(u))sin(v), (a + rcos(u))cos(v), 0)$$
$$x_{uu} = (-rcos(u)cos(v), -rcos(u)sin(v), -rsin(u))$$
$$x_{uv} = (rsin(u)sin(v), -rsin(u)cos(v), 0)$$
$$x_{vv} = (-(a + rcos(u))cos(v), -(a + rcos(u))sin(v), 0).$$

From this we obtain $E = \langle x_u, x_u\rangle = r^2, F = \langle x_u, x_v\rangle = 0, G = \langle x_v, x_v\rangle = (a + rcosu)^2$. Since $|x_u \times x_v| = (EG - F^2)^{-1/2}$,

$$e = \langle N, x_{uu}\rangle = \langle \frac{x_u \times x_v}{|x_u \times x_v|}, x_{uu}\rangle = \frac{(x_u, x_v, x_{uu})}{(EG - F^2)^{-1/2}} = \frac{r^2(a + rcos(u))}{r(a + rcos(u))} = r.$$

Following the same process for f and g we obtain, $f = \frac{(x_u, x_v, x_{uv})}{r(a+rcos(u))} = 0$ and $g = \frac{(x_u, x_v, x_{vv})}{r(a+rcos(u))} = cosu(a + rcos(u))$.

Therefore,

$$K = \frac{cos(u)}{r(a + rcos(u))}.$$

Note that as the orientation of the surface changes the sign of K will change as well. With this definition of Gaussian curvature we can see that each of these definitions of curvature are different yet related concepts used to understand the geometry of surfaces.

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