INTRODUCTION TO HODGE THEORY VIA THE EXAMPLE OF ELLIPTIC CURVES

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Abstract. In this paper, we focus on illustrating some fundamental concepts in Hodge Theory via the classic example of elliptic curves. We start our discussion with some preliminaries on complex manifolds and the Hodge decomposition theorem, and then go deeper into the definition, the period mapping and the period domain, as well as the Monodromy representation of the elliptic curves.

Contents

1. Introduction

Algebraic geometry and differential geometry are two principal branches of geometry. Intriguingly, Hodge theory serves as a bridge between the two. In the 1930s, British mathematician William Vallance Douglas Hodge started enriching the theories about de Rham cohomology to study algebraic geometry. Hodge's original motivation was to study complex projective varieties, which was later overshadowed by his theory's wide applications to Riemannian manifolds and Kähler manifolds. Almost one hundred years later, one of the Clay Mathematics Institute's Millennium Prize Problems, the Hodge conjecture, is still open.

In this paper, we will first introduce some preliminaries of complex smooth manifolds and state the Hodge decomposition theorem. After that, we will work on the classic example of Elliptic curves, the simplest interesting Riemann surface, as an application of Hodge theory. Hopefully, from this example our readers will come to appreciate Hodge theory and other theoretical tools in geometric studies.

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Before we begin, a word on background is in order. The reader is assumed to be familiar with linear algebra, real smooth manifolds (more precisely the first eleven chapters of *Introduction To Smooth Manifolds* by John M. Lee), and possess some basic knowledge of complex analysis and group theory.

2. Preliminaries

In this section, we start with the construction of the complex tangent bundle on a complex manifold based on the real tangent bundle, and the dual construction of complex cotangent bundle. Then we move on to the theoretical development of the space of complex-valued smooth p-forms and how the space decomposes into subspaces of (p, q)-forms. After that, we introduce the definition of Kähler manifolds with a few classic examples, and finally state the Hodge decomposition theorem for compact Kähler manifolds.

2.1. **Tangent Bundles on a Complex Manifold.**

Let X be a complex manifold of dimension n, $x \in X$ and

$$(U; z_1 = x_1 + iy_1, \dots, z_n = x_n + iy_n)$$

be a holomorphic chart for X around x.

Definition 2.1. The *real tangent bundle* $T_X(\mathbb{R})$ is the tangent bundle whose fiber $T_{X,x}(\mathbb{R})$ is the real span of $\mathbb{R}\langle \partial_{x_1}, \dots, \partial_{x_n}, \partial_{y_1}, \dots, \partial_{y_n} \rangle$.

Definition 2.2. The *complex tangent bundle* is defined as $T_X(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$. Its fiber $T_{X,x}(\mathbb{C}) := T_{X,x}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ is the complex span $\mathbb{C}\langle \partial_{x_1}, \dots, \partial_{x_n}, \partial_{y_1}, \dots, \partial_{y_n} \rangle$. Define

$$\partial_{z_j} := \frac{1}{2}(\partial_{x_j} - i\partial_{y_j})$$
$$\partial_{\overline{z}_j} := \frac{1}{2}(\partial_{x_j} + i\partial_{y_j})$$

Thus we can write

$$T_{X,x}(\mathbb{C}) = \mathbb{C}\langle \partial_{z_1}, \dots, \partial_{z_n}, \partial_{\overline{z}_1}, \dots, \partial_{\overline{z}_n} \rangle.$$

Remark 2.3. One should see that with such definitions, $\mathbb{C}\langle \partial_{z_1}, \dots, \partial_{z_n} \rangle$ and $\mathbb{C}\langle \partial_{\overline{z}_1}, \dots, \partial_{\overline{z}_n} \rangle$ are preserved under holomorphic change of coordinates.

Definition 2.4. The *holomorphic tangent bundle* T'_X has fiber

$$T'_{X,x} = \mathbb{C}\langle \partial_{z_1}, \dots, \partial_{z_n} \rangle.$$

The *anti-holomorphic tangent bundle* T'_X has fiber

$$T''_{X,x} = \mathbb{C}\langle \partial_{\overline{z}_1}, \dots, \partial_{\overline{z}_n} \rangle.$$

Now we have canonical direct sum decomposition

$$T_X(\mathbb{C}) = T'_X \oplus T''_X.$$

2.2. **Cotangent Bundles on a Complex Manifold.**

Let $\{d_{x_1}, \dots, d_{x_n}, d_{y_1}, \dots, d_{y_n}\}$ be the basis dual to $\{\partial_{x_1}, \dots, \partial_{x_n}, \partial_{y_1}, \dots, \partial_{y_n}\}$, $\{d_{z_1}, \dots, d_{z_n}\}$ the basis dual to $\{\partial_{z_1}, \dots, \partial_{z_n}\}$, and $\{d_{z_1}, \dots, d_{z_n}\}$ the basis dual to $\{\partial_{z_1}, \dots, \partial_{z_n}\}$.

Definition 2.5. The *real cotangent bundle* $T^*_X(\mathbb{R})$ is the cotangent bundle whose fiber $T^*_{X,x}(\mathbb{R})$ is the real span of $\mathbb{R}\langle d_{x_1}, \dots, d_{x_n}, d_{y_1}, \dots, d_{y_n}\rangle$.

Definition 2.6. The *complex cotangent bundle* is $T^*_X(\mathbb{C}) := (T_X(\mathbb{C}))^* = T^*_X(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$. Its fiber $T^*_{X,x}(\mathbb{C}) := T^*_{X,x}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ is the complex span $\mathbb{C}\langle d_{x_1}, \dots, d_{x_n}, d_{y_1}, \dots, d_{y_n}\rangle$.

Definition 2.7. The *holomorphic cotangent bundle* T'^*_X has fiber

$$T'^*_{X,x} = \mathbb{C}\langle d_{z_1}, \dots, d_{z_n}\rangle.$$

The *anti-holomorphic cotangent bundle* T''^*_X has fiber

$$T''^*_{X,x} = \mathbb{C}\langle d_{\overline{z}_1}, \dots, d_{\overline{z}_n}\rangle.$$

Now we have canonical direct sum decomposition

$$T^*_X(\mathbb{C}) = T'^*_X \oplus T''^*_X.$$

2.3. **Complex-valued Forms.**

Definition 2.8. Let M be a smooth manifold. Define the space of *complex-valued smooth p-forms* as

$$A^p := E^p(M) \otimes_{\mathbb{R}} \mathbb{C} \simeq C^\infty(M, T_M(\mathbb{C})).$$

where $E^p(M)$ denotes the space of real valued smooth p-forms on M.

The notion of exterior differentiation extends to complex-valued smooth forms:

$$d : A^{p-1} \to A^p$$

Definition 2.9. Let X be a complex smooth manifold. Pick $x \in X$, and (p, q) a pair of non-negative integers. Define complex vector spaces

$$\Lambda^{p,q}(T^*_{X,x}) := \Lambda^p(T'^*_{X,x}) \otimes \Lambda^q(T''^*_{X,x}) \subset \Lambda^{p+q}(T^*_{X,x}(\mathbb{C}))$$

There is a canonical internal direct sum decomposition of complex vector spaces:

$$\Lambda^l_{\mathbb{C}}(T^*_{X,x}) = \oplus_{p+q=l} \Lambda^{p,q}(T^*_{X,x})$$

Remark 2.10. The above decomposition is canonical because it is invariant under a holomorphic change of coordinates. In particular, we can define smooth vector bundles $\Lambda^{p,q}_X(T^*_X)$ and obtain

$$\Lambda^l_{\mathbb{C}}(T^*_{X,x}) = \oplus_{p+q=l} \Lambda^{p,q}(T^*_{X,x})$$

Definition 2.11. The space of (p, q)*-forms* on X:

$$A^{p,q}(X) := C^\infty(X, \Lambda^{p,q}(T^*_X)$$

is the complex vector space of smooth sections of the smooth complex vector bundle $\Lambda^{p,q}(T^*_X)$.

One should observe that

$$d(A^{p,q}) \subset A^{p+1,q} \oplus A^{p,q+1}.$$

Define natural projection $\pi^{p,q} : A^{p+q}(X) \to A^{p,q}(X)$, and define operators $d' : A^{p,q} \to A^{p+1,q}$ and $d'' : A^{p,q} \to A^{p,q+1}$ via

$$d' = \pi^{p+1,q} \circ d$$
$$d'' = \pi^{p,q+1} \circ d$$

Note that

$$d = d' + d''.$$

Definition 2.12. The *Bott-Chern cohomology groups* of X are defined as quotient spaces

$$H^{p,q}_{BC}(X) := \frac{A^{p,q}(X) \cap \operatorname{Ker} d}{d'd''(A^{p-1,q-1})}$$

The reader should take note on this definition since it will appear later in the statement of the Hodge decomposition theorem.

2.4. Kähler Manifold.

Definition 2.13. A *Hermitean form* on a finite dimensional complex vector space W is a $\mathbb{C}$-bilinear form

$$h : W \times \overline{W} \to \mathbb{C}$$

such that

$$h(v, w) = \overline{h(w, v)}, \quad \forall v, w \in W$$

A *Hermitean metric* on W is a positive definite Hermitean form, i.e. one for which

$$h(v, v) > 0, \quad \forall 0 \neq v \in W.$$

A Hermitean metric h on a complex manifold X is the assignment of a Hermitean metric

$$h(-,-)_x : T'_{X,x} \times \overline{T'}_{X,x} \to \mathbb{C}$$

to every $x \in X$ that varies smoothly with x, i.e. such that the functions

$$h_{jk}(z) := (\partial_{z_j}, \partial_{\bar{z}_k})$$

are smooth on charts.
Using a chart $(U; z)$, the Hermitean metric h can be expressed on U in tensor product form as

$$h = \sum_{jk} h_{jk}(z) dz_j \otimes d\overline{z}_k$$

Then on the same chart, the *associated* $(1,1)$*-form* of the Hermitean metric h is

$$\omega = \frac{1}{2} \sum_{jk} h_{jk}(z) dz_j \wedge d\overline{z}_k$$

Remark 2.14. Every complex manifold admits a Hermitean metric.

Definition 2.15. A Hermitean metric h on a complex manifold X is *Kähler* if its associated $(1,1)$-form is closed. A complex manifold X is *Kähler* if it admits a Kähler metric.

Next we provide a few classic examples of Kähler manifold:

Example 2.16. All complex curves are Kähler manifolds.

Proof. We've see earlier that all complex manifolds admit a Hermitean metric h. Let ω denote its associated $(1,1)$-form, which is also a complex valued smooth 2-form. Since a complex curve is a complex manifold with real dimension 2, it admits no 3-forms. Thus, $d\omega = 0$ and complex surfaces are all necessarily Kähler. □

Example 2.17. Projective spaces $\mathbb{P}^n$ are Kähler manifolds under the Fubini-Study metric. Moreover, it is a well-establised fact that any compact submanifold of a Kähler manifold is Kähler. Consequently, because projective varieties are closed subspaces of projective spaces, projective varieties are all Kähler. This explains why Hodge's original interest in projective varieties was replaced by the general Kähler manifolds.

Theorem 2.18 (**The Hodge Decomposition**). *Let X be a compact Kähler manifold. There are injective natural maps $H^{p,q}_{BC}(X) \to H^{p+q}_{dR}(X, \mathbb{C})$. We denote their images by $H^{p,q}_{BC}$. There is an internal direct sum decomposition*

$$H^l_{dR}(X, \mathbb{C}) = \oplus_{p+q=l} H^{p,q}(X).$$

When X is compact, then $\dim X < \infty$.

A proof to this theorem can be found in [1].

Remark 2.19. The Hodge Decomposition theorem concerns a fundamental property of the complex cohomology of compact Kähler manifolds. It tells us that for any $a_{dR} \in H^l_{dR}$, there exists a canonical decomposition $a_{dR} = \sum_{p+q=l} a^{p,q}_{dR}$ where each $a^{p,q}_{dR}$ is an element of the embedded $H^{p,q}_{BC}$ and does not depend on the choice of the Kähler metric. This canonical decomposition of $H^l_{dR}(X, \mathbb{C})$ and the canonical isomorphism $H^l_{dR}(X, \mathbb{C}) \simeq H^l(X, \mathbb{Z}) \otimes \mathbb{C}$ lead to the notion of pure Hodge structure of weight l.

Definition 2.20. A *pure Hodge structure* of integer weight l consists of an abelian group $H_{\mathbb{Z}}$ and a decomposition of its complexification H into a direct sum of complex subspaces $H_{p,q}$, where $p+q=l$, with the property that the complex conjugate of $H_{p,q}$ is $H_{q,p}$:

$$H := H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} = \oplus_{p+q=l} H^{p,q}$$
$$\overline{H^{p,q}} = H^{q,p}$$

In this paper, the $H_{\mathbb{Z}}$'s that concern us are the de Rham cohomology groups of elliptic curves.

3. Elliptic Curves

Definition 3.1. An *elliptic curve* is an algebraic variety given by

$$y^2 = P(x)$$

where $P(x)$ is a degree 3 separable polynomial with complex coefficients. After a change of variables, we may assume that the three roots of $P(x)$ are $0, 1$ and λ, where $\lambda \neq 0, 1$, i.e.

$$y^2 = x(x-1)(x-\lambda)$$

Denote the above Riemann surface $\mathscr{C}_\lambda$, and we shall call the family of all $\mathscr{C}_\lambda$ the *Legendre family.*

3.1. **Local Period Mapping and Local Period Domain.**

Consider $y = \sqrt{x(x-1)(x-\lambda)}$. The evaluation of y depends on the sheet of x. More specifically, in any simply connected set on the Riemann sphere which does not contain $0, 1, \lambda$ or ∞, y has two single-valued determinations. Thus, if we cut the Riemann sphere from 0 to 1 and from λ to ∞, each single-valued determination of y defines a holomorphic function. Now we slightly abuse our notation and denote $\mathscr{C}_\lambda$ as the Riemann surface of y. When we go around δ, we do not cross any cuts so we remain in one sheet of Riemann sphere. But when we go around γ we do cross cuts so γ lies in two sheets of Riemann spheres. Thus, $\mathscr{C}_\lambda$ consists of two copies of Riemann spheres minus the cuts. The specific pasting process is illustrated in the figures below. Basically we stretch the cuts into holes and glue the holes in two copies of Riemann spheres correspondingly.

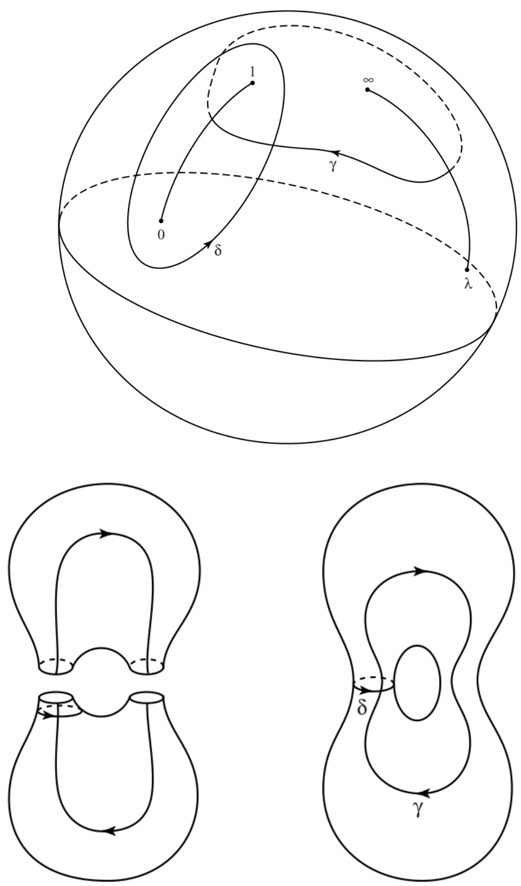

FIGURE 1. Cut(See [2])

Now we see that $\mathscr{C}_\lambda$ is topologically equivalent to a genus 1 torus with its first homology group generated by $[\gamma]$ and $[\delta]$. Let $[\gamma^*]$ and $[\delta^*]$ be the basis of first cohomology group of $\mathscr{C}_\lambda$ that is the dual to $[\gamma]$ and $[\delta]$. Consider

$$\omega = \frac{dz}{y} = \frac{dz}{\sqrt{x(x-1)(x-\lambda)}}.$$

Since on $\mathscr{C}_\lambda$, y admits single valued holomorphic determination, ω is holomorphic and thus closed by Cauchy's theorem. Thus, we can write the cohomology class of ω as

$$[\omega] = [\delta^*]\int_\delta \omega + [\gamma^*]\int_\gamma \omega$$

For ease of notation, denote $\int_\delta \omega$ as A and $\int_\gamma \omega$ as B. A and B are called *periods* of ω and the pair (A, B) is called the period vector of $\mathscr{C}_\lambda$.

Now we proceed to define our period map. First observe that neither A nor B can be zero:

$$\begin{aligned}[\omega] \cup [\overline{\omega}] &= (A[\delta^*] + B[\gamma^*]) \cup (\overline{A}[\delta^*] + \overline{B}[\gamma^*]) \\ &= (A\overline{B} - B\overline{A})[\delta^* \cup \gamma^*]\end{aligned}$$

Since $[\delta^* \cup \gamma^*]$ is dual to the fundamental class of $\mathscr{C}_\lambda$,

$$\int_{i\mathscr{C}_\lambda} \omega \wedge \overline{\omega} = 2\mathrm{I}mB\overline{A}$$

By the definition of ω, the integrand equals

$$i\left|\frac{1}{z(z-1)(z-\lambda)}\right| dz \wedge d\overline{z} = 2\left|\frac{1}{z(z-1)(z-\lambda)}\right| dx \wedge dy,$$

where $dx \wedge dy$ is the natural orientation defined by the holomorphic coordinate. Thus, $\mathrm{I}mB\overline{A} > 0$ and consequently neither A nor B can be zero. Hence, we define $\tau(\lambda)$ to be

$$\tau(\lambda, \gamma, \delta) = \frac{\int_\gamma \omega}{\int_\delta \omega}.$$

Moreover, multiplying ω by a nonzero constant if necessary, we may assume that $\int_\delta \omega = 1$ and then $\tau(\lambda, \gamma, \delta)$ is in the complex upper half plane, which we denote by $\mathbb{H}$. In the case of Elliptic curves, we call τ *local period mapping* and its codomain $\mathbb{H}$ *local period domain*. More generally, period maps and period domains satisfy the following definition:

Definition 3.2. A *period mapping* is a mapping which assigns to a point s of the base S of a family $\{X_s\}_{s\in S}$ of algebraic varieties over the field of complex numbers the cohomology spaces $H^*(X_s)$ of the fibres over this point, provided with a pure Hodge structure.

Definition 3.3. A *period domain* is the parameter space of a Hodge structure.

Now the reader might ask, how do we get the pure Hodge structure out of the period mapping τ and the period domain $\mathbb{H}$? Since $\mathscr{C}_\lambda$ is topologically equivalent to a genus one torus, by de Rham Isomorphism theorem, we have

$$\begin{aligned}H^2_{dR}(\mathscr{C}_\lambda, \mathbb{C}) &= \mathbb{C} \\ H^1_{dR}(\mathscr{C}_\lambda, \mathbb{C}) &= \mathbb{C}^2 \\ H^0_{dR}(\mathscr{C}_\lambda, \mathbb{C}) &= \mathbb{C}\end{aligned}$$

First consider $H^2_{dR}(\mathscr{C}_\lambda, \mathbb{C})$. We must have $H^2_{dR}(\mathscr{C}_\lambda, \mathbb{C}) = H^{1,1}(\mathscr{C}_\lambda, \mathbb{C})$. Otherwise, suppose that there is a non-trivial $(2,0)$-form $\omega_{2,0}$ on $\mathscr{C}_\lambda$. Then $\overline{\omega}_{2,0}$ is a non-trivial $(0,2)$-form. But according to theorem 2.18, this implies that $\mathbb{C}^2 \subset H^2_{dR}(\mathscr{C}_\lambda, \mathbb{C}) = \mathbb{C}$; contradiction! As we've seen before, as a Kähler manifold, the Kähler metric on $\mathscr{C}_\lambda$ induces an associated $(1,1)$-form. Since $H^{1,1}(\mathscr{C}_\lambda, \mathbb{C})$ is one dimensional, we know that the associated $(1,1)$-form represents the only class in $H^2_{dR}(\mathscr{C}_\lambda, \mathbb{C})$. This completes the discussion about $H^2_{dR}(\mathscr{C}_\lambda, \mathbb{C})$.

Now consider $H^0_{dR}(\mathscr{C}_\lambda, \mathbb{C})$. Again, this group is one dimensional, and by Liouville's theorem the constant functions represent the only equivalance class.

Finally, we come to the case of $H^1_{dR}(\mathscr{C}_\lambda, \mathbb{C})$. As we've shown before, $\omega = \frac{dz}{y}$ is holomorphic on $\mathscr{C}_\lambda$, so $[\omega] \in H^{1,0}(\mathscr{C}_\lambda, \mathbb{C})$, and consequently $[\overline{\omega}] \in H^{0,1}(\mathscr{C}_\lambda, \mathbb{C})$. Again, by theorem 2.18, we get that

$$H^1_{dR}(\mathscr{C}_\lambda, \mathbb{C}) = H^{1,0}(\mathscr{C}_\lambda, \mathbb{C}) \oplus H^{0,1}(\mathscr{C}_\lambda, \mathbb{C}),$$

where $H^{1,0}(\mathscr{C}_\lambda, \mathbb{C})$ is the span of $[\omega]$, and $H^{0,1}(\mathscr{C}_\lambda, \mathbb{C})$ is the span of $[\overline{\omega}]$.

Now to answer the question how to obtain pure Hodge structure from local period mapping τ and period domain $\mathbb{H}$, it suffices to clarify how to reconstruct ω from τ and $\mathbb{H}$. For each λ, fix basis $[\delta_\lambda]$ and $[\gamma_\lambda]$ for each $H_1(\mathscr{C}_\lambda)$ with $[\delta \cup \gamma]$ being the fundamental class. We may define

$$\omega = \delta^*_\lambda + \tau(\lambda, \gamma_\lambda, \delta_\lambda)\gamma^*_\lambda.$$

From here, reconstructing the pure Hodge structure follows from the previous paragraphs. One should notice that it is not canonical to get the desired pure Hodge structure from a pair of period mapping and period domain, because there is no canonical way to fix basis δ and γ for each element of the Legendary family. We will elaborate on this issue in the next section.

3.2. **Global Period Domain.**

In the previous subsection, we've seen constructions and calculations for a fixed $\mathscr{C}_\lambda$. In this subsection, we start to view each single $\mathscr{C}_\lambda$ as an element of the whole Legendre family. More specifically, we view τ as an invariant under complex isomorphism, a non-constant holomorphic function of λ and learn how to extend the concept of period mapping and period domain globally, which will shed light on the relation between the whole Legendre family and each single elliptic curve.

Proposition 3.4. *If $f : \mathscr{C}_\lambda \to \mathscr{C}_{\lambda'}$ is a complex isomorphism, i.e. a map that is biholomorphic in local coordinates, then $\tau(\lambda, \delta, \gamma) = \tau(\lambda', \delta', \gamma')$, where $\delta' = f_*\delta$ and $\gamma' = f_*\gamma$.*

Proof. Let $\omega = \frac{dz}{\sqrt{x(x-1)(x-\lambda)}}$ and $\omega' = \frac{dz}{\sqrt{x(x-1)(x-\lambda')}}$. We claim that

$$f^*\omega' = c\omega$$

for some nonzero complex number c. First of all, from the previous subsection, we know that $H^{1,0}(\mathscr{C}_\lambda, \mathbb{C})$ is one dimensional and the only equivalence class is represented by ω, and similarly $H^{1,0}(\mathscr{C}_{\lambda'}, \mathbb{C})$ is one dimensional and the only equivalence class is represented by ω'. Since f is a complex isomorphism, $f^*\omega' = c\omega$ for some c. To see that c is non-zero, one should notice that on one hand,

$$\int_{\mathscr{C}_\lambda} f^*\omega' \wedge f^*\overline{\omega}' = |c|^2 \int_{\mathscr{C}_\lambda} \omega \wedge \overline{\omega}.$$

On the other hand,

$$\int_{\mathscr{C}_\lambda} f^*\omega' \wedge f^*\overline{\omega}' = \int_{f_*\mathscr{C}_\lambda} \omega' \wedge \overline{\omega'} = \int_{\mathscr{C}_{\lambda'}} \omega' \wedge \overline{\omega}'$$

Since $\int_{\mathscr{C}_\lambda} \omega \wedge \overline{\omega} \neq 0$ and $\int_{\mathscr{C}_{\lambda'}} \omega' \wedge \overline{\omega}' \neq 0$, we have that $c \neq 0$. Therefore,

$$\begin{aligned}\tau(\lambda', \delta', \gamma') &= \frac{\int_{\delta'} \omega'}{\int_{\gamma'} \omega'} \\ &= \frac{\int_{f_*\delta} \omega'}{\int_{f_*\gamma} \omega'} \\ &= \frac{\int_{\delta} f^*\omega'}{\int_{\gamma} f^*\omega'} \\ &= \frac{c\int_{\delta} \omega}{c\int_{\gamma} \omega} \\ &= \tau(\lambda, \delta, \gamma)\end{aligned}$$

□

Proposition 3.5. *With respect to λ, τ is a holomorphic function.*

Proof. We've shown before that neither $A(\lambda)$ nor $B(\lambda)$ can be zero. Thus it suffices to show that both A and B are holomorphic functions of λ. Recall that by definition

$$\begin{aligned}A(\lambda) &= \int_\delta \frac{dz}{\sqrt{x(x-1)(x-\lambda)}} \\ B(\lambda) &= \int_\gamma \frac{dz}{\sqrt{x(x-1)(x-\lambda)}}\end{aligned}$$

Since δ is bounded and $\frac{1}{\sqrt{x(x-1)(x-\lambda)}}$ holomorphic,

$$\begin{aligned}\frac{\partial}{\partial\overline{\lambda}} A(\lambda) &= \frac{\partial}{\partial\overline{\lambda}} \int_\delta \frac{dz}{\sqrt{x(x-1)(x-\lambda)}} \\ &= \int_\delta \frac{\partial}{\partial\overline{\lambda}} \frac{dz}{\sqrt{x(x-1)(x-\lambda)}} \\ &= 0\end{aligned}$$

Similarly, $\frac{\partial}{\partial\overline{\lambda}} B(\lambda) = 0$. Therefore both A and B are holomorphic functions of λ, and so is τ. □

Proposition 3.6. *The value of τ depends on the choice of homology basis. More specifically, if $\{[\delta], [\gamma]\}$ and $\{[\delta'], [\gamma']\}$ are both bases of $H^1(\mathscr{C}_\lambda)$ such that both $[\delta' \cup \gamma']$ and $[\delta \cup \gamma]$ are the fundamental class of $\mathscr{C}_\lambda$, and τ is calculated with respect to $\{[\delta], [\gamma]\}$, while τ' with respect to $\{[\delta'], [\gamma']\}$, then $\tau = M\tau'$ for some $M \in SL(2, \mathbb{C})$.*

Proof. Suppose for some $a, b, c, d \in \mathbb{C}$,

$$\begin{aligned}\delta' &= a\delta + b\gamma \\ \gamma' &= c\delta + d\gamma\end{aligned}$$

Then the condition that $[\delta' \cup \gamma']$ is the fundamental class translates into

$$\int_{\mathscr{C}_\lambda} (ad - bc)\delta^* \wedge \gamma^* = 1$$

Hence, $ad - bc = 1$. With respect to δ' and γ', the new periods are

$$A' = aA + bB$$
$$B' = cA + dB$$

Thus,

$$\tau' = \frac{d\tau + c}{b\tau + a}.$$

□

Proposition 3.7. *Function τ is non-constant even with fixed δ and γ.*

Proof. We will show that τ approaches ∞ as λ tends to infinity along the real axis. To show this, first observe that for $\lambda \gg 2$,

$$\int_\delta \frac{dz}{\sqrt{x(x-1)(x-\lambda)}} \sim \int_\delta \frac{dz}{x\sqrt{-\lambda}} = \frac{2\pi}{\sqrt{\lambda}}.$$

By deforming γ into a path from λ to 1 on one copy of Riemann sphere and a path from λ to 1 on the other copy of the Riemann sphere, noting that we turned the other copy "inside-out" in the pasting process, we get

$$\int_\gamma \frac{dz}{\sqrt{x(x-1)(x-\lambda)}} = -2\int_1^\lambda \frac{dz}{\sqrt{x(x-1)(x-\lambda)}}.$$

The difference between the last integrand and $1/[x\sqrt{x-\lambda}]$ is $1/[2x^2]+$ some higher powers of $1/x$, and we know

$$-2\int_1^\lambda \frac{dx}{x\sqrt{x-\lambda}} = \frac{4}{\sqrt{\lambda}} \arctan \frac{\sqrt{1-\lambda}}{\lambda} \sim \frac{2i}{\sqrt{\lambda}} \log \lambda.$$

Thus,

$$\tau(\lambda) \sim \frac{i}{\pi} \log \lambda.$$

□

3.2.1. *Local Monodromy Representation.*
In this section, we further explore the features of the elliptic curves as algebraic varieties. In the studies about algebraic varieties, people are often interested in singular points. To understand the behavior of elliptic curves near the singular points, we start by looking into the local Monodromy representation on a subfamily of elliptic curves $\mathscr{C}_s$ defined by

$$y^2 = (x^2 - s)(x - 1).$$

First of all, what is *monodromy*? Here is an intuitive description from [3]:

> In mathematics, *monodromy* is the study of how objects from mathematical analysis, algebraic topology, algebraic geometry and differential geometry behave as they "run round" a singularity.

In order to understand monodromy more comprehensively, one needs to understand fiber bundles. Consider a topological space E and a continuous surjective map $\pi : E \to B$. Denote F_x as $\pi^{-1}(x)$ for $x \in B$, and denote $\pi' : U_x \times F_x \to U_x$ as the projection that ignores the F_x part. If for every $x \in B$, there exists a neighborhood $U_x \subset B$ such that $\pi^{-1}(U_x) \simeq U_x \times F_x$ and the following diagram commutes, then we call E the total space, B the base space, F the fiber, and the tuple (E, B, π, F) a fiber bundle.

$$\begin{array}{ccc} \pi^{-1}(U_x) & \cong & U_x \times F_x \\ & \searrow^{\pi} & \downarrow^{\pi'} \\ & & U_x \end{array}$$

Pick $x \in B$ and fix $\tilde{x} \in E$. Choose a loop in B and call it α. We can lift α to $\tilde{\alpha}$ such that $\tilde{\alpha}(0) = \tilde{x}$. There are theorems stating that by defining $\alpha \cdot \tilde{z} = \tilde{\alpha}(1)$, we get a well-defined action of $\pi_1(B, x)$ on F_x. Such action is called the *monodromy action.* To study such action, we usually look into the representations

$$\pi_1(B, x) \to \mathrm{Aut}\,(H_*(F_x)),$$

which are called the *monodromy representations.* The codomains of such homomorphisms are called the *monodromy groups.* Geometers are often curious about such representations because they yield important insight into the relation between the fiber F and the total space E.

We start our investigation into the monodromy of the subfamily of all $\mathscr{C}_s$. Since every element of $\mathscr{C}_s$ has real dimension 2, we will be primarily interested in the homomorphism

$$\pi_1(\mathbb{P} \setminus \{0, 1, \infty\}, s) \to \mathrm{Aut}\,(H_1(\mathscr{C}_s)),$$

which we define to be the local monodromy representation. The singularity points arise when $s \to 0$ or 1, and the two situations are equivalent up to change of variables. Thus, we only look into the case where $s \to 0$.

Fix small $r > 0$. Consider the smooth map $\phi : [0, 2\pi] \to \{\mathscr{C}_s : |s| = r\}$ via

$$t \to \mathscr{C}_{r \exp it}$$

Since $\phi(0) = \phi(2\pi)$, ϕ is actually a smooth embedding of a circle in $\mathscr{C}_s$. The figure below illustrates how the cycles δ and γ change smoothly as t goes from 0 to 2π with the left figure corresponding to $t = 0$, the middle one corresponding to $t = \pi$, and the right one corresponding to $t = 2\pi$.

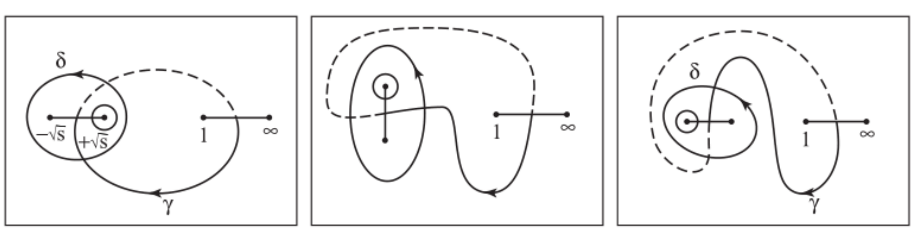

Figure 2. Picard-Lefschetz Transformation(See [2])

The reader should see that the leftmost and the rightmost figure represent the same surface, and we call the transformation that transforms the leftmost figure into the rightmost one the *Picard-Lefschetz* transformation. Denote the δ and γ at $t = 0$ as δ_0 and γ_0, and the pair at $t = 2\pi$ as $\delta_{2\pi}$ and $\gamma_{2\pi}$. Since δ_0 and γ_0 form a basis of the surface's first homology group, we wonder how to express $\delta_{2\pi}$ and $\gamma_{2\pi}$ in terms of δ_0 and γ_0. Clearly, $\delta_{2\pi} = \delta_0$. To express $\gamma_{2\pi}$ in terms of δ_0 and γ_0, we need the tool of intersection number, which is a well-defined anti-symmetric bilinear function on first homology classes of a fixed topological space. To calculate the intersection number of first homology classes $[\alpha]$ and $[\beta]$, denoted as $[\alpha] \cdot [\beta]$, we take representatives α and β and look at all the points where they intersect. Without loss of generality, we may assume that at each point of intersection, the oriented loops point in different directions. At each intersection point, if the direction α points is counterclockwise relative to the direction β points, then the intersection point is marked $+1$; otherwise, the intersection point is marked -1. The intersection number of $[\alpha]$ and $[\beta]$ is calculated by summing over all the labels over the intersection points. Now the reader should be able to verify that

$$[\delta] \cdot [\gamma] = 1,$$

noting that the doted part of γ does not intersect with δ since it lies on a different sheet. Similarly, by superimposing the leftmost figure and the rightmost figure, we obtain the intersection number of $[\gamma_{2\pi}]$ with $[\delta_0]$ and $[\gamma_0]$:

$$[\gamma_{2\pi}] \cdot [\delta_0] = -1$$
$$[\gamma_{2\pi}] \cdot [\gamma_0] = +1$$

By the bilinearity of intersection number, if $[\gamma_{2\pi}] = a[\delta_0] + b[\gamma_0]$, then

$$\begin{aligned}
-1 &= (a[\delta_0] + b[\gamma_0]) \cdot [\delta_0] \\
&= a[\delta_0] \cdot [\delta_0] + b[\gamma_0] \cdot [\delta_0] \\
&= 0 - b \\
&= -b \\
+1 &= (a[\delta_0] + b[\gamma_0]) \cdot [\gamma_0] \\
&= a[\delta_0] \cdot [\gamma_0] + b[\gamma_0] \cdot [\gamma_0] \\
&= a + 0 = a
\end{aligned}$$

Therefore,

$$[\delta_{2\pi}] = [\delta_0]$$
$$[\gamma_{2\pi}] = [\delta_0] + [\gamma_0]$$

With respect to the basis $\{[\delta_0], [\gamma_0]\}$, we express this result in the matrix form

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

3.2.2. *Global Monodromy Representation.*
Now we proceed to study the global monodromy representation of the Legendre family $y^2 = x(x-1)(x-\lambda)$,

$$\rho : \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, \lambda) \to \mathrm{Aut}\,(H_1(\mathscr{C}_\lambda))$$

For the ease of expressing the matrix formula of the above representation, let's define a and b, the generators of $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, \lambda)$, as the equivalence classes of loops shown in Figure 3: It is easy to see that singularity points arise only when $\lambda \to 0$ or $\lambda \to 1$. We will investigate the two cases separately. When $\lambda \to 0$, the

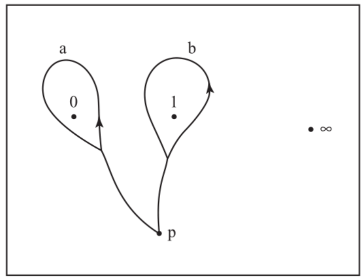

Figure 3. Parameter Space for the Legendre Family(See [2])

picture of the monodromy action is similar to Figure 2, except that λ completes a full circle instead of a half circle. Thus, naturally, with respect to δ and γ (see Figure 4), the matrix representation of $\rho(a)$ equals

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

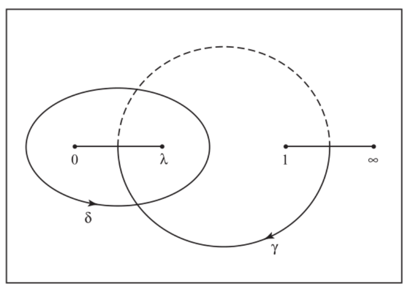

Figure 4. Standard Homology Basis(See [2])

Now we calculate $\rho(b)$. Since in this case λ should be close to 1, we modify our cuts and get figure 5, where the left panel shows the modified cuts and the right panel shows the surface after Picard-Lefschetz transformation, namely when λ completes a half circle. Similar to what we've done in the case of local Monodromy representation, we compute intersection numbers and get the matrix representation of the Picard-Lefschetz transformation with respect to $[\delta]$ and $[\gamma]$

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

Hence, the matrix representation of $\rho(b)$ equals

$$B = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

Let G denote the monodromy group of the Legendre family. From the above discussion, we know that $G = \langle A, B \rangle$. For better knowledge of the monodromy

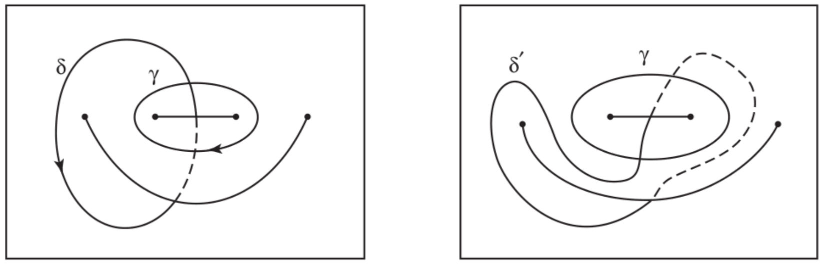

FIGURE 5. Monodromy for $\lambda \to 1$(See [2])

group, we first define

$$\Gamma(2) := \{M \in SL(2, \mathbb{C}) : M \equiv id \mod 2\}.$$

Since $A, B \in \Gamma(2)$, we know that G is a subgroup of $\Gamma(2)$. The following theorem gives us a precise group theoretic formula of G.

Theorem 3.8.

(1) $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\})$ *is a free group on two generators.*
(2) The global monodromy representation is injective.
(3) The image of the monodromy representation is $\Gamma(2)$.
(4) $\Gamma(2)$ *has index six in* $SL(2, \mathbb{C})$.

Proof. Here we only provide a sketch of the proof. (1) is straightforward and (4) is an exercise in linear algebra, which we leave to the reader. To prove (2), we first note that both A and B are automorphisms of the complex upper half plane, so by composing them with the cayley transform, which is a conformal map from the complex upper half plane to the unit disk, we can conveniently look at their action on the unit disk. Consider the set of vertices $V(E) := \{g(E) : E \in G\}$ where E denotes the center of the unit disk. For any $x, y \in V(E)$, put a geodesic edge between them if $x = gy$ where $g = A, A^{-1}, B$ or B^{-1}, and we end up with a Tree $\mathscr{T}$. [Here we use A to denote A composed with Cayley transform and similarly for B.] Define $\sigma : G \to \mathrm{Aut}(\mathscr{T})$ to be the identity map. Then we have the injective composition

$$\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, \lambda) \xrightarrow{\rho} G \xrightarrow{\sigma} \mathrm{Aut}(\mathscr{T})$$

Thus, we get that ρ is injective and (2) holds.
To prove (3), we simply compare the fundamental domain of the action of $\Gamma(2)$ with that of G and conclude that they are the same. If the reader is curious of more details, a more comprehensive proof with illustrative pictures can be found in [2]. □

Remark 3.9. We mentioned at the end of section 3.1 that for the Legendre family, local period mapping and local period domain depend on the choice of a basis of $H_1(\mathscr{C}_\lambda)$. From the discussions about the monodromy representations, the reader should see that the global period domain must be a quotient space of the quotient of the local period domain by the action of the monodromy group G. In fact, the only obstruction to the globalization of period mapping and period domain is the monodromy action, so the global period domain is just the quotient of the local period domain by the action of the monodromy group G, which is just $\mathbb{H}/\Gamma(2)$ in case of the Legendre family.

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