INTRODUCTION TO SCHRAMM-LOEWNER EVOLUTION

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Abstract. This paper serves as an introduction to the theory of Schramm-Loewner Evolution (SLE). The loop-erased random walk ($LERW$) is introduced as a discrete analogue to SLE, and its properties are examined. We then move to an explanation of the properties of compact $\mathbb{H}-$hulls and their mapping-out functions. Chordal Loewner theory is developed, and the paper culminates in a brief characterization of SLE. The Appendix provides a stochastic calculus based proof of the conformal invariance of Brownian motion.

Contents

1. Introduction

Schramm-Loewner evolution was developed as a scaling limit for the loop-erased random walk ($LERW$) and the uniform spanning tree, two related discrete probabilistic processes [1]. Since then, it has been conjectured to be or proven to be the scaling limit of a variety of discrete conformally invariant processes on the plane, having deep implications for statistical physics and probability [2]. This paper provides a mostly self-contained introduction to the theory of SLE. The goal is to provide the reader with both a technical understanding of the theory underlying this process as well as an intuitive understanding of the process itself.

The remainder of this paper provides an introduction to the theory of SLE. Section 2 provides an overview of the loop-erased random walk in order to provide a discrete analogue of SLE for the reader to keep in mind. Section 3 introduces compact $\mathbb{H}-$hulls and their mapping-out functions, presenting several of their properties. Section 4 provides an overview of Chordal Loewner theory independent of SLE. Section 5 contains a brief characterization of SLE and provides references for some major results in the field. Some of the proofs rely on the conformal invariance of Brownian motion, a proof of which is presented in the appendix. Section 2 is

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largely adapted from [3], while the remainder of this paper is largely adapted from [4].

2. Loop-Erased Random Walks

In this section, we present the Loop-erased random walk ($LERW$), a discrete probabilistic process known to converge to a Schramm-Loewner Evolution. We will find that this model satisfies a domain-Markov property essential to characterizing SLE, which we will revisit in that context in section 5. The goal of this section is to ground the theory of Schramm-Loewner Evolutions in an understanding of a discrete process that converges to an SLE, developing an intuition as to how SLE arises as a random process in its own right.

Let D be a bounded region in $\mathbb{R}^2$. Set $\varepsilon \in (0, \infty)$ and cover $\mathbb{R}^2$ with a grid $\varepsilon\mathbb{Z}^2$. Define δD as the set of points on $\varepsilon\mathbb{Z}^2$ which lie either on ∂D or are within length δ of ∂D but are not inside D. In other words, δD is the set of points "just outside" D. This setting is illustrated in figure 1.

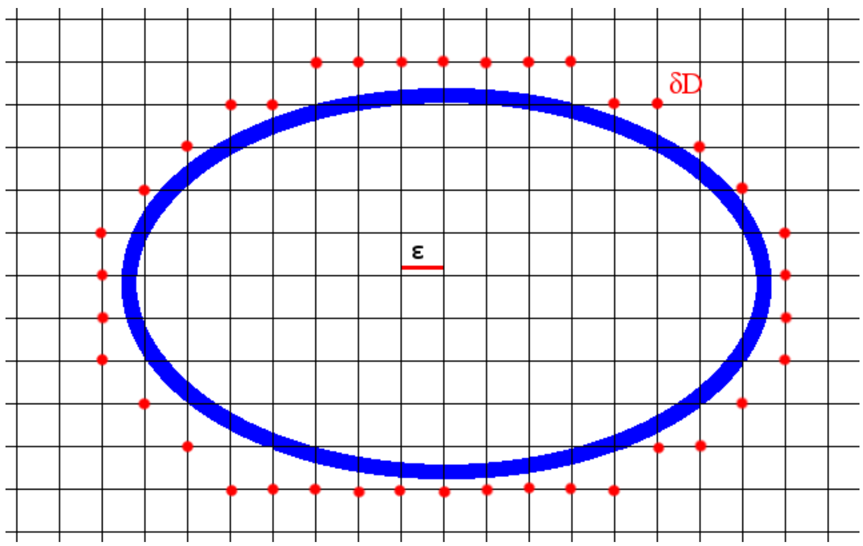

Figure 1: The setting of a LERW on $\varepsilon\mathbb{Z}^2 \cap D$.

Fix $x_0 \in D \cap \varepsilon\mathbb{Z}$ and run a standard two dimensional random walk on $\varepsilon\mathbb{Z}$ until the walk exits D. Recall that a simple random walk exists a bounded domain in finite time with probability 1 [5]. For any path $\mathbf{x} = (x_0, \ldots, x_m)$, where $x_m \in \delta D$, define the loop-erasure $L = L(\mathbf{x})$ of $\mathbf{x}$ inductively as follows. Set $L_0 = x_0$, and for all $j \geq 0$, define $n_j := \max\{n \leq m \colon x_n = L_j\}$. We then inductively define

$$L_{j+1}(\mathbf{x}) := x_{n_j+1} \tag{2.1}$$

until $L_\sigma = x_m$. We have constructed the following definition.

Definition 2.2. Given a bounded domain $D \subset \mathbb{R}^2$, the loop-erasure $L = L(\mathbf{x})$ of some path path $\mathbf{x}$ in $\varepsilon\mathbb{Z} \cap D$ is known as the **loop-erased random walk** ($LERW$) corresponding to $\mathbf{x}$. The number σ such that $L_\sigma = x_m$ is known as the **number of steps** of the $LERW$.

Figures 2 and 3 below illustrate this definition. It is important to note that in figure 3, the loops from figure 2 have been erased *in the order in which they were formed*.

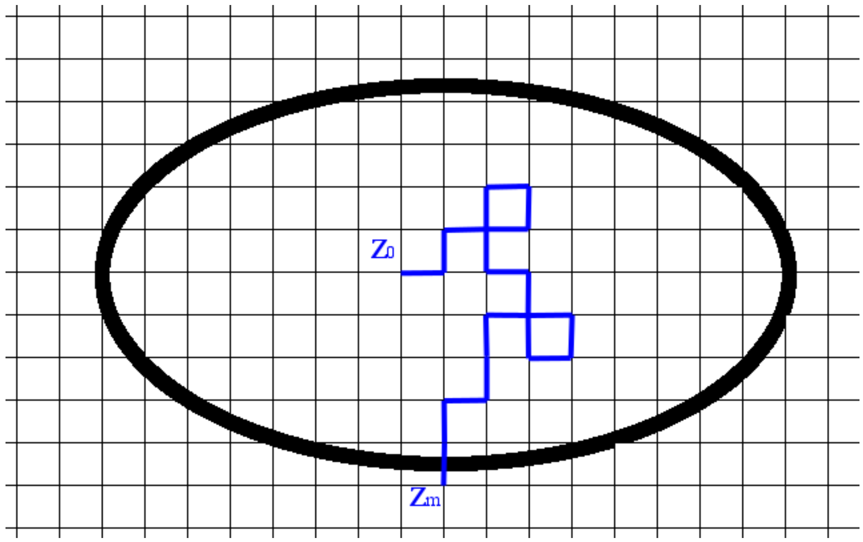

Figure 2: A planar simple random walk.

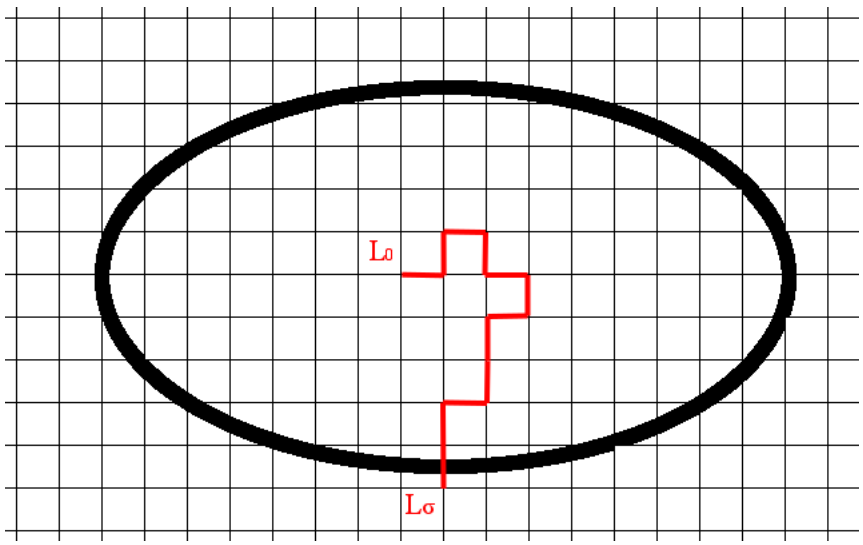

Figure 3: A loop-erased random walk (LERW) derived from figure 2

So far, I have constructed a $LERW$ exclusively in a grid overlaid on $\mathbb{R}^2$. This context was primarily for illustrative purposes; it is not necessary to restrict ourselves to this setting. Given any state space S and some $\mathbf{x} = (x_0, \ldots, x_m) \in S^{m+1}$, we may define the loop-erasure $L(\mathbf{x})$ exactly as in Equation (2.1) and Definition 2.2. We will now show that, given the assumption that a $LERW$ is performed on a recurrent Markov chain $(X_n)_{n\geq 0}$ on a discrete state-space $\mathcal{S}$, then the $LERW$ satisfies a domain Markov property that is central to the theory of SLE.

Before we define and prove the Domain Markov property for loop-erasures of recurrent Markov chains, we first establish the context in which we are working.

Let $(X_n)_{n\geq 0}$ be a recurrent Markov chain on some state space $\mathcal{S}$. Suppose that $A \subseteq \mathcal{S}$ is non-empty and let τ_A denote the hitting time of A by $(X_n)_{n\geq 0}$. Set $X[0,\tau_A] = (X_0, \ldots, X_{\tau_A})$ and define the loop-erasure $L = L(X[0,\tau_A])$ up to the hitting time τ_A. This is analogous to defining the $LERW$ in the case of a two dimensional simple random walk up to the time it hits the boundary of a bounded domain. The loop-erased random walk L run up to the time it hits $A \subseteq \mathcal{S}$ is referred to as $L^A = L^A(X[0,\tau_A])$. Let σ be the number of steps of L^A and, for $z, w \in \mathcal{S}$, let $p(z,w)$ denote the transition probabilities for the Markov chain $(X_n)_{n\geq 0}$. Assume that that $X_0 = x \in \mathcal{S}$. For $y \in A$ such that $L_\sigma = X_{\tau_A} = y$ with positive probability, let $\mathcal{L}(x, y; A)$ be the law of the loop-erasure of the Markov chain $(X_n)_{n\geq 0}$ conditioned on the event $\{X_{\tau_A} = y\}$. We now arrive at the statement and proof of the Domain Markov property in this context.

Proposition 2.3. ***(Domain Markov property of LERW on a recurrent Markov chain)*** *Take $y_0 \in A$ such that $L^A_\sigma = X_{\tau_A} = y_0$ with positive probability. Consider $y_1, \ldots, y_j \in S$ so that with positive probability for $\mathcal{L}(x, y_0; A)$,*

$$\{L^A_\sigma = y_0, \ldots, L^A_{\sigma-j} = y_j\}.$$

The conditional law of $L[0, \sigma - j]$ given this event is $\mathcal{L}(x, y_j; A \cup \{y_1, \ldots, y_j\})$.

Proof. For each $A \subseteq \mathcal{S}$ and $x \in A$, denote by $G(x, A)$ the expected number of visits to x by the Markov chain $(X)_{n\geq 0}$ before τ_A if $X_0 = x$. We claim that for all $n \geq 1, \mathbf{w} = (w_0, \ldots, w_n)$, with $w_0 = x$, $w_n \in A$, and $w_1, \ldots, w_{n-1} \in S \setminus A$,

$$\begin{aligned} P(L^A = \mathbf{w}) &= \sum_{\{\mathbf{x}:\, L(\mathbf{x}) = \mathbf{w}\}} P(X[0,\tau_A] = \mathbf{x}) \\ &= G(w_0, A) p(w_0, w_1) G(w_1, A \cup \{w_0\}) p(w_1, w_2) \cdots \\ &\qquad G(w_{n-1, A \cup \{w_0, w_1, \ldots, w_{n-2}\}}) p(w_{n-1}, w_n). \end{aligned} \tag{2.4}$$

This claim states that the probability that the loop-erasure of the Markov chain $(X_n)_{n\geq 0}$ equals the path $\mathbf{w}$ equals the product of the expected number of returns of $(X_n)_{n\geq 0}$ to each step in $\mathbf{w}$ times the transition probabilities from one step to the next. We prove this claim by induction on the length of $\mathbf{w}$. For $n = 1$, we have that

$$\begin{aligned} P(L^A = (w_0, w_1)) &= \sum_{\{\mathbf{x}:\, x_0 = X_0 = x,\ x_{\tau_A} \in A\}} P(L^A(X[0,\tau_A]) = \mathbf{w} \mid X[0,\tau_A] = \mathbf{x}) \times \\ &\qquad P(X[0,\tau_A] = \mathbf{x}) \\ &= \sum_{\{\mathbf{x}:\, L(\mathbf{x}) = \mathbf{w}\}} P(X[0,\tau_A] = \mathbf{x}) \\ &= p(w_0, w_1) \sum_{\{\mathbf{x}:\, L(\mathbf{x}) = \mathbf{w}\}} \frac{P(X[0,\tau_A] = \mathbf{x})}{p(w_0, w_1)} \\ &= G(w_0, A) p(w_0, w_1). \end{aligned}$$

The first line equals the second line because $L(\mathbf{x})$ is a deterministic function. Once we know the path $\mathbf{x}$, we know $L(\mathbf{x})$ with probability 1. This means that $P(L^A(X[0,\tau_A]) = \mathbf{w} \mid X[0,\tau_A] = \mathbf{x})$ equals 0 when $X_{\tau_A} \neq w_1$ and it equals 1 when

$X_{\tau_A} = w_1$, and we are left with the sum on the second line. We can factor $p(w_0, w_1)$ out of the sum on the second line because $p(w_0, w_1)$ is the final term in every path in the sum. Once we do this, we are left with a sum that counts the number of paths that return to w_0 at least once weighted by their probability.

For the induction step, assume that Equation (2.4) is true for paths with $n-1$ steps and assume that $\mathbf{w}$ is a path with n steps. We have the following calculation:

$$\begin{aligned} P(L^A = \mathbf{w}) &= \sum_{\{\mathbf{x}\colon L(\mathbf{x})=\mathbf{w}\}} P(X[0,\tau_A] = \mathbf{x}) \\ &= p(w_{n-1}, w_n) \sum_{\{\mathbf{x}\colon L(\mathbf{x})=\mathbf{w}\}} \frac{P(X[0,\tau_A] = \mathbf{x})}{p(w_{n-1}, w_n)} \\ &= p(w_{n-1}, w_n) \sum_{\{\mathbf{x}\colon L(\mathbf{x})=\mathbf{w}\}} P(X[0,\tau_A - 1] = (x_0, \ldots, w_{n-1})) \\ &= p(w_{n-1}, w_n) P(L^A = (w_0, \ldots, w_{n-1})) \\ &= G(w_0, A) p(w_0, w_1) G(w_1, A \cup \{w_0\}) p(w_1, w_2) \cdots \\ &\qquad\qquad G(w_{n-1, A\cup\{w_0, w_1, \ldots, w_{n-2}\}}) p(w_{n-1}, w_n). \end{aligned}$$

This completes the proof of the claim.

It is natural at this point to define the function,

$$F(w_0, \ldots, w_{n-1}; A) = \prod_{j=0}^{n-1} G(w_j, A \cup \{w_0, \ldots, w_{j-1}\}). \tag{2.5}$$

We wish to show that F is symmetric in its first n arguments, which will allow some recombinations of the product of the $G(w, \cdot)$ in equation (2.4). I claim that

$$G(y, A')G(y', A' \cup \{y\}) = G(y', A')G(y, A' \cup \{y'\}) \tag{2.6}$$

Equation (2.6) will demonstrate the symmetry of F in its first n arguments because, once it is proven, the product on the right side of Equation (2.5) will be the same no matter the order of the first n argument of F.

Observe that if N_y is the number times the Markov chain returns to y, then

$$E[N_y \mid X_0 = y] = \sum_{n=1}^{\infty} E[\mathbf{1}_{\{X_n = y\}} \mid X_0 = y] = \sum_{n=1}^{\infty} P\{X_n = y \mid X_0 = y\},$$

which justifies equality

$$G(y, A') = \sum_{\{\mathbf{x}\colon x_0 = y\}} P(X[0, \tau_{A'}] = \mathbf{x}). \tag{2.7}$$

We can categorize the left and right sides of (2.6) as follows. We will use the notation $y \mapsto A'$ to denote the collection of paths that start at y and end at A' without reaching y'. Similarly, the notation $y \mapsto y' \mapsto A'$ denotes the collection of paths that start at y, reach y', and then hit A'. The notation for the right side of (2.6) is identical. The following is an enumeration of the paths whose probabilities of occurring constitute the sum in Equation (2.7) for each $G(\cdot, \cdot)$.

$$G(y,A') = \begin{cases} y \mapsto A' \\ y \mapsto y' \mapsto A' \end{cases} \qquad G(y', A' \cup \{y\}) = \begin{cases} y' \mapsto y \\ y' \mapsto A' \end{cases}$$

$$G(y',A') = \begin{cases} y' \mapsto A' \\ y' \mapsto y \mapsto A' \end{cases} \qquad G(y, A' \cup \{y'\}) = \begin{cases} y \mapsto y' \\ y \mapsto A' \end{cases}$$

Everything in the top row appears in the bottom row, which implies that there is a one to one correspondence between the paths in the sums of the left side of (2.6) and the right side of (2.6). In particular, we see that $y \mapsto A'$ appears in $G(y', A' \cup \{y\})$ (on the left) and in $G(y, A' \cup \{y'\})$ (on the right). Moreover, $y' \mapsto A'$ appears in $G(y', A' \cup \{y\})$ (on the left) and in $G(y', A')$ (on the right). Finally, the multiplication of the probabilities of the paths $(y \mapsto y' \mapsto A') \times (y' \mapsto y)$ on the left and the paths $(y' \mapsto y \mapsto A') \times (y \mapsto y')$ on the right are equal.

We can now find the law of $L[0, \sigma - j]$ conditioned on the event $\{L^A_\sigma = y_0, \dots, L^A_{\sigma-j} = y_j\}$. Once more letting $(X_n)_{n\geq 0}$ be a recurrent Markov chain, we have the following equality:

$$P(L^A_0 = w_0, \dots, L^A_\sigma = w_n \mid L_\sigma = w_n, \dots, L_{\sigma-j} = w_{n-j}) = \tag{2.8}$$

$$\frac{p(w_{n-j}, w_{n-j+1}) G(w_{n-j}, A) \dots p(w_{n-1}, w_n) G(w_{n-1}, A \cup \{w_{n-j}, \dots, w_{n-2}\})}{P(L_\sigma = w_n, \dots, L_{\sigma-j} = w_{n-j})} \times$$

$$\prod_{k=0}^{n-j-1} p(w_k, w_{k+1}) G(w_k, A \cup \{w_{n-1} \cup \{w_0, \dots, w_{k-1}\}\})$$

Observe from Equation (2.4) that

$$\prod_{k=0}^{n-j-1} p(w_k, w_{k+1}) G(w_k, A \cup \{w_{n-1} \cup \{w_0, \dots, w_{k-1}\}\}) =$$

$$P(L^{A \cup \{w_{n-j}, \dots, w_{n-1}\}}(X[0, \tau_{A \cup \{w_{n-j}\}}]) = (w_0, \dots, w_{n-j}))$$

From Equation (2.4), we also see that

$$p(w_{n-j}, w_{n-j+1}) G(w_{n-j}, A) \dots p(w_{n-1}, w_n) G(w_{n-1}, A \cup \{w_{n-j}, \dots, w_{n-2}\})$$

is the probability that an independent loop-erased random walk beginning at w_{n-j} equals $(w_{n-j}, \dots, w_n)$. Putting these two observations together and letting $\widetilde{L}$ be an independent $LERW$ starting from w_{n-j}, we see that Equation (2.8) is equal to

(2.9)

$$\frac{p(w_{n-j}, w_{n-j+1}) G(w_{n-j}, A) \dots p(w_{n-1}, w_n) G(w_{n-1}, A \cup \{w_{n-j}, \dots, w_{n-2}\})}{P(L_\sigma = w_n, \dots, L_{\sigma-j} = w_{n-j})} \times$$

$$P(L^{A \cup \{w_{n-j}, \dots, w_{n-1}\}} = (w_0, \dots, w_{n-j})) =$$

$$\frac{P(\widetilde{L}^A = (w_{n-j}, \dots, w_n))}{P(L_\sigma = w_n, \dots, L_{\sigma-j} = w_{n-j})} P(L^{A \cup \{w_{n-j}, \dots, w_{n-1}\}} = (w_0, \dots, w_{n-j})) =$$

$$\frac{P(L_\sigma = w_n, \ldots, L_{\sigma-j} = w_{n-j} \mid L^{A\cup\{w_{n-j},\ldots,w_{n-1}\}} = (w_0, \ldots, w_{n-j}))}{P(L_\sigma = w_n, L_{\sigma-1} = w_{n-1})} \times$$

$$P(L^{A\cup\{w_{n-j},\ldots,w_{n-1}\}} = (w_0, \ldots, w_{n-j})) =$$

$$P(L^{A\cup\{w_{n-j},\ldots,w_{n-1}\}}(X[0, \tau_{A\cup\{w_{n-j}\}}]) = (w_0, \ldots, w_{n-j}) \mid L_\sigma = w_n, \ldots, L_{\sigma-j} = w_{n-j})$$

This proves the proposition.

□

We may contrast the domain Markov property from Proposition 2.3 with the Markov property for a simple random walk (SRW). The following discussion is inspired by [6] and [7]. In the case of the SRW, if we observe a path $(Y_n)_{n\geq 0}$ in a domain D up to some stopping time τ, the remainder of the path $(Y_n)_{n\geq\tau}$ follows the law of a SRW starting at Y_τ. In the case of the LERW, however, we build a "slit" in the domain D backwards from $L_\sigma(X_{\tau_A})$ as follows. At time j we replace D with $D \setminus \{L_{\sigma-1}, \ldots, L_{\sigma-j}\}$. The law of the LERW conditioning on exiting at $L_{\sigma-j}$ becomes the law of a LERW on the domain $D \setminus \{L_{\sigma-1}, \ldots, L_{\sigma-j}\}$ conditioned on exiting at $L_{\sigma-j}$.

As mentioned previously, for a bounded domain $D \subset \mathbb{R}^2$, as $\varepsilon \to 0$ the LERW on $\varepsilon\mathbb{Z}^2 \cap D$ converges to a Schramm-Loewner Evolution. However, it is not obvious how to construct this continuous version of a $LERW$. One idea for the construction would be to run a two dimensional Brownian motion in D until it exits the domain and then to "erase" the loops as they occur. This idea fails because the geometry of planar Brownian motion is incredibly complicated. It has points of any multiplicity (even infinite multiplicity) and loops at any scale (so there is no "first" loop to erase). Moreover, decisions about which loops to erase first might propagate to decisions about which macroscopic loops to erase.

The answer to these problems is SLE. The question remains: where might we search for the scaling limit of $LERW$? As it turns out, the law of $LERW$ is also conformally invariant [8]. These two properties - conformal invariance and the domain Markov property - turn out to characterize a family of continuous stochastic processes known as SLE, which we now begin to characterize.

3. Compact $\mathbb{H}-$Hulls and Their Mapping-Out Functions

The theory of Schramm-Loewner evolutions is based around the evolution of subsets of the upper half plane, $\mathbb{H}$, known as compact $\mathbb{H}-$hulls. This section describes the properties of these subsets. For the purposes of this paper, $\mathbb{R} \not\subseteq \mathbb{H}$, so any compact $\mathbb{H}-$hull must have nonzero Lebesgue measure. Closures of subsets of $\mathbb{H}$, however, are taken with respect to $\mathbb{C}$; if K is a compact $\mathbb{H}-$hull then $\mathbb{R} \cap \overline{K} \neq \emptyset$.

Definition 3.1. A subset K of the upper half plane is a **compact** $\mathbb{H}-$**hull** if K is bounded and $H = \mathbb{H} \setminus K$ is a simply connected domain.

Given a compact $\mathbb{H}-$hull K, we can associate it with a canonical conformal isomorphism $g_K : H \to \mathbb{H}$, known as the **mapping-out function** of K. Before we construct the mapping-out function, however, we must first examine some properties of conformal isomorphisms from $D \subseteq \mathbb{H}$ to $\mathbb{H}$.

Given a proper simply connected domain $D \subseteq \mathbb{H}$, define

$$D^0 = \overline{D} \cap \mathbb{R}, \qquad D^* = D \cup D^0 \cup \{\overline{Z} \colon z \in D\}.$$

For any open set $U \subseteq D^0$, define

$$D_U^* = D \cup U \cup \{\overline{z} \colon z \in D\}.$$

Furthermore, we say that a function $f^* : D_U^* \to \mathbb{C}$ is **reflection-invariant** if

$$f^*(\overline{z}) = \overline{f^*(z)}, \qquad \forall z \in D_U^*. \tag{3.2}$$

Let f be a continuous function on D. If there exists a continuous, reflection-invariant extension f^* on D_U^*, it is necessarily real-valued on U and unique by (3.2). We now show that, given some $D \subseteq \mathbb{H}$, there exists a unique, reflection-invariant conformal isomorphism from D to $\mathbb{H}$ that extends continuously to certain subsets of the real line.

Proposition 3.3. ***(Reflection-invariant conformal isomorphism)*** *Let $D \subseteq \mathbb{H}$ be a simply connected domain such that D^0 contains an interval. Let $I \subseteq D^0$ be a proper open subinterval of $\mathbb{R}$ and let $x \in I$. There exists a unique conformal isomorphism $\phi : D \to \mathbb{H}$ that extends to a homeomorphism $D \cup I \to \mathbb{H} \cup (-1, 1)$ and maps x to 0. Furthermore, ϕ extends to a reflection-invariant conformal isomorphism $\phi^* : D_I^* \to \mathbb{H}_{(-1,1)}^*$.*

Proof. Note that D_I^* and $\mathbb{H}_{(-1,1)}^*$ are proper simply connected domains. By the Riemann mapping theorem, there exists a unique conformal isomorphism $\phi^* : D_I^* \to \mathbb{H}_{(-1,1)}^*$ with $\phi^*(x) = 0$ and $\arg((\phi^*)'(x)) = 0$. Define $\rho : D_I^* \to \mathbb{H}_{(-1,1)}^*$ by $\overline{\rho(z)} = \phi^*(\overline{z})$. Then ρ is a conformal isomorphism with $\rho(x) = 0$ and $\arg(\rho'(x)) = 0$. By the uniqueness in the Riemann mapping theorem, $\rho = \phi^*$, and so ϕ^* is reflection-invariant. Reflection invariance implies that $\phi^*(I) \subseteq (-1, 1)$ and $(\phi^*)^{-1}(-1, 1) \subseteq I$, so $\phi^*(I) = (-1, 1)$.

Recall that $\arg((\phi^*)'(x)) = 0$. Taking the derivative of ϕ^* at x, we see that

$$\frac{d}{dz}\phi^*(z)\bigg|_{z=x} = \lim_{w \to 0} \frac{\phi^*(x + w)}{w}.$$

If $\phi^*(D) \not\subseteq \mathbb{H}$, by continuity and the fact that $\phi^*(I) = (-1, 1)$, it would be the case that $\phi^*(D) \subseteq \overline{\mathbb{H}}$ (the lower half-plane). If this were the case, then $\arg(\phi^*(x + w)) \in (\pi, 2\pi)$ and $\arg(w) \in (0, \pi)$. There is no way that $\arg((\phi^*)'(x)) = 0$, which contradicts the Riemann mapping theorem. We conclude that $\phi^*(D) \subseteq \mathbb{H}$, and it follows that $\phi(D) \subseteq \mathbb{H}$. On the other hand, if there were some $z \in \mathbb{H}$ such that $(\phi^*)^{-1}(z) \in \overline{H}$, it would follow that $(\phi^*)^{-1}(\mathbb{H}) \cap \mathbb{R} \neq \emptyset$, contradicting the fact that this map is an isomorphism. We conclude that $\phi(D) = \mathbb{H}$.

Conversely, suppose that $\psi : D \to \mathbb{H}$ is a conformal isomorphism that extends to a homeomorphism from $D \cup I$ to $\mathbb{H} \cup (-1, 1)$, maps x to 0, and has a continuous extension ψ^* by reflection to D_I^*. We see that ϕ^* is a bijection to $\mathbb{H}_{(-1,1)}^*$, and that it is holomorphic by the Schwarz reflection principle. Moreover, $\psi^*(x) = 0$, and $\arg(\psi^*)'(x) = 0$ since $\psi^*(I) = (-1, 1)$. It follows once more by the uniqueness from the Riemann mapping theorem that $\psi^* = \phi^*$, so $\psi = \phi$.

$\square$

Now that we have established that there exists a unique, reflection-invariant conformal isomorphism from a domain $D \subseteq \mathbb{H}$ to $\mathbb{H}$, it is time to use Proposition 3.3 to construct the mapping-out function from a compact $\mathbb{H}-$hull K to the upper half-plane. In Proposition 3.4 we will introduce a constant, a_K, associated with the mapping-out function for a compact $\mathbb{H}-$hull. For now we will not investigate the origins of a_K, but later in this section we will define the half-plane capacity of a compact $\mathbb{H}-$hull $\text{hcap}(K)$ (Definition 3.8) and show that a_K is equal to $\text{hcap}(K)$ (Proposition 3.9).

Proposition 3.4. ***(The Mapping-out function)*** *Let K be a compact $\mathbb{H}-$hull and set $H = \mathbb{H} \setminus K$. There exists a unique conformal isomorphism $g_K : \mathbb{H} \setminus K \to \mathbb{H}$ such that $g_K(z) - z \to 0$ as $|z| \to \infty$. Moreover, $g_K(z) - z$ is bounded uniformly in $z \in H$. Finally, for some $a_K \in \mathbb{R}$, we have*

$$g_K(z) = z + \frac{a_K}{z} + O(|z|^{-2}), \;\; |z| \to \infty. \tag{3.5}$$

Proof. Set

$$D = \{z\colon -z^{-1} \in H\}$$

Then, $D \subseteq \mathbb{H}$ is a simply connected domain which is a neighborhood of 0 in $\mathbb{H}$. Choose a bounded open interval $I \subseteq D^0$ containing 0. By Proposition 2.2, there is a conformal isomorphism $\phi : D \to \mathbb{H}$ which extends to a reflection-invariant conformal isomorphism ϕ^* on D_I^*, with $\phi(0) = 0$ and $\arg(\phi^*)'(0) = 0$. Consider the Taylor expansion of ϕ^* at 0. Since ϕ^* maps I into $\mathbb{R}$, the coefficients must all be real. As $z \to 0$, we have

$$\phi^*(z) = az + bz^2 + cz^3 + O(|z|^4)$$

for some $a \in (0, \infty)$ and $b, c \in \mathbb{R}$.

Define g_K in H by $g_K(z) = -a\phi(-z^{-1}) - b/a$. This function is a conformal isomorphism from H to $\mathbb{H}$, and it has the required expansion at infinity with $a_K = (b/a)^2 - c/a$. We see from Equation (3.5) that $g_K(z) - z$ is bounded near infinity. Moreover, because $\phi(0) = 0$ and ϕ is an isomorphism, ϕ^* is a homeomorphism of neighborhoods of 0. We see that g_K can only take bounded sets to bounded sets, and so $g_K - z$ is uniformly bounded on H.

Finally, if $g : H \to \mathbb{H}$ is a conformal isomorphism such that $g(z) - z \to 0$ as $|z| \to \infty$, then $f = g \circ g_K^{-1}$ is a conformal automorphism of $\mathbb{H}$ with $f(z) - z \to 0$ as $|z| \to \infty$. We see that $f(\infty) = \infty$, so $f(z) = \sigma z + \mu$ for some $\mu \in \mathbb{R}$ and $\sigma \in (0, \infty)$. The condition that $f(z) - z \to 0$ as $|z| \to \infty$ forces the choices of $\sigma = 1$ and $\mu = 0$, and we conclude that g_K is unique.

□

Now that we have constructed the mapping-out function g_K, it is time to see how g_K changes in relation to a translation or a scaling of K. This information will be used later in this section to establish continuity and differentiability estimates for the mapping-out function.

Proposition 3.6. ***(Properties of the mapping-out function)*** *Let K be a compact $\mathbb{H}-$hull. Let $r \in (0, \infty)$ and $x \in \mathbb{R}$. Set*

$$rK = \{rz\colon z \in K\}, \qquad K + x = \{z + x\colon z \in K\}$$

Then rK and $K+x$ are compact $\mathbb{H}-$hulls with mapping out functions

$$g_{rK}(z) = rg_K(z/r), \qquad g_{K+x}(z) = g_K(z-x) + x$$

respectively.

Proof. The fact that both rK and $K+x$ are compact $\mathbb{H}-$hulls is immediate from the fact that they are bounded, connected, and their closures intersect the real line. We now turn to calculating the mapping-out functions of $K+x$ and rK, respectively.

To compute the mapping-out function of $K+x$, we first recall

$$g_K(z-x) + x = z + \frac{a_K}{z-x} + O(|z|^{-2}).$$

We now observe that

$$z + \frac{a_K}{z-x} - z - \frac{a_K}{z} = \frac{x}{z(z-x)} = O(|z|^{-2})$$

It follows that $g_K(z-x)+x$ satisfies the requirements of a mapping-out function in Proposition 3.4, and the uniqueness of mapping-out functions implies that $g_{K+x}(z) = g_K(z-x)+x$.

To compute the mapping-out function of rK, we first calculate

$$\begin{aligned} rg_K(z/r) &= r\left(\frac{z}{r} + r\frac{a_k}{z} + O(|z|^{-2})\right) \\ &= z + r^2\frac{a_k}{z} + O(|z|^{-2}). \end{aligned}$$

Just as in the case of a translation, it follows that $rg_K(z/r)$ satisfies the requirements of a mapping-out function in Proposition 3.4, and the uniqueness of mapping-out functions implies that $g_{rK} = rg_K(z/r)$.

$\square$

Proposition 3.6 allows us to reduce many problems involving compact $\mathbb{H}-$hulls and their mapping-out functions to the often simpler case where $K \subseteq \mathbb{D}$, the unit disk. We now turn to a composition property of mapping-out functions which allows us to encode nested families of compact $\mathbb{H}-$hulls as families of conformal isomorphisms. This property will be invaluable when we consider nested families of compact $\mathbb{H}-$hulls in Section 4.

Proposition 3.7. ***(Properties of compact $\mathbb{H}-$hulls)*** *Let K_0 and K_1 be compact $\mathbb{H}-$hulls. Set $K = K_0 \cup g_{K_0}^{-1}(K_1)$. Then K is a compact $\mathbb{H}-$hull containing K_0 and we have*

$$g_K = g_{K_1} \circ g_{K_0}, \qquad a_k = a_{K_0} + a_{K_1}.$$

Moreover, all compact $\mathbb{H}-$hulls are obtained in this way.

Proof. Set $H_0 = \mathbb{H} \setminus K_0$ and $H = \mathbb{H} \setminus K$. We know that H is simply connected because $g_{K_0}^{-1}$, as a conformal isomorphism, maps boundaries to boundaries. We can define a conformal isomorphism $g : H \to \mathbb{H}$ by $g = g_{K_1} \circ g_{K_0}$. Consider a sequence of points (z_n) in H_0 with $|z_n| \to \infty$. Then $|g_{K_0}(z_n)| \to \infty$ and, specifically, $g_{K_0}(z_n)/z_n \to 1$. Hence there exists N such that for all $n > N$, $g_{K_0}(z_n) \notin K_1$. Then,

$$z_n(g(z_n) - z_n) = z_n(g_{K_1}(g_{K_0}(z_n)) - g_{K_0}(z_n)) + z_n(g_{K_0}(z_n) - z_n) \to a_{k_1} + a_{k_0}.$$

It follows that $g = g_K$ and $a_K = a_{K_0} + a_{K_1}$. Moreover, we know that K is bounded because both g_{K_1} and g_{K_2} are isomorphisms that take bounded sets to bounded sets by Equation (3.5). We now know that K is a compact $\mathbb{H}-$hull and g_K is its mapping-out function.

On the other hand, suppose that K is a compact $\mathbb{H}-$hull containing K_0. Define $K_1 = g_{K_0}(K \setminus K_0)$ and $H_1 = g_{K_0}(\mathbb{H} \setminus K)$. We see that K_1 is bounded. We also see that $K = K_0 \cup g_{K_0}^{-1}(K_1)$ and $H_1 = \mathbb{H} \setminus K_1$. Moreover, H_1 is a simply connected domain because g_{K_0} maps $\overline{K_0} \cap (\overline{K \setminus K_0})$ to an interval in $\mathbb{R}$. It follows that K_1 is a compact $\mathbb{H}-$hull, as required.

□

We now revisit the constant a_K associated with the mapping out function of a compact $\mathbb{H}-$hull K. We will define the half-plane capacity of a compact $\mathbb{H}-$hull, hcap(K), and show that a_K is equal to the half-plane capacity of K.

Definition 3.8. Let K be a compact $\mathbb{H}-$hull and set $H = \mathbb{H} \setminus K$. Define $T(H)$ as the exit time of a standard planar Brownian motion starting at some $z \in H$. We define the **half-plane capacity of** K as

$$\text{hcap}(K) = \lim_{y \to \infty} y\mathbb{E}[\text{Im}(B_{T(H)})]. \tag{3.9}$$

Proposition 3.10. ***(Half-plane capacity equals*** a_K***.)*** *Given a compact* $\mathbb{H}-$*hull* K*, the half-plane capacity hcap*(K) *exists and is equal to the constant* a_K *in equation (3.5).*

Proof. To see that the limit in equation (3.9) exists, observe, by equation (3.5), that as $|z| \to \infty$

$$z(g_K(z) - z) \xrightarrow{|z| \to \infty} a_K \in \mathbb{R}.$$

Let $G_t = g_K(B_t)$. Set $M_t = G_t - B_t$. By the conformal invariance of Brownian motion (see Appendix), we see that $(M_t)_{t<T(H)}$ is a continuous local martingale. Moreover, M_t is bounded because $G_t \to B_t$ if $|B_t| \to \infty$ in H. We also see that, $M_t \to G_{T(H)} - B_{T(H)}$ as $t \nearrow T(H)$. It follows by optional stopping that, if $B_0 = z \in H$, then

$$g_K(z) - z = \mathbb{E}_z[G_T - B_T]. \tag{3.11}$$

Taking $z = iy$, we have the following calculation, which completes the proof:

$$\begin{aligned} y\mathbb{E}_{iy}[\text{Im}(B_{T(H)})] &= -y\text{Im}(\mathbb{E}_z[G_T - B_T]) \\ &= \text{Re}(z(g_K(z) - z)) \\ &= \text{Im}(-y\mathbb{E}_Z[G_T - B_T]) \\ &= \text{Im}(i^2 y(g_K(z) - z)) \\ &= \text{Im}(i(z(g_K(z) - z))) \\ &= \text{Re}(z(g_K(z) - z)) \xrightarrow{|z| \to \infty} a_K. \end{aligned}$$

□

Now that we know that a_K in Equation (3.5) equals the half-plane capacity, we may deduce that $\text{hcap}(K)$ exhibits the following properties under translation and dilation. Corollary 3.12 is the result of a direct application of Propositions 3.6 and 3.10.

Corollary 3.12. ***(Scaling half-plane capacity)*** *Let $r \in (0, \infty)$ and $x \in \mathbb{R}$. It follows that*

$$hcap(rK) = r^2 hcap(K), \qquad hcap(K + x) = hcap(K).$$

We may similarly apply Propositions 3.7 and 3.10 to deduce the behavior of a_K under the composition of mapping-out functions. Corollary 3.13 will be useful in examining the properties of families of nested compact $\mathbb{H}-$hulls in Section 4.

Corollary 3.13. ***(Half-plane capacity in nested compact $\mathbb{H}-$hulls)*** *Let K and K' be compact $\mathbb{H}-$hulls. Assume that $K \subseteq K'$ and set $\widetilde{K} = g_K(K' \setminus K)$. It follows that*

$$hcap(K) \leq hcap(K) + hcap(\widetilde{K}) = hcap(K').$$

Now that we have introduced compact $\mathbb{H}-$hulls, their mapping-out functions, and half-plane capacity, we will end this section with four estimates on the behavior of mapping-out functions. In Section 4, these estimates will allow us to examine the behavior of nested collections of compact $\mathbb{H}-$hulls.

Before we present these estimates, we must first introduce the concept of the radius of a compact $\mathbb{H}-$hull. First off, note that the mapping-out function for $K = \overline{\mathbb{D}} \cap \mathbb{H}$, where $\mathbb{D}$ is the unit disk, is

$$g_{\overline{\mathbb{D}} \cap \mathbb{H}} = z + 1/z. \tag{3.14}$$

We see from Corollary 3.12 that if $K \subseteq r\mathbb{D}$, it follows that $\text{hcap}(K) \leq \text{hcap}(r\overline{D} \cap \mathbb{H}) = r^2$. This fact motivates the following definition.

Definition 3.15. The **radius of a compact $\mathbb{H}-$hull** K is the following quantity:

$$\text{rad}(K) = \inf\{r > 0 \colon \text{there exists some } x \in \mathbb{R} \text{ such that } K \subseteq r\mathbb{D} + x\}.$$

It follows from this definition and Corollary 3.12 that

$$\text{hcap(K)} \leq \text{rad}(K)^2. \tag{3.16}$$

Now that we have defined the radius of a compact $\mathbb{H}-$hull, it is time to present a boundary estimate for the mapping-out function.

Proposition 3.17. ***(Boundary estimate of mapping-out function)*** *Let $S \subseteq H$ be measurable. Then*

$$\lim_{y\to\infty, x/y\to 0} \pi y P_{x+iy}(\widehat{B}_{T(H)} \in S) = \lambda(g_K(S))$$

where λ is the Lebesgue measure on $\mathbb{R}$.

Proof. Let $g_K(x+iy) = u+iv$. Then, because g behaves like the identity as z goes to infinity by Equation (3.5), we know that $u/y \to 0$ and $v/y \to 1$ as $y \to \infty$ with $x/y \to 0$.

By the conformal invariance of Brownian motion (see Appendix), we have that

$$P_{x+iy}(B_{T(H)} \in S) = P_{u+iv}(B_{T(\mathbb{H})} \in g_K(S)) = \int_{g_K(S)} \frac{v}{\pi((t-u)^2+v^2)}\, dt \tag{3.18}$$

where we have used the harmonic measure for the upper half-plane in the second equality. See [9] for a derivation of the Harmonic measure.

At this point, multiply both sides of Equation (3.18) by πy and let $y \to \infty$ and $x/y \to 0$. The numerator under the integral converges to v^2 which causes the integrand to converge to 1.

□

Proposition 3.17 lets us see that

$$\lim_{y\to\infty} \pi y P_{iy}(B_{T(H)} \in K) = \lim_{y\to\infty} \pi y P_{iy}(B_{T(H)} \notin H^0) = \lambda(\mathbb{R} \setminus g_K(H^0))$$

where the first equality is true because $\partial H \setminus (K \cup H^0)$ is at most countable and so of Lebesgue measure 0. We now use proposition 3.17 to get the following estimate of the value of the mapping-out function on the real line.

Proposition 3.19. ***(Estimate for the mapping-out function on*** $\mathbb{R}$***)*** *Let K be a compact $\mathbb{H}-$hull and let $x \in \mathbb{R}$. Suppose that $[x,\infty) \cap \overline{K} = \emptyset$. Then $g_K(x) \geq x$. If in addition $K \subseteq \mathbb{D}$ and $x \in (1,\infty)$, then $g_K(x) \leq x + 1/x$.*

Proof. For $b > x$ and $y > \mathrm{rad}(K)$, note that

$$P_{iy}(B_{T(H)} \in (x,b)) = P_{iy}(B_{T(\mathbb{H})} \in (x,b)) - P_{iy}(B_{T(\mathbb{H})} \text{ hits } K \text{ before } \mathbb{R},\, B_{T(\mathbb{H})} \in (x,b)).$$

We immediately see that

$$P_{iy}(B_{T(H)} \in (x,b)) \leq P_{iy}(B_{T(\mathbb{H})} \in (x,b)). \tag{3.20}$$

Multiply both sides of Equation (3.20) by πy and then let $y \to \infty$. By Proposition 3.16, we obtain

$$g_K(b) - g_K(x) \leq b - x. \tag{3.21}$$

Subtract b from both sides of Equation (3.21) and let $b \to \infty$ to see that $g_K(x) \geq x$.

If we also have that $K \subseteq \mathbb{D}$ and $x \in (1,\infty)$, then

$$P_{iy}(B_{T(\mathbb{H}\setminus\overline{\mathbb{D}})} \in (x,b)) \leq P_{iy}(B_{T(H)} \in (x,b)) \tag{3.22}$$

Multiply both sides of Equation (3.22) by πy and let $y \to \infty$. Using the known form of the mapping out function for $\overline{\mathbb{D}} \cap \mathbb{H}$ from Equation (3.14), we see that

$$(b + 1/b) - (x + 1/x) \leq g_K(b) - g_K(x). \tag{3.23}$$

Subtract b from both sides of Equation (3.23) and take $b \to \infty$ to get the result.

□

Having presented an estimate of the mapping-out function on the real line, we now present a continuity estimate for the mapping-out function of a compact $\mathbb{H}-$hull K on $H = \mathbb{H} \setminus K$.

Proposition 3.24. ***(Continuity estimate of the mapping-out function)*** *Let K be a compact $\mathbb{H}-$hull. Then*

$$|g_K(z) - z| \leq 3rad(K), \qquad \forall z \in H$$

Proof. Using proposition 3.6, we can dilate and translate K to consider the case where $K \subseteq \overline{\mathbb{D}}$ and $\mathrm{rad}(K) = 1$. Fix $z \in H$ and consider a complex Brownian motion B_t starting from z. For $t < T = T(H)$, set $G_t = g_K(B_t)$. By the conformal invariance of Brownian motion, G_t converges almost surely as $t \nearrow T$ to a limit $G_T \in \mathbb{R}$. Moreover, $G_T \in g_K(H^0)$ if and only if $B_T \in H^0$, in which case $G_T = g_K(B_T)$.

Recall that $g_K(z) - z$ is a bounded holomorphic function on H. For $t < T$, set $M_t = g_K(B_t) - B_t = G_t - B_t$. Then $(M_t)_{t<T}$ is a continuous and bounded local martingale, and $M_t \to G_T - B_T$ as $t \nearrow T$. Hence, by optional stopping:

$$g_K(z) - z = E_z(G_T - B_T) \tag{3.25}$$

Note that, by Proposition 3.19, $\{|x| > 1\} \subseteq H^0$ and $\{|x| > 2\} \subseteq g_K(\{|x| > 1\})$. If $|B_T| > 1$, then $B_T \in H^0$. Once more using Proposition 3.19, we have that $|G_T - B_T| = |G_K(B_T) - B_T| \leq 1/|B_T| \leq 1$. On the other hand, if $|B_T| \leq 1$, then $G_T \notin g_K(\{|x| > 1\})$, so $|G_T| \leq 2$. In any case $|G_T - B_T| \leq 3$. We conclude that $|g_K(z) - z| \leq 3$.

□

We end this section with a differentiability estimate for the mapping-out function. Equation (3.5) states that for every compact $\mathbb{H}-$hull K, there are constants C_K and R_K, depending on K, such that

$$\left| g_K(z) - z - \frac{a_K}{z} \right| < \frac{C_K}{|z|^2}, \qquad \text{for all } |z| > R_K.$$

The following Proposition improves upon this estimate. It states that if $K \subseteq \overline{\mathbb{D}}$, then we can take $C_K = Ca_K$, where C does not depend on K, and $R_K = 2$.

Proposition 3.26. ***(Differentiability estimate for the mapping-out function)*** *There is an absolute constant $C < \infty$ with the following properties. For all $r \in (0, \infty)$ and all $\xi \in \mathbb{R}$, for any compact $\mathbb{H}-$hull $K \subseteq r\overline{\mathbb{D}} + \xi$,*

$$\left| g_K(z) - z - \frac{a_k}{z - \xi} \right| \leq \frac{Cra_k}{|z - \xi|^2}, \qquad |z - \xi| \geq 2r$$

Proof. We shall prove the case when $r = 1$ and $\xi = 0$, when $K \subseteq \overline{\mathbb{D}}$. The general case follows by scaling and translation using Proposition 3.6. Let $D = \mathbb{H} \setminus \overline{\mathbb{D}} = \{z \in \mathbb{H} \colon |z| > 1\}$. Write $T = T(H)$ and define for $\theta \in [0, \pi]$

$$a(\theta) = E_{e^{i\theta}}(\mathrm{Im}(B_T))$$

Using the strong Markov property and Equation (3.24), we have that

$$\mathrm{Im}(z - g_K(z)) = E_z(\mathrm{Im}(B_T)) = \int_0^\pi h_D(z,\theta)a(\theta)\,d\theta,$$

where h_D is the harmonic measure on D. In order to find an expression for h_D, let g be the conformal isomorphism from D to $\mathbb{H}$ given by $g(z) = z + z^{-1}$ and note that $g(e^{i\theta}) = 2\cos\theta$. Then, for $z \in D$ and $w = g(z)$, by the conformal invariance of Brownian motion (see Appendix):

$$h_D(z,\theta) = h_{\mathbb{H}}(w, 2\cos\theta)\frac{d}{d\theta}g(e^{i\theta}) = Im\left(\frac{1}{2\cos\theta - w}\right)\frac{2\sin\theta}{\pi}$$

Hence:

$$\mathrm{Im}(z - g_K(z)) = \int_0^\pi \mathrm{Im}\left(\frac{1}{2\cos\theta - w}\right)\frac{2\sin\theta}{\pi}a(\theta)\,d\theta.$$

Set

$$a = \int_0^\pi \frac{2\sin\theta}{\pi}a(\theta)\,d\theta.$$

Now, consider the holomorphic function$f$ on $H^* \setminus \{0\}$ given by

$$f(z) = g_k^*(z) - z - a/z$$

and set $v(z) = Im(f(z))$. Observe that there is a constant $C < \infty$ such that, for all $|z| \geq 3/2$ and $\theta \in [0,\pi]$:

$$\left|\frac{1}{w - 2\cos\theta} - \frac{1}{z}\right| = \frac{|2\cos\theta - z^{-1}|}{|z||z + z^{-1} - 2\cos\theta|} \leq \frac{C}{|z|^2}$$

Hence, for $z \in \mathbb{H}$ with $|z| \geq 3/2$,

$$|v(z)| = |E_z(\mathrm{Im}(B_T) - 1/z)| \leq \int_0^\pi \left|\frac{1}{w - 2\cos\theta} - \frac{1}{z}\right|\frac{2\sin\theta}{\pi}a(\theta)\,d\theta \leq \frac{Ca}{|z|^2}$$

Since $v(\overline{z}) = -v(z)$, the same bound holds for all $z \in \mathbb{C}$ with $|z| \geq 3/2$. At this point, we would like to obtain bounds on $|\partial v/\partial x|$ and $\partial v/\partial y$. In order to do so, we must use the fact that, if u is a harmonic function in a region R, then

$$\left|\frac{\partial u}{\partial x}(z)\right|, \left|\frac{\partial u}{\partial y}(z)\right| \leq \frac{4||u||_\infty}{\pi \mathrm{dist}(z, \partial D)}, \qquad \text{for all } z \in R. \tag{3.27}$$

We refer the reader to [4] for a proof of this fact. We now see that for $|z| \geq 2$, we can apply Equation (3.27) to the region $D_z = \{w \in \mathbb{C} \colon |w| > (3/4)|z|\}$ to obtain, for a new constant $C < \infty$:

$$\left|\frac{\partial v}{\partial x}(z)\right|, \left|\frac{\partial v}{\partial y}(z)\right| \leq \frac{Ca}{|z|^3}.$$

By the Cauchy-Riemann equations, the same bound holds for $|f'(z)|$ for all $|z| \geq 3/2$, and it follows that $f(z) \to 0$ as $|z| \to \infty$. We now have for $|z| \geq 2$,

$$|f(z)| = \left|\int_1^\infty f'(tz)z\,dt\right| \leq \frac{Ca}{|z|^2}\int_1^\infty t^{-3}\,dt = \frac{Ca}{|z|^2}. \tag{3.28}$$

It follows that $zf(z) \to 0$ as $|z| \to \infty$. Using Equation (3.5), we calculate that, for some constant $D \in \mathbb{R}$,

$$|zf(z)| \leq |a_K - a| + \frac{D}{|z|} \leq \frac{C}{|z|} \xrightarrow{|z|\to\infty} 0.$$

We conclude that $a = a_k$. It follows that Equation (3.28) is the desired estimate, because we have rescaled to make $r = 1$ and translated to make $\xi = 0$.

□

4. Chordal Loewner Theory

Having examined compact $\mathbb{H}-$hulls and their mapping-out functions, it is time to turn to Loewner theory. The goal is to establish a one-to-one correspondence between continuous real-valued paths $(\xi_t)_{t\geq 0}$ and increasing families $(K_t)_{t\geq 0}$ of compact $\mathbb{H}-$hulls that have a certain local growth property. The term "chordal" refers to the fact that the family of hulls $(K_t)_{t\geq 0}$ evolve to the point ∞ (that is, $\sup\{|z| \colon z \in K_t\} \to \infty$ as $t \to \infty$). In the alternative, closely related *radial* variant of the theory, an interior point of the domain serves as the point to which the hulls evolve. This paper will only present chordal Loewner theory. For an overview of Radial SLE, we refer the reader to [4]. For a complete proof that the hulls evolve to the point at ∞ in Chordal SLE, we refer the reader to [10]. For now, we must formalize the concept of an increasing family of compact $\mathbb{H}-$hulls.

Definition 4.1. Let $(K_t)_{t\geq 0}$ be a family of compact $\mathbb{H}-$hulls. We say that $(K_t)_{t\geq 0}$ is an **increasing** collection of compact $\mathbb{H}-$hulls if $K_s \subset K_t$ whenever $s < t$ where the inclusion is strict.

In Chordal SLE, it is not enough to know that a family of hulls is increasing. The hulls must satisfy the following growth property.

Definition 4.2. Set $K_{t^+} = \bigcap_{s>t} K_s$. For $s < t$, define $K_{s,t} = g_{K_s}(K_t \setminus K_s)$. We say that the collection $(K_t)_{t\geq 0}$ satisfies the **local growth property** if

$$\text{rad}(K_{t,t+h}) \xrightarrow{h \searrow 0} 0 \text{ uniformly on compacta in } t.$$

Figure 4 illustrates $K_{s,t}$. Below figure 4, we will show that the local growth property is a type of continuity condition for an increasing collection of compact $\mathbb{H}-$hulls and proves the existence of the Loewner transform, a real-valued process associated with $(K_t)_{t\geq 0}$.

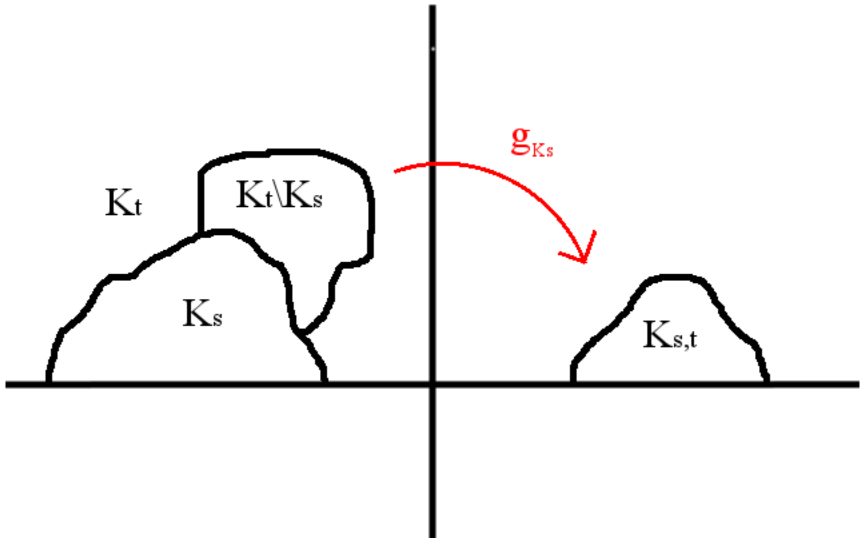

Figure 4: A representation of $K_{s,t}$

Proposition 4.3. ***(Local growth property and Loewner transform)*** *Let $(K_t)_{t\geq 0}$ be an increasing family of compact $\mathbb{H}-$hulls with the local growth property. Then $K_{t^+} = K_t$ for all t and the map $t \mapsto hcap(K_t)$ is continuous and strictly increasing on $[0, \infty)$. Moreover, for all $t \geq 0$, there is a unique $\xi_t \in \mathbb{R}$ such that $\xi_t \in \overline{K_{t,t+h}}$ for all $h > 0$, and the process $(\xi_t)_{t\geq 0}$ is continuous.*

Proof. Set $K_{t,t^+} = g_{K_t}(K_{t^+} \setminus K_t)$. For all $t \geq 0$ and $h > 0$, by Corollary 3.13, we have that

$$\text{hcap}(K_{t+h}) = \text{hcap}(K_t) + \text{hcap}(K_{t,t+h}).$$

Equation (3.16) tell us that $\text{hcap}(K_{t,t^+}) \leq \text{hcap}(K_{t,t+h}) \leq \text{rad}(K_{t,t+h})^2$. It now follows from the local growth property that $t \mapsto \text{hcap}(K_t)$ is continuous and that $\text{hcap}(K_{t,t^+}) = 0$.

Now that we have established that the map $t \mapsto \text{hcap}(K_t)$ is continuous, it is time to show that $K_{t^+} = K_t$. Because g_{K_t} takes bounded sets to bounded sets and boundaries to boundaries, we know that $K_{t,t+h}$ is either a compact $\mathbb{H}-$hull or the empty set. By the local growth property, it follows that that K_{t,t^+} is either the empty set of a single point. If a compact $\mathbb{H}-$hull K were a single point, then $H = \mathbb{H} \setminus K$ would not be simply connected, contradicting the fact that K is a compact $\mathbb{H}-$hull. We conclude that $K_{t^+} = K_t$.

We proceed to establish that the map $t \mapsto \text{hcap}(K_t)$ is increasing. We know that $K_{t,t+h} \neq \emptyset$ and cannot be a single point. We may dilate and translate $\overline{\mathbb{D}} \cap \mathbb{H}$ until $r(\overline{\mathbb{D}} \cap \mathbb{H}) + \xi \subseteq K_{t,t+h}$ for some $r, \xi \in \mathbb{R}$. By corollary 3.12, we see that $\text{hcap}(K_{t,t+h}) > 0$, and it follows that $t \mapsto \text{hcap}(K_t)$ is strictly increasing on $[0, \infty)$.

We now show that there is a unique $\xi_t \in \mathbb{R}$ such that $\xi_t \in \overline{K_{t,t+h}}$ for all $h > 0$, and that the process $(\xi_t)_{t\geq 0}$ is continuous. For fixed $t \geq 0$, the sets $\overline{K_{t,t+h}}$ are compact and decreasing in $h > 0$, By the local growth property, it follows that this collection has a unique common element $\xi \in \mathbb{R}$. For $t \geq 0$ and $h > 0$, choose $z \in K_{t+2h} \setminus K_{t+h}$ and set $w = g_{K_t}(z)$ and $w' = g_{K_{t+h}}(z)$. Then, $w \in K_{t,t+2h}$ and $w' \in K_{t+h,t+2h}$. By Proposition 3.7, we see that $w' = g_{K_{t,t+h}}(w)$. It follows that

$|\xi_t - w| \leq 2\text{rad}(K_{t,t+h})$, $|\xi_{t+h} - w'| \leq 2\text{rad}(K_{t+h,t+2h})$, and $|w - w'| \leq 3\text{rad}(K_{t,t+h})$. The first two inequalities are true because $\xi_t, w \in K_{t,t+h}$ and $\xi_{t+h}, w' \in K_{t+h,t+2h}$, respectively, while the third inequality is true by proposition 3.24. We conclude that

$$|\xi_{t+h} - \xi_t| \leq 2\text{rad}(K_{t+h,t+2h}) + 3\text{rad}(K_{t,t+h}) + 2\text{rad}(K_{t,t+h}) \xrightarrow{h \to 0} 0.$$

$\square$

Definition 4.4. The process $(\xi_t)_{t\geq 0}$ is the **Loewner transform** of $(K_t)_{t\geq 0}$.

The Loewner transform is one of the most important concepts in the theory of SLE. Later, we will see that an increasing family $(K_t)_{t\geq 0}$ of compact $\mathbb{H}-$hulls satisfying the local growth property can be recovered from its Loewner transform under some mild assumptions. Before we do so, however, we will show that the parametrization of the compact $\mathbb{H}-$hulls is largely arbitrary, which will allow us to inroduce a natrual prametrization.

Proposition 4.5. ***(Parametrization of compact $\mathbb{H}-$hulls)*** *Let $T, T' \in (0, \infty]$ and let $\tau : [0, T') \to [0, T)$ be a homeomorphism. Let $(K_t)_{t\in[0,T)}$ be an increasing family of compact $\mathbb{H}-$hulls with the local growth property and with Loewner transform $(\xi_t)_{t\in[0,T)}$. Set $K'_t = K_{\tau(t)}$ and $\xi'_t = \xi_{\tau(t)}$. Then $(K'_t)_{t\in[0,T')}$ is an increasing family of compact $\mathbb{H}-$hulls with the local growth property and with Loewner transform $(\xi'_t)_{t\in[0,T')}$.*

Proof. Note that the homeomorphism τ is increasing. It follows immediately that the re-parametrized collection of compact $\mathbb{H}-$hulls is increasing as well. The continuity of τ allows us to observe that $t + h \to t$ implies $\tau(t+h) \to \tau(t)$, which implies the local growth property. The Loewner transform is immediate from Definition 4.4 and τ.

$\square$

We now introduce a natural parametrization of $(K_t)_{t\geq 0}$. By proposition 4.3 the map $\eta : 2t \mapsto \text{hcap}(K_t)$ is a homeomorphism on $[0, T)$. Let $\tau : \text{hcap}(K_t) \to 2t$ be the inverse homeomorphism. By Proposition 4.5, we obtain a collection $(K'_t)_{t\in[0,T')}$ such that $\text{hcap}(K'_t) = 2t$ for all t. This leads us to Definition 4.6.

Definition 4.6. If $(K_t)_{t\geq 0}$ is an increasing family of compact $\mathbb{H}-$hulls satisfying the local growth property such that $\text{hcap}(K'_t) = 2t$ for all t, we say that $(K'_t)_{t\in[0,)}$ is **parametrized by half-plane capacity**.

As mentioned previously, we know that an increasing family $(K_t)_{t\geq 0}$ of compact $\mathbb{H}-$hulls satisfying the local growth property can be recovered from its Loewner transform. An important tool that allows this to happen is the **Loewner differential equation**. We now show that if a family $(K_t)_{t\geq 0}$ of compact $\mathbb{H}-$hulls satisfies the local growth property, then their mapping out functions $(g_t)_{t\geq 0}$ satisfy the Loewner differential equation.

Proposition 4.7. ***(Loewner differential equation)*** *Let $(K_t)_{t\geq 0}$ be an increasing family of compact $\mathbb{H}-$hulls, that satisfy the local growth property and are prametrized* *by half-plane capacity. Let* $(\xi_t)_{t\geq 0}$ *be the Loewner transform of* $(K_t)_{t\geq 0}$*. Set* $g_t = g_{K_t}$ *and* $\zeta(z) = \inf\{t \geq 0\colon z \in K_t\}$*. Then, for all* $z \in \mathbb{H}$*, the function* $(g_t(z)\colon t \in [0, \zeta(z)))$ *is differentiable and satisfies Loewner's differential equation*

$$\dot{g}_t(z) = \frac{2}{g_t(z) - \xi_t}. \tag{4.8}$$

Moreover, if $\zeta(z) < \infty$*, then* $g_t(z) - \xi_t \to 0$ *as* $t \to \zeta(z)$*.*

Proof. Let $0 \leq s < t < \zeta(z)$, and set $z_t = g_t(z)$. By Corollary 3.13, we see that

$$\mathrm{hcap}(K_s) + \mathrm{hcap}(K_{s,t}) = \mathrm{hcap}(K_t).$$

Because the hulls are parametrized by half-plane capacity, we see that $\mathrm{hcap}(K_{s,t}) = 2(t-s)$. Moreover, by Proposition 3.7 we have $g_{K_{s,t}}(z_s) = z_t$. Because $\xi_s \in K_{s,t}$, we also see that $K_{s,t} \subseteq \xi_s + 2\mathrm{rad}(K_{s,t})\overline{\mathbb{D}}$. We now apply Proposition 3.24 to the compact $\mathbb{H}-$hull $K_{s,t}$ to obtain

$$|z_t - z_s| \leq 3\mathrm{rad}(K_{s,t}). \tag{4.9}$$

Applying Proposition 3.26, provided that $|z_s - \xi_s| \geq 4\,\mathrm{rad}(K_{s,t})$, we have

$$\left| z_t - z_s - \frac{2(t-s)}{z_s - \xi_s} \right| \leq \frac{4C\mathrm{rad}(K_{s,t})(t-s)}{|z_s - \xi_s|^2}. \tag{4.10}$$

We use equation (3.5) along with the local growth property to see that $(z_t)_{z\in[0,\zeta(z))}$ is continuous, so we know that the map $t \mapsto |z_t - \xi_t|$ is positive and continuous on $[0, \zeta(z))$. Equation (4.10) along with the local growth property show that $(z_t\colon t \in [0, \zeta(z)))$ is differentiable with $\dot{z}_t = 2/(z_t - \xi_t)$.

Finally, if $\zeta(z) < \infty$, then for $z < \zeta(z) < t$, we have that $z \in K_t \setminus K_s$. So $z_s \in K_{s,t}$. It follows that $|z_s - \xi_s| \leq 2\mathrm{rad}(K_{s,t})$, so, by the local growth property, $|z_s - \xi_s| \to 0$ as $s \to \zeta(s)$.

□

We have shown that the mapping-out functions of an increasing family of compact $\mathbb{H}-$hulls with the local growth property satisfy the Loewner differential equation. If we are to recover a family of compact $\mathbb{H}-$hulls from the Loewner differential equation, however, we must characterize solutions of the equation. In order to do so, we introduce, for $t \geq 0$ and $z \in \mathbb{C} \setminus \xi_0$, the function

$$b(t,z) = \frac{2}{z - \xi_t} = \frac{2(\overline{z} - \xi_t)}{|z - \xi_t|^2}. \tag{4.11}$$

Observe that $b(t,\cdot)$ is holomorphic on $\mathbb{C} \setminus \{\xi_t\}$. Moreover, for $z, z' \in \mathbb{C} \setminus \{\xi_t\}$, if $|z - \xi_t|, |z' - \xi_t| \geq 1/n$, then

$$\begin{aligned} |b(t,z) - b(t,z')| &= \frac{2(\overline{z} - \xi_t)}{|z - \xi_t|^2} - \frac{2(\overline{z'} - \xi_t)}{|z \;\; - \xi_t|^2} \\ &\leq 2n^2 |\overline{z} - \overline{z'}| \\ &= 2n^2 |z - z'|. \end{aligned} \tag{4.12}$$

It follows that $b(t, z)$ is locally Lipschitz continuous in the second argument. We may now characterize solutions to the Loewner differential equation. Proposition 4.13 is the result of some basic theory in Ordinary Differential Equations (see [11]) and the fact that $b(t, z)$ is locally Lipschitz in z.

Proposition 4.13. ***(Solution of the Loewner differential equation)*** *Let $(\xi_t)_{t\geq 0}$ be a real-valued continuous process. For all $z \in \mathbb{C} \setminus \{\xi_0\}$, there is a unique $\zeta(z) \in (0, \infty]$ and a unique continuous map $(g_t(z)\colon t \in [0, \zeta(z)))$ in $\mathbb{C}$ such that, for all $t \in [0, \zeta(z))$, we have $g_t(z) \neq \xi_t$ and*

$$g_t(z) = z + \int_0^t \frac{2}{g_s(z) - \xi_s}\, ds$$

and such that $|g_t(z) - \xi_t| \to 0$ as $t \to \zeta(z)$ whenever $\zeta(z) < \infty$. Set $\zeta(\xi_0) = 0$ and define $C_t = \{z \in \mathbb{C}\colon \zeta(z) > t\}$. Then, for all $t \geq 0$, C_t is open, and $g_t : C_t \to \mathbb{C}$ is holomorphic.

Definitions 4.14. The process $(g_t(z))_{t<\zeta(z)}$ is the **maximal solution** starting from z, and $\zeta(z)$ is the **lifetime** of the solution. The continuous real-valued process $(\xi_t)_{t\geq 0}$ is the **driving function**, and the collection $\{(g_t)_{t\geq 0}(z)\colon z \in \mathbb{H} \cup \mathbb{R}) \setminus \xi_0\}$ is the **Loewner flow** in $\mathbb{H}$ with driving function $(\xi_t)_{t\geq 0}$.

We now show that, given a driving function $(\xi_t)_{t\geq 0}$, we can recover an increasing family of compact $\mathbb{H}-$hulls satisfying the local growth property from the Loewner differential equation. Define the sets

$$K_t = \{z \in \mathbb{H}\colon \zeta(z) \leq t\}, \qquad H_t = \mathbb{H} \setminus K_t. \tag{4.15}$$

We wish to show that $g_t(H_t) \subseteq \mathbb{H}$ so we can work in the upper half plane when defining SLE. Fix $z \in \mathbb{H}$ and $s \leq t < \zeta(z)$. Let $y_s = \text{Im}(g_s(z))$ and $\delta = \inf_{s\leq t} |z_s - \xi_s|$. Because $t < \zeta(z)$, we know that $\delta > 0$, and so, using Equation (4.11), we see that

$$\dot{y}_s \geq \frac{-2y_s}{\delta^2}.$$

Solving this differential equation, we find the formula

$$y_t \geq e^{-2t/\delta^2} > 0,$$

which implies that $g_t(H_t) \subseteq \mathbb{H}$. This allows us to restrict the definitions of $\zeta(\cdot)$ and $g_t(\cdot)$ to $\mathbb{H}$ as opposed to all of $\mathbb{C}$, which is the natural setting of Chordal SLE. Proposition 4.16 demonstrates that $(K_t)_{t\geq 0}$, as defined in Equation (4.15), is an increasing collection of compact $\mathbb{H}-$hulls with the local growth property.

Proposition 4.16. ***(Generating compact $\mathbb{H}-$hulls from a continuous driving function)*** *The family of sets $(K_t)_{t\geq 0}$ as defined in Equation (4.15) is an increasing family of compact $\mathbb{H}-$hulls with the local growth property. Moreover, $hcap(K_t) = 2t$ and $g_{K_t} = g_t$ for all t, and the driving function $(\xi_t)_{t\geq 0}$ is the Loewner transform of $(K_t)_{t\geq 0}$.*

Proof. For $t \geq 0$ and $w \in \mathbb{H}$, we know that $\text{Im}(b(t, w)) < 0$ by Equation (4.11). It follows that Loewner's differential equation has a unique solution $(w_s\colon s \in [0, t])$ in $\mathbb{H}$ with given terminal value $w_t = w$, and that w is the unique point in $\mathbb{H}$

with these properties. (This is a consequence of the invertability of g_t, which is a particular case of a more general result in ODE theory. See [11].) Because w was arbitrary, it follows that $g_t : H_t \to \mathbb{H}$ is a bijection. We already know, by Proposition 4.13, that g_t is holomorphic, so we have shown that $g_t : H_t \to \mathbb{H}$ is a conformal isomorphism. This implies that H_t is simply connected. To see why, consider the inverse isomorphism from $\mathbb{H} \to H_t$. It has to map a simply connected region to another simply connected region because it maps the boundary of one to the boundary of the other in a continuous manner.

Now, we work to obtain some basic estimates of the Loewner flow. Observe from the definition of K_t that g_0 is the identity. Fix $T \geq 0$, set $r = \sup_{t \leq T} |\xi_t - \xi_0| \vee \sqrt{T}$. Now, fix $R \geq 4r$, and take $z \in \mathbb{H}$ with $|z - \xi_0| \geq R$. Define

$$\tau = \inf\{t \in [0, \zeta(z)) \colon |g_t(z) - z| = r\} \wedge T. \tag{4.17}$$

Note that $\tau < \zeta(z)$. Moreover, for all $t \leq \tau$,

$$|g_t(z) - \xi_t| = |(g_t(z) - z) + (z - \xi_0) + (\xi_0 - \xi_t)| \geq R - 2r$$

because $|(z - \xi_0)| \geq R$ and $(g_t(z) - z), (\xi_0 - \xi_t) \leq r$. We see that

$$g_t(z) - z = \int_0^t \frac{2}{g_s(z) - \xi_s}\, ds, \qquad \text{and} \qquad z(g_t(z) - z) - 2t = 2\int_0^t \frac{z - g_s(z) + \xi_s}{g_s(z) - \xi_s}\, ds.$$

It follows that

$$|g_t(z) - z| \leq \frac{2t}{R - 2r} \leq \frac{t}{r} \tag{4.18}$$

and

$$|z(g_t(z) - z) - 2t| \leq \frac{(4r + 2|\xi_0|)t}{R - 2r}. \tag{4.19}$$

If $\tau < T$, Equation (4.18) implies that $|g_\tau(z) - z| \leq \tau/r \leq T/r \leq r$, which is a contradiction. We conclude that $\tau = T$. It now follows by Equation (4.17) that $T < \zeta(z)$, so $z \in H_T$. Setting $R = 4r$, we see that

$$|z - \xi_0| > 4r \qquad \forall z \in H_T = \mathbb{H} \setminus K_T$$

which implies

$$|z - \xi_0| \leq 4r \qquad \forall z \in K_T.$$

It follows that K_T is bounded and hence is a compact $\mathbb{H}-$hull. On the other hand, by considering the limit $R \to \infty$ in Equation (4.19), we see that $z(g_t(z) - z) \to 2t$ as $|z| \to \infty$. Moreover, $g_t(z) - z \to 0$ as $|z| \to \infty$, so $g_t = g_{K_t}$ and $\operatorname{hcap}(K_t) = 2t$ for all t.

It remains to prove the local growth property and identify the Loewner transform. Fix $s \geq 0$. Define for $t \geq 0$

$$\widetilde{\xi}_t = \xi_{t+s}, \qquad \widetilde{H}_t = g_s(H_{s+t}), \qquad \widetilde{K}_t = \mathbb{H} \setminus \widetilde{H}_t, \qquad \text{and} \qquad \widetilde{g}_t = g_{s+t} \circ g_s^{-1}.$$

We can differentiate $\widetilde{g}_t$ in t to see that $(\widetilde{g}_t)_{t\geq 0}$ is the Loewner flow driven by $(\widetilde{\xi}_t)_{t\geq 0}$, that $\widetilde{H}_t$ is the domain of $\widetilde{g}_t$, and that $\widetilde{K}_t = g_s(K_{s+t} \setminus K_s) = K_{s,s+t}$. The estimate $|z - \xi_0| \leq 4r$ for all $z \in K_t$ applies to give

$$|z - \xi_s| \leq 4\left(\sup_{s\leq u\leq s+t} |\xi_u - \xi_s| \vee \sqrt{t}\right) \text{ for all } z \in K_{s,s+t}$$

Taking $t \to 0$, we see that $(K_t)_{t\geq 0}$ has the local growth property and has Loewner transform $(\xi_t)_{t\geq 0}$.

□

Thus far, we have shown that there is *some* family of increasing compact $\mathbb{H}-$hulls that satisfies the local growth property associated with a given continuous real-valued process $(\xi_t)_{t\geq 0}$. We now show that there is actually a one to one correspondence between the collection of all continuous real processes and the collection of all families $(K_t)_{t\geq 0}$ of increasing compact $\mathbb{H}-$hulls satisfying the local growth property.

Let $\mathcal{K}$ be the collection of all compact $\mathbb{H}-$hulls. This collection can be made into a metric space as follows. Let d be the metric on $L^\infty(\mathbb{C})$ defined by $d(f,g) = ||f-g||_\infty$. We can induce a metric on $\mathcal{K}$ by

$$d_{\mathcal{K}}(K_1, K_2) = d(g_{K_1}^{-1}, g_{K_2}^{1}).$$

Let $\mathcal{L}$ be the collection of increasing families $(K_t)_{t\geq 0}$ of compact $\mathcal{H}-$hulls that have the local growth property and that are parametrized by half-plane capacity. Then, $\mathcal{L} \subseteq C([0,\infty), \mathcal{K})$. Fix a metric of uniform convergence on compact time intervals on $C([0,\infty), \mathcal{K})$. For example, we could use the metric

$$\rho(\eta, \theta) = \sup_t \operatorname{dist}(z \in \eta_t \setminus \theta_t, w \in \theta_t \setminus \eta_t),$$

where $\rho(\eta,\theta) = 0$ if $\eta = \theta$. We end this section by describing a homeomorphism from $C([0,\infty),\mathbb{R})$ to $\mathcal{L}$, establishing the aforementioned one to one correspondence between these sets.

Proposition 4.20. *There is a homeomorphism $L : C([0,\infty),\mathbb{R}) \to \mathcal{L}$ given by*

$$L((\xi_t)_{t\geq 0}) = (K_t)_{t\geq 0}, \qquad \text{where} \qquad K_t = \{z \in \mathbb{H} \colon \zeta(z) \leq t\}$$

and where $\zeta(z)$ is the lifetime of the maximal solution to Loewner's differential equation

$$\dot{z}_t = \frac{2}{z_t - \xi_t}$$

starting from z.

Proof. Simply note that as $\rho(\xi_t^n, \xi_t) \to 0$, it follows from Equation (4.11) that $b(t,z;\xi_t^n) \to b(t,z;\xi_t)$ for all t. This implies that the solutions to Loewner's differential equation converge as $\xi_t^n \to \xi_t$ uniformly, which in turn implies that the compact $\mathbb{H}-$hulls generated by the solutions converge as well.

On the other hand, assume that $(K_t)_{t\geq 0}^n \to (K_t)_{t\geq 0}$ in the ρ metric. By Equation (3.5), we see that

$$g_{K_t^n}(z) - g_{K_t}(z) = \frac{a_{K^n} - a_K}{z} + O(|z|^{-2}).$$

As $(K_t)^n_{t\geq 0} \to (K_t)_{t\geq 0}$, it follows that $g_{K^n_t} \to g_{K_t}$ uniformly in t. We see that $d_{\mathcal{K}}(K^n_t, K_t) \to 0$ for all t, which means that the Loewner transforms of $(K_t)^n_{t\geq 0}$ and $(K_t)_{t\geq 0}$ converge for all t as well.

□

5. Schramm-Loewner Evolution

Thus far in the paper, we have given a continuous analogue to SLE, we have examined the properties of compact $\mathbb{H}-$hulls, and we have developed the basic theory of Loewner evolution. We have not, however, talked about SLE. This section will immediately introduce the definition of a Schramm-Loewner evolution, and we will proceed to examine some of its basic properties.

Definition 5.1. A random variable $(K_t)_{t\geq 0}$ is a Schramm-Loewner evolution, or an SLE_κ, if its Loewner transform equals $\sqrt{\kappa}W_t$, where W_t is a standard Brownian motion and $\kappa \in [0, \infty)$.

Thanks to Proposition 3.10, we can construct an SLE_κ as $K_t = \{z \in \mathbb{H}\colon \zeta(z) \leq t\}$, where $\zeta(z)$ is the lifetime of the solution to Loewner's differential equation

$$\dot{g}_t(z) = \frac{2}{g_t(z) - \xi_t}$$

starting at z with $\xi_t = \sqrt{\kappa}W_t$.

Now that we know the definition of SLE_κ, we will characterize it in terms of a scaling property and a domain Markov property. Definitions 5.2 and 5.3 introduce these properties.

Definition 5.2. There is a natural scaling map on $\mathcal{L}$. Let $\lambda \in (0, \infty)$ and take $(K_t)_{t\geq 0} \in \mathcal{L}$. Define $K^\lambda_t = \lambda K_{\lambda^{-2}t}$. Because $\text{hcap}(\lambda K_t) = \lambda^2 \text{hcap}(K_t)$, and because $(K_t)_{t\geq 0}$ is parametrized by half-plane capacity, we see that $K^\lambda_t \in \mathcal{L}$. We say that a random variable $(K_t)_{t\geq 0} \in \mathcal{L}$ is **scale invariant** if K^λ_t has the same distribution as K_t for all $\lambda \in (0, \infty)$.

Definition 5.3. There is also a natural time shift on $\mathcal{L}$. Let $s \in [0, \infty)$ and $(K_t)_{t\geq 0} \in \mathcal{L}$. Define $K^{(s)}_t = g_{K_s}(K_{s+t} \setminus K_s) - \xi_s = K_{s+t,s} - \xi_s$. A random variable $(K_t)_{t\geq 0} \in \mathcal{L}$ has the **domain Markov property** if $K^{(s)}_t$ has the same distribution as K_t for all $s \in [0, \infty)$ and is independent of $\mathcal{F}_s = \sigma((\xi_r)_{r\leq s})$.

The relationship between this (continuous) planar Markov property and the (discrete) planar Markov property presented in Section 2 requires some explanation. The discrete model can be thought of as follows. If we run a LERW at x_0 conditioned to end at x_m, the distribution of the initial j steps of the LERW remains unchanged as we "cut" the final $\sigma - j$ steps from the domain. Essentially, "cutting" the domain by a path does not affect the distribution of the first "uncut" steps of the path. In this continuous variant, the mapping-out function serves as the "cutting" mechanism. If we remove part of the domain (and then conformally map the domain back to $\mathbb{H}$), the distribution of the paths remains unchanged.

We now characterize SLE_κ in terms of scale invariance and the domain Markov property.

Proposition 5.4. ***(Scale invariance and the domain markov property of Schramm-Loewner evolutions)*** *Let $(K_t)_{t\geq 0} \in \mathcal{L}$ be a random variable. Then*

$(K_t)_{t\geq 0}$ is an SLE if and only if it is scale invariant and has the domain Markov property.

Proof. Let $(\xi_t)_{t\geq 0}$ be the Loewner transform of $(K_t)_{t\geq 0}$. Recall from Proposition 3.1 that $(\xi_t)_{t\geq 0}$ is continuous. For $\lambda \in (0,\infty)$ and $s \in [0,\infty)$, define $\xi_t^\lambda = \lambda\xi_{\lambda^{-2}t}$ and $\xi_t^{(s)} = \xi_{s+t} - \xi_s$. By proposition 4.3, we see that ξ_t^λ is the Loewner transform of K_t^λ and that $\xi_t^{(s)}$ is the Loewner transform of $K_t^{(s)}$.

It follows that $(K_t)_{t\geq 0}$ is scale invariant if and only if $(\xi_t)_{t\geq 0}$ is scale invariant, and, similarly, that $(K_t)_{t\geq 0}$ has the domain Markov property if and only if $(\xi_t)_{t\geq 0}$ has stationary independent increments. By the Lévy-Khinchin Theorem, we see that $(\xi_t)_{t\geq 0}$ is a Brownian motion of some diffusivity $\kappa \in [0,\infty)$ with some constant drift. The scaling invariance forces that drift to be 0.

□

A continuous path $(\gamma_t)_{t\geq 0}$ is said to *generate* an increasing family of compact $\mathbb{H}-$hulls if $H_t = \mathbb{H} \setminus K_t$ is the unbounded component of $\mathbb{H} \setminus \gamma([0,t])$. The following result, which is fundamental to the theory of SLE, was proven by Schramm and Rohde in [12] for all κ except for $\kappa = 8$. The case of $\kappa = 8$ was later proven by Lawler, Schramm, and Werner in [8].

Proposition 5.5. ***(The compact $\mathbb{H}-$hulls are generated by a curve)*** *Let $(K_t)_{t\geq 0}$ be an SLE_κ for some $\kappa \in [0,\infty)$. Let $(g_t)_{t\geq 0}$ be the associated Loewner flow and let $(\xi_t)_{t\geq 0}$ be the associated Loewner transform. The function $g_t^{-1} : \mathbb{H} \to H_t$ extends continuously to $\overline{\mathbb{H}}$ for all $t \geq 0$ almost surely (with respect to the probability space for the driving function). Furthermore, if we define $\gamma_t = g_t^{-1}(\xi_t)$, it follows that $(\gamma_t)_{t\geq 0}$ is continuous and generates $(K_t)_{t\geq 0}$ almost surely.*

Definition 5.6. The curve $(\gamma_t)_{t\geq 0}$ is known as an SLE_κ **path**. It is often simply referred to as an SLE_κ, letting the context signal that we refer to the path rather than the compact $\mathbb{H}-$hulls.

Schramm-Loewner evolutions exhibit several astounding properties that are beyond the scope of this paper. For example, as hinted at in Section 1, SLE_κ is conformally invariant. Moreover, SLE_κ exhibits remarkably different properties for different values of κ. Specifically, if $0 \leq \kappa \leq 4$, then SLE_κ paths are simple curves. For $4 < \kappa < 8$, SLE_κ paths exhibit a behavior known as "swallowing", in which every point in the domain is eventually inside the K_t generated by γ_t. If $\kappa \geq 8$, then SLE_κ paths are space-filling curves. Furthermore, as mentioned in Section 1, SLE_κ paths are the scaling limits of several planar stochastic processes. Specifically, SLE_2 is the scaling limit of the loop-erased random walk [8]. For an extensive overview of SLE_κ we refer the reader to [10].

6. Appendix

In this appendix, I present a proof from [4] for the conformal invariance of Brownian motion.

Proposition 6.1. ***(Conformal Invariance of Brownian Motion)*** *Let D and D' be domains in $\mathbb{C}$, and let $\phi : D \to D'$ be a conformal isomorphism. Fix $z \in D$, and set $z' = \phi(z)$. Let $(B_t)_{t\geq 0}$ and $(B'_t)_{t\geq 0}$ be complex Brownian motions starting at z and z', respectively. Define*

$$T = \inf\{t \geq 0\colon B_t \notin D\} \quad \text{and} \quad T' = \inf\{t \geq 0\colon B'_t \notin D'\}.$$

Set $\widetilde{T} = \int_0^T |\phi'(B_t)|^2\, dt$, *and define, for* $t < \widetilde{T}$

$$\tau(t) = \int \left\{ s \geq 0\colon \int_0^s |\phi'(B_r)|^2\, dr = t \right\}, \qquad \text{and} \qquad \widetilde{B}_t = \phi(B_{\tau(t)}).$$

Then, $(\widetilde{T}, (\widetilde{B}_t)_{t \leq \widetilde{T}})$ *and* $(T', (B'_t)_{t \leq T'})$ *have the same distribution.*

Proof. Assume for now that D is bounded and that ϕ has a C^1 extension to $\overline{D}$. Then, $T < \infty$ almost surely. We work with the natural filtration of Brownian motion unless otherwise specified. We may define the adapted process Z_t (with respect to which the Itô integral can be defined) and the adapted process A_t as follows:

$$Z_t = \phi(B_{T \wedge t}) + (B_t - B_{T \wedge t})$$

$$A_t = \int_0^{T \wedge t} |\phi'(B_s)|^2\, ds + (t - T \wedge t)$$

Note that, almost surely, A is an increasing homeomorphism of $[0, \infty)$ with an inverse that is an extension of τ. Denote this inverse homeomorphism also by τ.

Write $\phi = u + iv$, $B_t = X_t + iY_t$, and $Z_t = M_t + iN_t$. Then, for $t < T$, $M_t = u(B_t) = u(X_t, Y_t)$, and $N_t = v(B_t) = v(X_t, Y_t)$. By Ito's formula (the second derivative terms disappear because all complex differentiable functions are harmonic), we have

$$dM_t = \frac{du}{dx}(B_t)\, dX_t + \frac{du}{dy}(B_t)\, dY_t$$

and

$$dN_t = \frac{dv}{dx}(B_t)\, dX_t + \frac{dv}{dy}(B_t)\, dY_t.$$

Thanks to the Cauchy-Riemann equations:

$$dM_t dM_t = |\phi'(B_t)|^2\, dt = dA_t = dN_t dN_t, \qquad dM_t dN_t = 0$$

Now, for $t \geq T$:

$$dM_t = dX_t, \quad dN_t = dY_t, \quad dM_t dM_t = dt = dA_t = dN_t d_N t, \quad M_t dN_t = 0$$

We see that $(M_t)_{t \geq 0}$, $(N_t)_{t \geq 0}$, $(M_t^2 - A_t)_{t \geq 0}$, and $(N_t^2 - A_t)_{t \geq 0}$ are all continuous local martingales. Let $\widetilde{M}_s = M_{\tau(s)}$ and $\widetilde{N}_s = N_{\tau(s)}$. By Doob's optional stopping theorem (treating $\tau(s)$ as a stopping time) and by Itô calculus, $\widetilde{M}_s, \widetilde{N}_s, (\widetilde{M}_s^2 - s)$, and $(\widetilde{N}_s^2 - s)$ are all local martingales.

Define $\widetilde{Z}_t = \widetilde{M}_t + i\widetilde{N}_t$. By the above paragraph and by Levy's characterization of Brownian Motion (see [13]), $\widetilde{Z}_t$ is a Brownian motion starting from $z' = \phi(z)$. Note that $\widetilde{B}_t = \widetilde{Z}_t$ for $t \leq \widetilde{T}$. Because ϕ is a bijection, we see that $\widetilde{T} = \inf\{t \geq 0\colon B_t \notin D_n\}$. It follows that $(\widetilde{T}, (\widetilde{B}_t)_{t \leq \widetilde{T}})$ and $(T', (B'_t)_{t \leq T'})$ have the same distribution as claimed.

When D is not bounded or when ϕ does not have a C^1 extension to $\overline{D}$, choose a sequence of bounded open sets $D_n \nearrow D$ with $\overline{D}_n \subseteq D$ for all n. Set $D'_n = \phi(D_n)$ and set

$$T_n = \inf\{t \geq 0 \colon B_t \notin D_n\}, \quad \text{and} \quad T'_n = \inf\{t \geq 0 \colon B'_t \notin D'_n\}.$$

Let $\widetilde{T}_n = \int_0^{T_n} |\phi'(B_t)|^2 \, dt$. We see that $\widetilde{T}_n \nearrow \widetilde{T}$ and $T'_n \nearrow T'$ almost surely as $n \to \infty$, meaning that $\widetilde{B}_t$ and B'_t converge to some $\widetilde{B}_{\widetilde{T}}$ and some $B'_{T'}$, respectively. Since ϕ is C^1 on $\overline{D}_n$, we know that $(\widetilde{T}_n, (\widetilde{B}_t)_{t \leq \widetilde{T}_n})$ and $(T', (B'_t)_{t \leq T'_N})$ have the same distribution for all n. Taking $n \to \infty$, we conclude that $(\widetilde{T}, (\widetilde{B}_t)_{t \leq \widetilde{T}})$ and $(T', (B'_t)_{t \leq T'})$ have the same distributions.

□

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