STABLE MINIMAL SURFACES

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Abstract. We examine properties of stable minimal surfaces. Of importance, we prove a theorem involving compactness of stable minimal hypersurfaces. An exposition on a generalization of this theorem, Sharp's compactness theorem, is given.

Contents

1. Introduction

A minimal surface is a critical point of the area functional. A basic question is to ask what the space of minimal surfaces in a manifold M looks like. A priori, there is no clear answer, and indeed it is in general a very hard question to tackle.

We do have some understanding of what certain subsets of this space look like though. In particular, we consider the space of so-called stable minimal surfaces. These are minimal surfaces which, loosely speaking, are area-minimizing. More precisely, a minimal surface is stable if there are no directions which can decrease the area; thus, it is a critical point with Morse index zero.

Stable minimal surfaces have many important properties. Notably, due to results by Schoen-Simon and Schoen-Simon-Yau, the integral of $|A|^2$ can be bounded above. This can be used to prove certain curvature estimates. These curvature estimates, it turns out, lead to a compactness theorem for stable minimal surfaces. This theorem is proved in Section 7.

The aforementioned compactness theorem is an important technical ingredient in Almgren-Pitts min-max theory. Recall that Birkhoff [CL03] used a min-max theory argument to prove that every Riemannian 2-sphere has a nontrivial closed geodesic.

Almgren-Pitts min-max theory extends this to minimal hypersurfaces of higher dimension. The theory has been used to prove several important conjectures. For example, in 2014 it was used by F. C. Marques and André Neves [MN14] to prove the Willmore conjecture.

One can ask whether the previous results necessarily require the Morse index to be zero, or if this hypothesis can be relaxed to simply have bounded index. Sharp's compactness theorem, introduced in Section 8, gives some insight into the space of minimal surfaces with higher index.

We assume basic knowledge of minimal surfaces in this paper. Sections 1.1 and 1.4 of [CM11] should be sufficient. Working knowledge of Riemannian Geometry is also assumed; the author recommends [dC92].

2. The Monotonicity Formula

Before stating and proving the monotonicity formula, we first recall the coarea formula, which will be used throughout the proof.

Theorem 2.1. *(The Coarea Formula) If Σ is a manifold and $h : \Sigma \to \mathbb{R}$ is a proper Lipschitz function on Σ, then for all locally integrable functions f on Σ and $r \in \mathbb{R}$,*

$$\int_{\{h\leq r\}} f|\nabla_\Sigma h| = \int_{-\infty}^{r}\int_{\{h=\tau\}} f\,d\tau.$$

A proof of this can be found in [Sim14].

Remark 2.2. Here is some intuition for the coarea formula: consider the equation

$$\int_{\{0\leq h\leq 1\}} |\nabla h| = \int_0^1\int_{\{h=\tau\}} 1 d\tau,$$

which is the coarea formula applied between 0 and 1 with $f = 1$. Expanding out the right hand side into a Riemann sum gives

$$\int_{\{0\leq h\leq 1\}} |\nabla h| = \lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{n}\int_{\{h=\frac{1}{j}\}} 1 \approx \sum_{j=1}^{N}\frac{1}{N}\int_{\{h=\frac{1}{j}\}} 1$$

for large enough N. Now, what is $1/N\int_{\{h=1/j\}} 1$? It is clear that $\int_{\{h=1/j\}} 1$ measures the area of $\{h = 1/j\}$; multiplication by $1/N$ has the effect of measuring the area of the $1/N$ tube around the level set $\{h = 1/j\}$. So, $1/N\int_{\{h=1/j\}} 1$ measures the volume of the $1/N$ tube around the level set $\{h = 1/j\}$.

If the level sets $\{h = 1/j\}$ were spaced exactly $1/N$ apart, then the sum of the volume of the $1/N$ tubes would be precisely the volume of $\{0 \leq h \leq 1\}$. However, this need not be the case in general. If two level sets are close together, then these $1/N$ tubes may overlap and hence overcount. On the other hand, if the level sets are far apart, then the $1/N$ tubes will not even be close to overlapping. Thus, the quantity being measured is precisely something that is large in magnitude for closely-spaced level sets and small in magnitude for far level sets. The gradient has exactly this property. The following figure shows an example of this with $h(x, y) = xe^{-x^2-y^2}$.

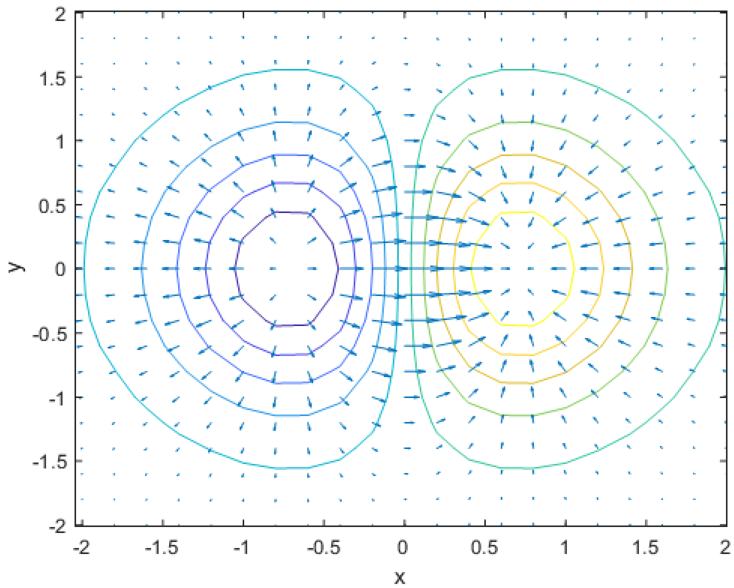

Figure 2.3. *Level sets and gradient vector field for* $h(x,y) = xe^{-x^2-y^2}$.

Observe that when the level sets are close together, the gradient is indeed large, and when the level sets are far apart, the gradient is small.

Theorem 2.4. *(The Monotonicity Formula) Let* $\Sigma^k \subset \mathbb{R}^n$ *be a minimal submanifold and* $x_0 \in \mathbb{R}^n$. *Then for all* $0 < s < t$ *we have*

$$\frac{\operatorname{Vol}(B_t(x_0) \cap \Sigma)}{t^k} - \frac{\operatorname{Vol}(B_s(x_0) \cap \Sigma)}{s^k} = \int_{(B_t(x_0) \setminus B_s(x_0)) \cap \Sigma} \frac{|(x-x_0)^N|^2}{|x-x_0|^{k+2}}.$$

Proof. We first prove an important fact involving coordinate functions on a k-dimensional minimal surface. Let x_i be the ith coordinate function on $\mathbb{R}^n$ restricted to Σ. Then,

$$\nabla_\Sigma |x|^2 = \nabla_\Sigma \Big(\sum_{i=1}^n x_i^2 \Big) = 2 \sum_{i=1}^n x_i \nabla_\Sigma x_i.$$

To compute $\nabla_\Sigma x_i$, note that $\nabla_\Sigma f = (\nabla f)^T$, where $(\nabla f)^T$ is the tangential portion of ∇f to Σ. So,

$$\nabla_\Sigma x_i = (\nabla x_i)^T = (e_i)^T.$$

Combining this with the above, we get

$$\nabla_\Sigma |x|^2 = 2 \sum_{i=1}^n x_i (e_i)^T = 2x^T.$$

Recall that $\operatorname{div}(X)$ is the trace of the linear map $Y \to \nabla_Y X$, where X, Y are vector fields on Σ [dC92]. It follows that

$$\operatorname{div}_\Sigma(x) = \sum_{i=1}^k \langle \nabla_{e_i} x, e_i \rangle = \sum_{i=1}^k \langle e_i, e_i \rangle = k,$$

since Σ is k dimensional and for all $v \in \mathbb{R}^n$ we have

$$\langle \nabla_v x, e_i \rangle = \sum_{j=1}^{n} \langle v[x_j] e_j, e_i \rangle = v[x_i] = (dx_i)(v) = \langle v, e_i \rangle.$$

Applying all of the above gives

$$\Delta_\Sigma |x|^2 = \operatorname{div}_\Sigma(\nabla_\Sigma |x|^2) = 2 \operatorname{div}_\Sigma x^T = 2 \operatorname{div}_\Sigma x = 2k,$$

where the third equality comes from the fact that $\operatorname{div}_\Sigma Y^N = 0$ for any vector field Y on a minimal surface Σ.

Let $f = |x - x_0|$ on Σ. Then by the above, $\Delta_\Sigma f^2 = \Delta_\Sigma x^2 = 2k$. By Stokes' theorem,

$$2k \operatorname{Vol}(\{f \leq r\}) = \int_{\{f \leq r\}} \Delta_\Sigma f^2 = \int_{\{f=r\}} |\nabla_\Sigma f^2| = 2 \int_{\{f=r\}} |(x - x_0)^T|. \tag{2.5}$$

Using the coarea formula, it follows that

$$\operatorname{Vol}(\{f \leq r\}) = \int_{\{f \leq r\}} |\nabla_\Sigma f|^{-1} |\nabla_\Sigma f| = \int_0^r \int_{f=\tau} |\nabla_\Sigma f|^{-1} d\tau.$$

Thus,

$$\begin{aligned}
\frac{d}{dr}(r^{-k} \operatorname{Vol}(\{f \leq r\})) &= -kr^{-k-1} \operatorname{Vol}(\{f \leq r\}) + r^{-k} \frac{d}{dr} \operatorname{Vol}(\{f \leq r\}) \\
&= -kr^{-k-1} \operatorname{Vol}(\{f \leq r\}) + r^{-k} \frac{d}{dr} \int_0^r \int_{\{f=\tau\}} |\nabla_\Sigma f|^{-1} d\tau \\
&= -kr^{-k-1} \operatorname{Vol}(\{f \leq r\}) + r^{-k} \int_{\{f=r\}} |\nabla_\Sigma f|^{-1}.
\end{aligned}$$

By the chain rule, we get

$$\nabla_\Sigma f^2 = 2f \nabla_\Sigma f$$

and so

$$\nabla_\Sigma f = \frac{\nabla_\Sigma f^2}{2f} = \frac{(x - x_0)^T}{|x - x_0|}.$$

Thus,

$$\begin{aligned}
\frac{d}{dr}(r^{-k} \operatorname{Vol}(\{f \leq r\})) &= -r^{-k-1} \int_{\{f=r\}} |(x - x_0)^T| + r^{-k} \int_{\{f=r\}} \frac{|x - x_0|}{|(x - x_0)^T|} \\
&= -r^{-k-1} \int_{\{f=r\}} |(x - x_0)^T| + r^{-k-1} \int_{\{f=r\}} \frac{|x - x_0|^2}{|(x - x_0)^T|} \\
&= r^{-k-1} \int_{\{f=r\}} \frac{|x - x_0|^2 - |(x - x_0)^T|^2}{|(x - x_0)^T|} \\
&= \int_{\{f=r\}} \frac{|(x - x_0)^N|^2}{|(x - x_0)^T| |x - x_0|^{k+1}},
\end{aligned}$$

where in the first line we applied (2.5) and in the second and last lines we have used the fact that $|x - x_0| = r$ on $\{f = r\}$.

Integrating and applying the coarea formula again gives

$$\begin{aligned}
r^{-k}\operatorname{Vol}(\{f\le r\})\Big|_s^t &= \int_s^t\int_{\{f=r\}}\frac{|(x-x_0)^N|^2}{|(x-x_0)^T||x-x_0|^{k+1}}\\
&= \int_{\{s\le f\le t\}}\frac{|(x-x_0)^N|^2}{|(x-x_0)^T||x-x_0|^{k+1}}|\nabla_\Sigma f|\\
&= \int_{\{s\le f\le t\}}\frac{|(x-x_0)^N|^2}{|(x-x_0)^T||x-x_0|^{k+1}}\frac{|(x-x_0)^T|}{|x-x_0|}\\
&= \int_{\{s\le f\le t\}}\frac{|(x-x_0)^N|^2}{|x-x_0|^{k+2}},
\end{aligned}$$

which completes the proof, since $\{f\le r\}=\Sigma\cap B_r(x_0)$. □

In particular, since $|(x-x_0)^N|^2/|x-x_0|^{k+2}$ is nonnegative, we get the following easy corollary:

Corollary 2.6. *Under the conditions of Theorem 2.4, if* $0<r\le R$ *then*

$$\frac{\operatorname{Vol}(B_r(x_0)\cap\Sigma)}{r^k}\le\frac{\operatorname{Vol}(B_R(x_0)\cap\Sigma)}{R^k}.$$

Remark 2.7. There is a simpler proof of the above corollary for area minimizing minimal surfaces.

Consider an area minimizing minimal surface $\Sigma^k\subset\mathbb{R}^n$ with $\partial\Sigma\cap B_r(x_0)=\emptyset$. Now consider the cone, C, with vertex x_0 and base $\Sigma\cap\partial B_r(x_0)$. Since Σ is area minimizing,

$$\operatorname{Vol}(\Sigma\cap B_r(x_0))\le\operatorname{Vol}(C).$$

This may be rewritten as follows:

$$\begin{aligned}
\operatorname{Vol}(\Sigma\cap B_r(x_0)) &\le \operatorname{Vol}(C)\\
&= \int_0^r \operatorname{Area}(C\cap\partial B_s(x_0))ds\\
&= \int_0^r\left(\frac{s}{r}\right)^{k-1}\operatorname{Area}(C\cap\partial B_r(x_0))ds\\
&= \frac{s^k}{kr^{k-1}}\Big|_0^r\operatorname{Area}(C\cap\partial B_r(x_0))\\
&= \frac{r}{k}\operatorname{Area}(C\cap\partial B_r(x_0))\\
&= \frac{r}{k}\operatorname{Area}(\Sigma\cap\partial B_r(x_0))=\frac{r}{k}\frac{\partial}{\partial r}\operatorname{Vol}(\Sigma\cap B_r(x_0))
\end{aligned}$$

where the third inequality uses the area scaling of the cone and the last line uses the fact that $C\cap\partial B_r(x_0)=\Sigma\cap\partial B_r(x_0)$.

It follows that

$$\begin{aligned}
\frac{d}{dr}\frac{\operatorname{Vol}(\Sigma\cap B_r(x_0))}{r^k} &= \frac{\partial/\partial r\operatorname{Vol}(\Sigma\cap B_r(x_0))}{r^k}-\frac{k}{r^{k+1}}\operatorname{Vol}(\Sigma\cap B_r(x_0))\\
&= \frac{1}{r^k}\Big(\frac{\partial}{\partial r}\operatorname{Vol}(\Sigma\cap B_r(x_0))-\frac{k}{r}\operatorname{Vol}(\Sigma\cap B_r(x_0))\Big)\ge 0
\end{aligned}$$

which proves the monotonicity inequality for area minimizing minimal surfaces.

Another important result, the mean value inequality, can be obtained using a generalized monotonicity formula (see Proposition 1.15 in [CM11]).

Corollary 2.8. *(The Mean Value Inequality) Let $\Sigma^k \subset \mathbb{R}^n$ be a minimal submanifold, $x_0 \in \Sigma$, and $s > 0$ with $B_s(x_0) \cap \partial\Sigma = \emptyset$. If f is a nonnegative function on Σ with $\Delta_\Sigma f \geq -\lambda s^{-2} f$, then*

$$f(x_0) \leq c(\lambda, s, k) \int_{B_s(x_0)\cap\Sigma} f.$$

The proof is omitted, but can be found in [CM11]. In fact, the precise theorem in [CM11] is stronger, and the value of c can be computed.

3. Simons' Inequality

We would ultimately like to show that the second fundamental form of a stable minimal surface is well controlled. The heuristic is as follows: since the second fundamental form dictates curvature, then having a controlled second fundamental form means the surface is not curving too badly. Thus, it makes sense to express the surface as the graph of a function with small derivatives. The following inequality becomes especially useful for this task.

Theorem 3.1. *(Simons' inequality). Let $\Sigma \subset \mathbb{R}^n$ be a minimal hypersurface. Then*

$$\Delta_\Sigma |A|^2 \geq -2|A|^4 + 2\Big(1 + \frac{2}{n-1}\Big)|\nabla_\Sigma |A||^2.$$

A generalized version of this can be found in [SSY75], but the above can be found in [CM11]. Expanding out the laplacian term, rearranging some terms, and dividing through by 2 gives the following alternate form:

$$|A|\Delta_\Sigma |A| + |A|^4 \geq \frac{2}{n-1}|\nabla_\Sigma |A||^2.$$

The proof is omitted, but is thoroughly presented in [CM11], page 66. The proof is a clever calculation using the Gauss equation, Codazzi equation, and the symmetry of $|A|^2$.

Remark 3.2. Since $|\Delta_\Sigma |A||^2 \geq 0$, we see that

$$\Delta_\Sigma |A|^2 \geq -2|A|^4.$$

Observe further that if $|A|^2 \leq 1$, then

$$\Delta_\Sigma |A|^2 \geq -2|A|^4 \geq -2|A|^2$$

and hence the mean value inequality can be applied to estimate $|A|^2$.

4. Stability of Minimal Surfaces

We know that minimal surfaces can be viewed as critical points of the area functional. For functions on $\mathbb{R}^n$, critical points may be minima, maxima, or saddle points (depending on the index), and these are distinguished by the second derivative. Minimal surfaces can be categorized similarly by looking at the second derivative (or, second variation) of the area functional.

We begin by presenting the second variation formula.

Definition 4.1. Let Σ be an orientable hypersurface with normal vector N. Then for smooth functions η we define the stability operator L by

$$L\eta = \Delta_\Sigma \eta + |A|^2\eta + \mathrm{Ric}_M(N,N)\eta$$

We may also regard the stability operator as acting on normal vector fields. Indeed, we can identify a normal vector field $X = \eta N$ by a smooth function η, and then define $LX = L\eta$.

Proposition 4.2. *(The Second Variation Formula) Let $\Sigma^k \subset M^n$ be an orientable, minimal submanifold. Let F be a normal variation on Σ with compact support. Then,*

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathrm{Vol}(F(\Sigma,t)) = -\int_\Sigma \langle F_t, LF_t\rangle.$$

Definition 4.3. We say that Σ is *stable* if for all variations F which fix $\partial\Sigma$ we have

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathrm{Vol}(F(\Sigma,t)) = -\int_\Sigma \langle F_t, LF_t\rangle \geq 0.$$

That is, Σ is area minimizing (technically we need it to be strictly positive, but this is the idea one should have in mind). Equivalently, a minimal surface is stable if L has Morse index zero.

The following stability inequality follows easily from the definition of stability.

Theorem 4.4. *Let $\Sigma \subset M^n$ be a stable, orientable, minimal hypersurface. Then for all Lipschitz functions η with compact support,*

$$\int_\Sigma (\inf_M \mathrm{Ric}_M + |A|^2)\eta^2 \leq \int_\Sigma |\nabla_\Sigma \eta|^2.$$

Proof. By integration by parts (which follows from the Leibniz formula for divergence and Stokes' theorem), we have

$$\int_\Sigma -\eta\Delta_\Sigma\eta = \int_\Sigma |\nabla_\Sigma\eta|^2.$$

Since Σ is stable,

$$0 \leq -\int_\Sigma \langle \eta, L\eta\rangle = -\int_\Sigma \eta\Delta_\Sigma\eta + |A|^2\eta^2 + \mathrm{Ric}_M(N,N)\eta^2.$$

Applying integration by parts and rearranging the above proves the theorem. □

For Ricci flat manifolds (in particular, $\mathbb{R}^n$), we obtain the following easy corollary.

Corollary 4.5. *Let $\Sigma \subset \mathbb{R}^n$ be a stable, orientable, minimal hypersurface. Then for all Lipschitz functions η with compact support,*

$$\int_\Sigma |A|^2\eta^2 \leq \int_\Sigma |\nabla_\Sigma\eta|^2.$$

One can ask if there is an L^p analog of this stability inequality for $p \neq 2$. It turns out that this is the case for certain p due to the following result by Schoen-Simon-Yau.

Theorem 4.6. *Let $\Sigma \subset \mathbb{R}^n$ be a stable, orientable minimal hypersurface. Then for all compactly supported nonnegative Lipschitz functions ϕ and $p \in [2, 2+\sqrt{2/(n-1)})$ we have*

$$\int_\Sigma |A|^{2p}\phi^{2p} \leq C(n,p)\int_\Sigma |\nabla \phi^{2p}|.$$

The full proof is omitted, but can be found in [SSY75] or [CM11]. The key is to combine the stability inequality with Simons' inequality. Using the stability inequality with $\eta = |A|^{1+q}f$ for $0 \leq q \leq \sqrt{2/(n-1)}$ gives

$$\begin{aligned}\int |A|^{4+2q}f^2 \quad &\leq \quad (1+q)^2\int f^2|\nabla|A||^2|A|^{2q} + \int |A|^{2+2q}|\nabla f|^2 \\ &\quad +2(1+q)\int f|A|^{1+2q}\langle \nabla f, \nabla |A|\rangle.\end{aligned}$$

One then bounds the first term on the right hand side using Simons' inequality. This yields

$$\begin{aligned}\frac{2}{n-1}\int f^2|\nabla|A||^2|A|^{2q} \quad &\leq \quad \int |A|^{4+2q}f^2 - 2\int |A|^{2+2q}|\nabla f|^2 \\ &\quad -(1+2q)\int f|A|^{1+2q}\langle \nabla f, \nabla|A|\rangle.\end{aligned}$$

Combining these two inequalities and using the Cauchy-Schwarz inequality to absorb some of the terms gives

$$\int f^2|A|^{2q}|\nabla|A||^2 \leq a\int |\nabla f|^2|A|^{2+2q}$$

for an appropriate choice of $a > 0$. Applying Cauchy-Schwarz to the cross term in the stability inequality and using the above bound gives

$$\int |A|^{4+2q}f^2 \leq c\int |A|^{2+2q}|\nabla f|^2.$$

We then set $p = 2 + q$ and $f = \phi^p$ and use Hölder's inequality to complete the proof.

5. The Log Cutoff Trick

As shown in Corollary 4.5 and Theorem 4.6, we are able to estimate the integral of A against some compactly supported nonnegative Lipschitz function ϕ by computing the integral of $\nabla\phi$. It would be nice if we had a certain such function whose gradient integrated to something small. The smaller the integral, the better the bound. It turns out that using a logarithmic cutoff function provides a better bound than using a linearly decaying cutoff function.

Definition 5.1. Let (M^n, g) be a Riemannian manifold. Define the *log cutoff function*, $\eta : M \to \mathbb{R}$, by

$$\eta = \begin{cases} 1 & 0 \leq r \leq R_1 \\ \log(r/R_2)/\log(R_1/R_2) & R_1 < r < R_2 \\ 0 & R_2 \leq r \end{cases}$$

where $0 < R_1 < R_2$ and $r = d_M(x_0, x)$ for some fixed $x_0 \in M$.

Remark 5.2. We may compute $\nabla\eta$. Clearly it is 0 outside $\mathrm{cl}(B_{R_2}(x_0)\setminus B_{R_1}(x_0))$ and via the chain rule is given by

$$\nabla\eta = \frac{\nabla r}{r\log(R_1/R_2)}$$

in $B_{R_2}(x_0)\setminus B_{R_1}(x_0)$. Since $|\nabla r| = 1$ (follows from first variation of length) we have

$$\sup_{B_{R_2}(x_0)\setminus B_{R_1}(x_0)} |\nabla\eta| = \frac{1}{R_1|\log(R_1/R_2)|}.$$

We can now prove the following theorem.

Theorem 5.3. *Let $\Sigma\subset\mathbb{R}^n$ be a stable, minimal, hypersurface with $3\le n\le 5$. Further suppose that $\mathrm{Vol}(\Sigma\cap B_r)\le ar^n$ for all $r>0$. Let $k>1$ and $R>0$. Then*

$$\int_{B^\Sigma_{kR}} |A|^n \le \frac{C}{\log(k)^{n-1}}.$$

(note: each ball is assumed to be centered at x_0)

Proof. Let $R_2 = kR$ and $R_1=\sqrt{k}R$. Applying the log cutoff function we have

$$\int_{B^\Sigma_{R_1}} |A|^{2p} \le \int_{B^\Sigma_{R_1}} |A|^{2p}\eta^{2p} \le \int_\Sigma |A|^{2p}\eta^{2p}.$$

By Theorem 4.6 it follows that

$$\int_\Sigma |A|^{2p}\eta^{2p} \le C(n,p)\int_\Sigma |\nabla\eta|^{2p} \le \frac{2^{2p}C(n,p)}{\log(k)^{2p}}\int_{B^\Sigma_{R_2}\setminus B^\Sigma_{R_1}} \frac{1}{r^{2p}}$$

where $C(n,p)$ is given as in Theorem 4.6. Now split up the domain of integration across several annuli. This gives

$$\int_\Sigma |A|^{2p}\eta^{2p} \le \frac{2^{2p}C(n,p)}{\log(k)^{2p}}\sum_{l=\log(k)/2+1}^{\log(k)}\int_{B^\Sigma_{e^lR}\setminus B^\Sigma_{e^{l-1}R}} \frac{1}{r^{2p}}.$$

On each annulus, $1/r^{2p}$ attains a maximum at the lower radius. Thus we have

$$\int_\Sigma |A|^{2p}\eta^{2p} \le \frac{2^{2p}C(n,p)}{\log(k)^{2p}}\sum_{l=\log(k)/2+1}^{\log(k)}\frac{1}{e^{2pl-2p}R^{2p}}\int_{B^\Sigma_{e^lR}\setminus B^\Sigma_{e^{l-1}R}} 1.$$

Expanding the domain of integration once more and using the area bound gives

$$\begin{aligned}\int_\Sigma |A|^{2p}\eta^{2p} &\le \frac{2^{2p}C(n,p)}{\log(k)^{2p}}\sum_{l=\log(k)/2+1}^{\log(k)}\frac{1}{e^{2pl-2p}R^{2p}}\int_{B^\Sigma_{e^lR}} 1\\ &\le \frac{2^{2p}C(n,p)}{\log(k)^{2p}}\sum_{l=\log(k)/2+1}^{\log(k)}\frac{ae^{nl}R^n}{e^{2pl-2p}R^{2p}}.\end{aligned}$$

Now set $2p=n$ to get

$$\int_\Sigma |A|^{2p}\eta^{2p} \le \frac{C(n,a)}{\log(k)^n}\sum_{l=\log(k)/2+1}^{\log(k)} 1 \le \frac{C(n,a)}{\log(k)^n}\sum_{l=1}^{\log(k)} 1 = \frac{C(n,a)}{\log(k)^{n-1}}$$

(where for convenience $\log(k)/2$ is assumed to be an integer and the constant C varies from line to line). □

Remark 5.4. The center of the ball in Theorem 5.3 was some arbitrary point x_0, so it makes sense to define r in the proof as $d_M(x, x_0)$ for this particular x_0. Also, technically the proof does not work for $n = 3$, since $2p \geq 4$. However, using Corollary 4.5 instead of Theorem 4.6 immediately fixes this.

6. Curvature Estimates for Minimal Surfaces

Theorem 6.1. *Let $\Sigma^{n-1} \subset \mathbb{R}^n$ be a stable, minimal hypersurface with $\partial\Sigma \subset \partial B_R(0)$, $\mathrm{Vol}(\Sigma \cap B_R(0)) \leq aR^{n-1}$ for some $a > 0$, and $3 \leq n \leq 6$. Then there exists some $C(n,a)$ (independent of Σ!) such that*

$$\sup_{B_R(0)\cap\Sigma} |A|^2 \leq \frac{C(n,a)}{d(x, \partial B_R(0))^2}.$$

The proof is an example of a blow-up argument, which leads to a contradiction.

Proof. First, to fix some notation, whenever a center is not given for a ball, assume it is centered at the origin. Furthermore, let $r(x)$ be the distance from x to the origin.

Now, suppose Theorem 6.1 is not true. Then there exists a sequence of minimal surfaces Σ_k such that

$$\sup_{B_R\cap\Sigma_k} |A_k|^2 (R - r(x))^2 \geq k^2.$$

Fix a $k \in \mathbb{N}$. Let $p_k \in B_R \cap \Sigma_k$ be such that the above sup is attained. Now, define σ_k by

$$\sigma_k = \frac{k}{2|A_k(p_k)|}.$$

Observe that $B_{2\sigma}(p_k) \subset B_R$. To see this, note that by definition of σ_k and p_k,

$$(R - r(p_k))^2 \geq \frac{k^2}{|A_k(p_k)|^2} = 4\sigma_k^2,$$

and because $R - r(p_k) > 0$ and $\sigma_k > 0$, we conclude that

$$R - r(p_k) \geq 2\sigma_k. \tag{6.2}$$

In particular, this implies that $R \geq 2\sigma_k$.

Now, by triangle inequality, for $y \in B_\sigma(p_k)$ we have

$$r(y) = d(y, 0) \leq d(y, p_k) + d(p_k, 0) \leq \sigma_k + r(p_k).$$

Thus,

$$\frac{R - r(y)}{R - r(p_k)} \geq \frac{R - r(p_k) - \sigma_k}{R - r(p_k)} \geq 1 - \frac{\sigma_k}{R - r(p_k)} \geq \frac{1}{2} \tag{6.3}$$

where the last inequality follows from (6.2).

This estimate implies that $|A_k|^2$ is well controlled on $B_{\sigma_k}(p_k) \cap \Sigma_k$. Indeed, for any $y \in B_{\sigma_k}(p_k) \cap \Sigma_k$ we have

$$|A_k(y)|^2 = \frac{|A_k(y)|^2 (R - r(y))^2}{(R - r(y))^2} \leq \frac{|A_k(p_k)|^2 (R - r(p_k))^2}{(R - r(y))^2} \leq 4|A_k(p_k)|^2$$

where the first inequality follows by definition of p_k and the second follows by (6.3).

To begin the dilation argument, define $\mu_k : \Sigma_k \to \mathbb{R}^n$ by $\mu_k(x) = 2|A_k(p_k)|x$. Let $\tilde{\Sigma}_k = \mu_k(B_{\sigma_k}(p_k) \cap \Sigma_k - p_k)$ (here the $-p_k$ is interpreted as a translation). Then $B_{\sigma_k}(p_k)$ has been mapped to $B_k(0)$. Note that due to this dilation, we have for $y \in \tilde{\Sigma}_k$,

$$|\tilde{A}_k(y)| \leq 1,$$

and in particular, since our scaling divides the second fundamental form by $2|A_k(p_k)|$, we have

$$|\tilde{A}_k(0)| = \frac{1}{2}.$$

As noted in Remark 3.2, we expect that this puts us in a position where we can apply the mean value inequality.

Recall that, for a positive function f on a manifold M^n and $p > 1$, we have that $\operatorname{grad} f^p = pf \operatorname{grad} f^{p-1}$. Thus,

$$\begin{aligned} \Delta f^p &= \operatorname{div}\operatorname{grad} f^p \\ &= \operatorname{div}(pf \operatorname{grad} f^{p-1}) = p \operatorname{div}(f \operatorname{grad} f^{p-1}) \\ &= p[\langle \operatorname{grad} f, \operatorname{grad} f^{p-1}\rangle + f^{p-1} \operatorname{div}\operatorname{grad} f] \\ &= p[\langle \operatorname{grad} f, (p-1)f \operatorname{grad} f\rangle + f^{p-1} \operatorname{div}\operatorname{grad} f] \\ &= p(p-1)f|\operatorname{grad} f|^2 + pf^{p-1}\Delta f. \end{aligned}$$

Recall Simons' inequality in $\mathbb{R}^n$:

$$\Delta|A|^2 \geq -2|A|^4 + 2\Big(1 + \frac{2}{n-1}\Big)|\nabla|A|^2|^2 \geq -2|A|^4.$$

Taking $f = |\tilde{A}_k|^2$ on $\tilde{\Sigma}_k$ and using the above two formulas gives

$$\Delta|\tilde{A}_k|^{2p} \geq p|\tilde{A}_k|^{2p-2}\Delta|\tilde{A}_k|^2 \geq -2p|\tilde{A}_k|^{2p+2}.$$

For $y \in \tilde{\Sigma}_k$ we have $|\tilde{A}_k(y)|^2 \leq 1$, so $|\tilde{A}_k(y)|^{2p+2} \leq |\tilde{A}_k(y)|^{2p}$. Therefore, on $\tilde{\Sigma}_k$, we obtain

$$\Delta|\tilde{A}_k|^{2p} \geq -2p|\tilde{A}_k|^{2p}$$

and so the mean value inequality applies.

Now define a cutoff function ϕ_k where $\phi_k|_{B_k} \equiv 1$, $\phi_k|_{B_{2k}} \equiv 0$, and ϕ_k decreases radially from k to $2k$ (that is, $\nabla\phi_k = 1/r$). Applying Theorem 4.6 with this cutoff function and $2p = 4 + \sqrt{7/5}$ gives

$$\begin{aligned} \int_{B_1 \cap \tilde{\Sigma}_k} |\tilde{A}_k|^{2p} &\leq \int_{B_k \cap \tilde{\Sigma}_k} |\tilde{A}_k|^{2p} = \int_{B_k \cap \tilde{\Sigma}_k} |\tilde{A}_k|^{2p}\phi_k^{2p} \leq \int_{B_{2k} \cap \tilde{\Sigma}_k} |\tilde{A}_k|^{2p}\phi_k^{2p} \\ &\leq C(n)\int_{B_{2k} \cap \tilde{\Sigma}_k} |\nabla\phi_k|^{2p} \leq C(n)k^{-4-\sqrt{7/5}}\operatorname{Vol}(B_{2k} \cap \tilde{\Sigma}_k) \end{aligned}$$

where the constant $C(n)$ varies from line to line (e.g., in the last inequality, it absorbs the 2^{-2p}). It remains to estimate $\operatorname{Vol}(B_{2k} \cap \tilde{\Sigma}_k)$ using the volume bound. By scaling,

$$\begin{aligned} \operatorname{Vol}(B_{2k} \cap \tilde{\Sigma}_k) &= (2|A_k(p_k)|)^{n-1}\operatorname{Vol}(B_{2\sigma_k}(p_k) \cap \Sigma_k) = \frac{2^{n-1}\operatorname{Vol}(B_{2\sigma_k}(p_k) \cap \Sigma_k)}{(2\sigma_k)^{n-1}} \\ &\leq \frac{(2k)^{n-1}\operatorname{Vol}(B_R \cap \Sigma_k)}{R^{n-1}} \leq a(2k)^{n-1} \end{aligned}$$

where the first inequality uses the monotonicity formula and the second inequality uses the volume bound. Therefore, we obtain

$$\int_{B_1 \cap \tilde{\Sigma}_k} |\tilde{A}_k|^{2p} \leq C(n,a) k^{-4-\sqrt{7/5}+n-1}.$$

By the mean value inequality, we have

$$\frac{1}{2^{2p}} = |\tilde{A}_k(0)|^{2p} \leq c(n) \int_{B_1 \cap \tilde{\Sigma}_k} |\tilde{A}_k|^{2p}$$

where $c(n)$ is the constant given in the mean value inequality. Combining these two bounds, we get

$$\frac{1}{2^{4+\sqrt{7/5}}} \leq C(n,a) k^{-4-\sqrt{7/5}+n-1}.$$

But for $3 \leq n \leq 6$, we have $-4 - \sqrt{7/5} + n - 1 < 0$ and so for large k we get a contradiction. □

An important corollary follows.

Corollary 6.4. *Let $\Sigma \subset \mathbb{R}^n$ be a stable, minimal hypersurface with $\partial\Sigma \subset \partial B_R$, $\mathrm{Vol}(\Sigma) \leq aR^{n-1}$, and $0 \in \Sigma$. Then*

$$|A|^2(0) \leq \frac{C(n,a)}{R^2}.$$

Remark 6.5. The curvature estimate is also true for $n = 7$, as shown in [SS81]. The proof is much more difficult than the above and is omitted. Curiously, such a result fails for $n \geq 8$. The fact that it holds for $n \leq 7$ closely relates to the fact that there are no non-trivial stable minimal cones. In some sense, these cones classify the possible blow ups one can get in the above argument.

We conclude this section with an important theorem. Essentially, the second fundamental form dictates how "curved" a surface is – the closer it is to 0 in magnitude, the closer the surface is to a plane at that point. Thus it makes sense if $|A|$ is small then we can represent a definite portion of our surface as a graph. To do this, we need the following definitions, which establish the precise notion of what it means to be a graph.

Definitions 6.6. Let (M^n, g) be a manifold embedded in $\mathbb{R}^{n+1}$, $x \in M$, and $r > 0$. We say that $B_r^M(x) \subset M$ *is graphical over* T_xM if there exists a function $u : \Omega \subset T_xM \to \mathbb{R}$ such that $u(\Omega) = B_r^M(x)$. The *graph of u over* Ω is defined as

$$\mathrm{Graph}(u) = \{x + u(x)N(x) \mid x \in \Omega\}$$

where N is the normal vector field on T_xM. Rephrased, $B_r^M(x)$ is graphical over T_xM if there exists an Ω and a u such that $\mathrm{Graph}(u) = B_r^M(x)$.

Definition 6.7. We say that M is *graphical* if for every $x \in M$ there exists an $r > 0$ such that $B_r(x)$ is graphical over T_xM.

Remark 6.8. In general, the r given in Definition 6.7 will depend on x. This motivates the following definition:

Definition 6.9. We say that M is *uniformly graphical* if there exists an $r > 0$ such that for every $x \in M$ the ball $B_r^M(x)$ is graphical over T_xM.

Theorem 6.10. *(Small Curvature Implies Graphical) Let $\Sigma \subset \mathbb{R}^n$ be an immersed surface with*
$$16s^2 \sup_{\Sigma} |A_\Sigma|^2 \leq 1.$$
If $x \in \Sigma$ and $d_\Sigma(x, \partial\Sigma) \geq 2s$, then there is a function u defined on $B_s^T(x) = B_s(x) \cap T_x\Sigma$ such that $B_s^\Sigma(x) \subset \mathrm{Graph}(u) \subset B_{2s}^\Sigma(x)$. Moreover, $|\nabla u| \leq 1$ and $\sqrt{2}s|\operatorname{Hess}_u| \leq 1$.

Proof. Some parts of the proof closely follow [CM11], with extra details added.

Define $d_{x,y} = d_{S^{n-1}}(N(x), N(y))$ where N is the Gauss map on Σ. Let $\alpha(t) : [0,1] \to \Sigma$ be a geodesic from x to y. Then, by the chain rule,
$$d_{x,y} \leq \int_0^1 |(N \circ \alpha)'(t)| dt = \int_0^1 |dN_{\alpha(t)}(\alpha'(t))| dt.$$
Let e_i be an orthonormal frame for Σ along $\alpha(t)$. Expanding $\alpha'(t)$ and $dN(e_i)$ in this frame gives
$$\begin{aligned} |dN(\alpha'(t))| &= \Big|\sum_i \langle \alpha'(t), e_i\rangle dN(e_i)\Big| = \Big|\sum_{i,j} \langle \alpha'(t), e_i\rangle \langle dN(e_i), e_j\rangle e_j\Big| \\ &\leq |\alpha'(t)| \Big|\sum_{i,j} \langle dN(e_i), e_j\rangle e_j\Big| \end{aligned}$$
where the Cauchy-Schwarz inequality was applied on the first factor. We now compute $\langle dN(e_i), e_j\rangle$. Recall that $A_{ij} = A(e_i, e_j) = \langle \nabla_{e_i} e_j, N\rangle$ and $\langle N, e_j\rangle = 0$. So,
$$e_i[\langle N, e_j\rangle] = \langle \nabla_{e_i} N, e_j\rangle + \langle N, \nabla_{e_i} e_j\rangle = 0$$
which implies that $\langle dN(e_i), e_j\rangle = -A_{ij}$. Using this, we obtain
$$|dN(\alpha'(t))| \leq |\alpha'(t)| \sqrt{\sum_{i,j} |A_{ij}|^2} = |\alpha'(t)||A|.$$
And so,
$$d_{x,y} \leq |A| \int_0^1 |\alpha'(t)| dt = |A| d_\Sigma(x, y)$$
Suppose now that $y \in B_{2s}^\Sigma(x)$. Then,
$$d_{x,y} \leq \sup_{B_{2s}^\Sigma} |A| d_\Sigma(x, y) \leq \frac{1}{4s}\Big(2s\Big) \leq \frac{1}{2} < \frac{\pi}{4}.$$
That is, the normal vectors are not close to horizontal since $d_{x,y} < \pi/2$ and, in fact, lie in some cone.

We now show that $\partial B_{2s}^\Sigma(x)$ is outside the cylinder $B_s^T(x) \times \mathbb{R}$. Let $\gamma : [0, 2s] \to \Sigma$ be an intrinsic geodesic parameterized by arclength such that $\gamma(0) = x$. Observe that the other endpoint of γ is on $\partial B_{2s}^\Sigma(x)$. Now,
$$|\partial_t \langle \gamma'(t), \gamma'(0)\rangle| \leq |A|(\gamma(t)) \leq \frac{1}{4s}$$
since $|A|^2 \leq 1/16s^2$. Integrating the above from 0 to t for any $t \leq 2s$ gives
$$\langle \gamma'(t), \gamma'(0)\rangle - \langle \gamma'(0), \gamma'(0)\rangle \geq \int_0^t -\frac{1}{4s} d\tau \geq -\frac{t}{4s}.$$

Because $\gamma'(0)$ is a unit vector, this implies that

$$\langle \gamma'(t), \gamma'(0) \rangle \geq 1 - \frac{t}{4s}.$$

Integrating the above from 0 to $2s$ gives

$$\langle [\gamma(2s) - \gamma(0)], \gamma'(0) \rangle \geq \int_0^{2s} \Big(1 - \frac{t}{4s}\Big) dt = \frac{3s}{2}.$$

Observe that

$$s < \frac{3s}{2} \leq |\langle [\gamma(2s) - \gamma(0)], \gamma'(0) \rangle| = |\gamma(2s) - \gamma(0)| \cos\theta$$

where θ is the angle between $\gamma(2s) - \gamma(0)$ and $\gamma'(0)$. The quantity $|\gamma(2s) - \gamma(0)| \cos\theta$ is exactly the distance from x to $\pi(\gamma(2s))$, where $\pi : B_{2s}^{\Sigma}(x) \to T_x\Sigma$ is the projection onto $T_x\Sigma$. Hence $\partial B_{2s}^{\Sigma}(x)$ lies outside $B_s^T(x) \times \mathbb{R}$.

By the above, the connected component of $(B_s^T(x) \times \mathbb{R}) \cap \Sigma$ containing x is a subset of $B_{2s}^{\Sigma}(x)$. Suppose there exists a $y \in [\pi(B_{2s}^{\Sigma}(x))]^c \cap B_s^T(x)$. Let $\tilde{y} \in \mathrm{cl}(B_{2s}^{\Sigma}(x))$ be such that $d(\tilde{y}, \{y\} \times \mathbb{R})$ is at a minimum.

Observe that $\tilde{y}$ cannot belong to $\partial B_{2s}^{\Sigma}(x)$. If it did, we would have that $(\{y\} \times \mathbb{R}) \cap B_{2s}^{\Sigma}(x) = \emptyset$, a contradiction. So $\tilde{y}$ belongs to the interior of $B_{2s}^{\Sigma}(x)$. Therefore $\nabla_{\Sigma} d(-, \{y\} \times \mathbb{R}) = 0$ at $\tilde{y}$. But this implies that $N(\tilde{y})$ is horizontal, a contradiction. It follows that $B_s^T(x) \subset \pi(B_{2s}^{\Sigma}(x))$.

Now consider $\pi : \pi^{-1}(B_s^T(x)) \to B_s^T(x)$. Since the unit normal of Σ is not too horizontal, the map π is a covering map. By a standard covering space argument it follows that π is injective. Moreover, by the above, π is surjective. This implies that $B_{2s}^{\Sigma}(x)$ is contained in the graph of a function u over $B_s^T(x)$. Recall that the normal vector on Σ can be written as

$$N = \frac{(-\nabla u, 1)}{\sqrt{1 + |\nabla u|^2}}.$$

By identifying $T_x\Sigma$ with $\mathbb{R}^k$, where k is the dimension of Σ, we have that $N(x) = (0, 1)$. Thus,

$$1 + |\nabla u|^2 = \langle N(x), N(y) \rangle^{-2} = |N(x)|^{-2} |N(y)|^{-2} \cos^{-2}\theta = \cos^{-2}\theta,$$

where θ is the angle between $N(x)$ and $N(y)$. Since $N(x)$ and $N(y)$ are on a unit sphere, the arc subtended by them has length $d_{x,y} = \theta$. So,

$$1 + |\nabla u|^2 = \cos^{-2} d_{x,y} \leq \cos^{-2}(\pi/4) \leq 2$$

from which $|\nabla u| \leq 1$ follows. To prove the Hessian bound, recall the following inequality (found in [CM11])

$$\frac{|\operatorname{Hess}_u|^2}{(1 + |\nabla u|^2)^3} \leq |A|^2.$$

Using the above gradient bound and the hypothesis on $|A|^2$ gives

$$|\operatorname{Hess}_u|^2 \leq (1 + |\nabla u|^2)^3 |A|^2 \leq (1 + 1)^3 \Big(\frac{1}{16s^2}\Big) = \frac{1}{2s^2}$$

from which the Hessian bound follows. $\square$

7. Compactness of Stable Minimal Hypersurfaces

The main goal of this paper is to prove a compactness result of stable minimal hypersurfaces [SS81]. In it, we talk about convergence of a sequence of minimal surfaces. A priori, it is not clear what it means for a sequence of surfaces to converge (or, even, converge smoothly). What we do instead is translate this problem into one of functions. Recall that every surface in $\mathbb{R}^n$ can be written as the union of graphs of functions. So, we hope to write the minimal surfaces as graphs of functions.

Indeed, given a sequence of minimal surfaces Σ_k in $\mathbb{R}^n$ satisfying the conditions of Theorem 6.10 (i.e., not curving too badly), we can locally write them as the graphs of smooth functions. By applying Theorem 6.10 to $B^{\Sigma_k}_{s_k}(p_k)$ for $p_k \in \Sigma_k$, we will obtain $C^{1,\alpha}$ functions u_k such that $\text{Graph}(u_k) \subset B^{\Sigma_k}_{2s_k}(p_k)$ is graphical over $B_{s_k}(p_k) \cap T_{p_k}\Sigma_k$. Since the minimal surface equation in $\mathbb{R}^n$ is an elliptic PDE, by elliptic regularity, it follows that each u_k is actually smooth.

Then we will say that the minimal surfaces converge if the u_k converge to some function u, and we define the limit to be the graph of u. This only works in a local sense, so it will remain to show that the local graphs can be "glued" together to achieve global convergence.

We now arrive at the following result.

Theorem 7.1. *Let $U \subset \mathbb{R}^n$ be open, and $K \subset U$ compact with $3 \leq n \leq 6$. Suppose $\Sigma_k \subset U$ is a sequence of stable minimal hypersurfaces with $\text{Vol}(\Sigma_k) \leq a$. Then there exists a subsequence converging to Σ, a stable minimal hypersurface in U (possibly with multiplicity).*

The following discussion provides a heuristic for this proof. First, we use stability, the area bound, and Theorem 6.1 to get a uniform bound on $|A_k|^2$ for each Σ_k. This tells us that the each Σ_k is not curving "wildly". Importantly, it implies that each is uniformly graphical. At this stage, we transition to a local argument. We use Arzela-Ascoli to show that there exists a convergent subsequence of the minimal graphs. From this, we patch together the local arguments and get global convergence.

Proof. **Step 1:** Establishing a uniform bound on $|A_k|^2$.

Fix a $k \in \mathbb{N}$. Since $d(x, \partial U) : K \to \mathbb{R}$ is a continuous function on a compact set, it has an absolute minimum, r. Fix an $x \in \Sigma_k \cap K$. Now $\text{Vol}(\Sigma_k) \leq a$ and,

$$\frac{\text{Vol}(\Sigma_k \cap B_r(x))}{r^{n-1}} \leq \frac{a}{r^{n-1}} = \tilde{a}(K)$$

where $\tilde{a}$ is a constant depending only on K. From this, it is clear that Theorem 6.1 applies. Thus for all $x \in \Sigma_k \cap K$,

$$\sup_{B_r(x) \cap \Sigma_k} |A_k|^2 \leq \frac{C}{r^2}$$

follows where C is a constant depending on the dimension of M and the area bound of Σ_k. Since C and r do not depend on x, it follows that $|A_k|^2$ is uniformly bounded.

That is,

$$\sup_{\Sigma_k \cap K} |A_k|^2 \leq \frac{C}{r^2}.$$

Step 2: Showing each $\Sigma_k \cap K$ is uniformly graphical.

Fix a $k \in \mathbb{N}$. Choose $s > 0$ small enough so that for all $x \in \Sigma_k \cap K$ we have $d(x, \partial U) \geq 2s$. Now set $s < \min\{r/2, r/(4\sqrt{C})\}$. Then,

$$d(x, \partial\Sigma_k) \geq d(x, \partial U) \geq r > 2s.$$

By choice of s, we also have

$$\sup_{\Sigma_k \cap K} |A_k|^2 \leq \frac{C}{r^2} < \frac{C}{16Cs^2} = \frac{1}{16s^2}$$

and so

$$16s^2 \sup_{\Sigma_k \cap K} |A_k|^2 \leq 1.$$

Now apply Theorem 6.10 to conclude that for each $x \in \Sigma_k \cap K$ there exists a u_k such that $\mathrm{Graph}(u_k) \subset B_{2s}^{\Sigma_k}(x)$ is graphical over the s-ball in $T_x\Sigma_k$. Importantly, $\Sigma_k \cap K$ is uniformly graphical since s does not depend on x.

Step 3: Transition to a local argument.

Let $p_k \in \Sigma_k \cap K \to p \in K$. Let $\tilde{\eta_k}$ be the unit normal at p_k of Σ_k. Let $\eta_k = N(\tilde{\eta_k})$, where $N : \Sigma_k \to S^{n-1}$ is the Gauss map. It follows that η_k is a sequence of points in a compact set, and hence has a subsequence converging to some $\eta \in S^{n-1}$. Define the candidate unit normal at p of the limit of Σ_k to be η. From this, it follows that the $T_{p_k}\Sigma_k$ converge to some plane Q, namely the plane passing through p with unit normal η.

Now let u_k be as given in Step 2, that is $\mathrm{Graph}(u_k) \subset B_{2s}^{\Sigma_k}(p_k)$ is the graph of u_k over the s-ball in $T_{p_k}\Sigma_k$. From Theorem 6.10 it follows that $|\nabla u_k| \leq 1$ (that is, the ∇u_k are pointwise bounded) and $\sqrt{2}s|\,\mathrm{Hess}_{u_k}| \leq 1$. We may align $T_{p_k}\Sigma_k$ so that it is parallel to $\mathbb{R}^{n-1}$ and p_k is the origin. Now, applying MVT for functions from $\mathbb{R}^{n-1}$ to $\mathbb{R}$ for $x, y \in B_s(p_k) \cap T_{p_k}\Sigma_k$ we get

$$|\nabla u_k(x) - \nabla u_k(y)| \leq \int_0^1 \Big|\frac{d}{dt}\nabla u_k((1-t)x + ty)\Big| dt$$

$$= \int_0^1 \Big|\,\mathrm{Hess}_{u_k}((1-t)x + ty)\Big|\Big|y - x\Big| dt \leq |\,\mathrm{Hess}_{u_k}||x - y|.$$

Since $\sqrt{2}s|\,\mathrm{Hess}_{u_k}| \leq 1$, there is a bound, b, on $|\,\mathrm{Hess}_{u_k}|$. So, let $\epsilon > 0$ be sufficiently small. Then there exists a δ (namely, $\delta = \epsilon/b$) such for $x, y \in B_\delta^{\Sigma_k}(p_k)$ we have

$$|\nabla u_k(x) - \nabla u_k(y)| < b\delta = \epsilon$$

Thus ∇u_k is uniformly continuous for every k. Since δ did not depend on k, the set $\{\nabla u_k\}$ is equicontinuous.

With this in mind, we may apply Arzela-Ascoli. The above discussion showed that each ∇u_k is pointwise bounded and that $\{\nabla u_k\}$ is equicontinuous. Hence, there exists a convergent subsequence converging to some u. This function u will be a graph over Q. Moreover, since divergence is a continuous function, and

$$\operatorname{div}\left(\frac{\nabla u_k}{\sqrt{1+|\nabla u_k|^2}}\right) = 0$$

for each k, the limit u is a solution to the minimal surface equation, and the graph of u over Q is a minimal surface. Observe that the estimate on ∇u_k implies that the u_k are uniformly bounded in $C^{1,\alpha}$, and hence $C^{1,\alpha}$. At first, it appears that u is also only $C^{1,\alpha}$. But, u is smooth due to elliptic regularity.

So far, a couple of technicalities have been ignored. These follow from the fact that Arzela-Ascoli applies for functions with the same domain. To rectify this, we should represent the sets Graph(u_k) as graphs over Q rather than $T_{p_k}\Sigma_k$. To do so, we will need to construct new functions u_k' which will have comparable gradient to the u_k. This will be addressed in Step 5.

Step 4: Global convergence.

Let $r > 0$ and $\{p_k^i\}$, $1 \leq i \leq l_k$, l_k maximal, be a collection of points in Σ_k such that the $B_r^{\Sigma_k}(p_k^i)$ are disjoint. Such l_k exist due to the area bound on Σ_k. Then by a Vitali covering argument $\cup_i B_{5r}^{\Sigma_k}(p_k^i)$ covers Σ_k. Since this works for any r, we may choose r so that $10r < 2s$. Without loss of generality we can assume that $l_k = l$ is constant. Take $p_k^i \to p^i$ (after possibly passing to a subsequence). Running the local argument on each of these subsequences with $B_{5r}^{\Sigma_k}(p_k^i)$ almost completes the proof.

It remains to show that the graphs over intersecting balls converge to the same graph. Let $S_k^i = \text{Graph}(u_k^i)$ and $S^i = \text{Graph}(u^i)$. In Step 5, we will see that these sets should be defined using the functions u_k' which have modified domains. We showed in Step 3 that $S_k^i \to S^i$ in k and that each S^i is a minimal hypersurface. To show the graphs over intersecting balls converge to the same graph, we show that $\cup S^i$ is a smooth manifold. Note that we need only check that this occurs pairwise. So, select S^i and S^j for $i \neq j$. We may further assume that these hypersurfaces intersect; otherwise, they are disjoint, and by the remark at the end of Step 3, they are smooth manifolds. The hypersurfaces may intersect multiple times; if this occurs, focus only on one intersection. This intersection may be tangent or not. If it is, then S^i is locally on one side of S^j, and so the maxiumum principle applies. Hence $S^i = S^j$. If not, then S^i crosses through S^j. However, this implies for some k large enough that S_k^i and S_k^j also cross. But each Σ_k is embedded, hence this is a contradiction.

Step 5: Addressing technicalities.

Let us first construct the functions u_k' described at the end of Step 3. We claim that there exists an $R' > 0$ so that the Graph(u_k) are graphical over $B_{R'}(p) \cap Q$ for all k.

Let $\epsilon > 0$. Then there exists a κ such that for $k > \kappa$ we have $|g_k - g| < \epsilon$. It follows that the $u_k(B_s(p_k) \cap T_{p_k}\Sigma_k)$ are eventually graphs over Q. Recall that, since the u_k are graphs, the normal vector for each $B_{2s}^{\Sigma_k}(p_k)$ lies within some cone. It follows that $u_k(B_s(p_k) \cap T_{p_k}\Sigma_k)$ is a graph over Q if for all $x \in u_k(B_s(p_k) \cap T_{p_k}\Sigma_k)$, we have $\langle N(x), \eta\rangle \neq 0$. Evidently this is the case for sufficiently large k, since the normal vectors of $T_{p_k}\Sigma_k$ will eventually be close to η.

Let $\Omega_k' \subset Q$ and u_k' a function on Ω_k' be such that $\text{Graph}(u_k') = u_k(B_s(p_k) \cap T_{p_k}\Sigma_k)$. Similar to how each Ω_k contained a ball of radius s, the Ω_k' will also. This new radius will be slightly smaller, so call it R'. By ensuring that $10r < 2s$ in Step 4, when using R', we guarantee that $5R' < 2s$. So, we can run the argument in Step 4. Observe that, since $u_k'(B_{R'}(p) \cap Q)$ is not in general $u_k(B_s(p_k) \cap T_{p_k}\Sigma_k)$, we should define these graphs in a different way. Thus, in Step 4, the definition of S_k^i should be using these smaller sets $B_{R'}(p) \cap Q$ and u_k' rather than u_k.

Finally, it remains to show that $|\nabla(u_k')|$ does not vary much (eventually) from $|\nabla(u_k)|$. Recall earlier that eventually we get $\langle N(x), \eta\rangle \neq 0$. That is, $\langle N(x), \eta\rangle$ does not change sign. So, we may assume (up to a change in orientation) that $\langle N(x), \eta\rangle > a$ for some $1 > a > 0$. Now, recall we view these graphs in $\mathbb{R}^{n-1}$, so we have

$$N = \frac{(-\nabla u_k', 1)}{\sqrt{1 + |\nabla u_k'|^2}}, \quad \eta = (0, 1).$$

Thus,

$$\langle N, \eta\rangle = \frac{1}{\sqrt{1 + |\nabla u_k'|^2}} > a$$

from which it follows that $|\nabla u_k'|$ is bounded. □

Remark 7.2. In Step 2, the $\Sigma_k \cap K$ could have several connected components. For example, take a sequence $\Sigma_k \subset \mathbb{R}^3$ such that each Σ_k is the union of the planes $z = -1/(2k)$ and $z = 1/(2k)$. It is clear that the Σ_k converge to the plane $z = 0$, albeit with multiplicity two.

Remark 7.3. The majority of the above theorems involve minimal surfaces in $\mathbb{R}^n$. We will still, however, want to be able to apply these theorems for general manifolds.

Note that, by definition, a manifold of dimension n is locally like $\mathbb{R}^n$. From this, it is intuitively obvious that similar results should hold up to some small error term.

For example, in the dilation argument in Theorem 6.1, the ball (B_R, g) is sent to $(B_1, \mu^* g)$ by the dilation $\mu = x/R$. The point of the argument is to not have an isometry. Thus, we should multiply our metric by some conformal factor, in particular, R^2. We claim that $R^2\mu^* g \to g_{\text{Euc}}$, where g_{Euc} is the usual Euclidean metric. Thus we reach the same conclusion as above: by "zooming" in enough, we can decrease the error to an arbitrarily small amount. Thus the above theorems should hold for general manifolds.

8. Sharp's Compactness Theorem

The previous theorem proved a statement about compactness of stable minimal hypersurfaces. Recall that stable means the Morse index of the area functional is zero; that is, there are no directions which decrease the area. It would be nice if we could relax this condition to simply having bounded index, rather than necessarily index zero. This is precisely what Sharp's theorem accomplishes.

Theorem 8.1. *(Sharp's Compactness Theorem) Let $3 \leq n \leq 7$ and let M^n be a smooth closed Riemannian manifold. If $\{\Sigma_k\} \subset M$ is a sequence of closed, connected and embedded minimal hypersurfaces with*

$$\mathcal{H}^{n-1}(\Sigma_k) \leq \Lambda < \infty \quad \text{and} \quad \text{index}(\Sigma_k) \leq I$$

for some fixed constants $\Lambda \in \mathbb{R}$, $I \in \mathbb{N}$ independent of k, then up to subsequence, there exists a closed connected and embedded minimal hypersurface $\Sigma \subset M$ where $\Sigma_k \to \Sigma$ (possibly with multiplicity) with

$$\mathcal{H}^{n-1}(\Sigma) \leq \Lambda < \infty \quad \text{and} \quad \text{index}(\Sigma) \leq I.$$

Moreover, Σ is smooth everywhere. The Σ_k converge to Σ smoothly away from finitely many points. [Sha17]

Remark 8.2. A natural question is why should we even predict that a bounded index would give such a theorem. This example shows that an index bound is necessary for such a theorem. Consider the Costa-Hoffman-Meeks surfaces, shown below:

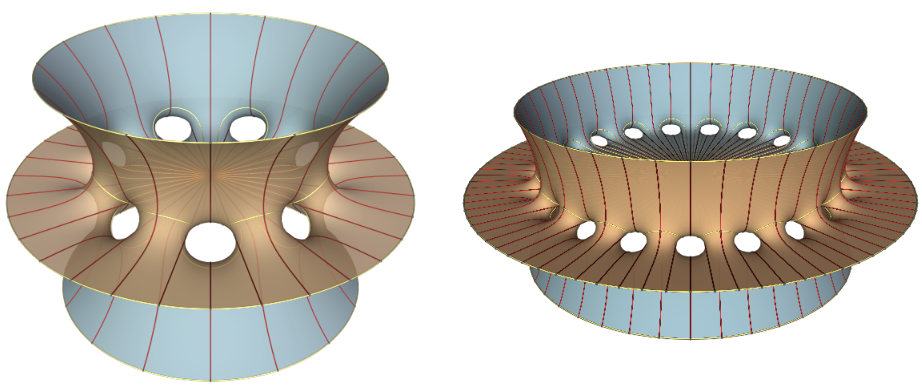

Figure 8.3. *Costa-Hoffman-Meeks surfaces with varying handles, models found at* [Web13]

Take, for example, the left surface. Consider any two consecutive holes. Imagine pinching the neck in this region. It is intuitively obvious that doing so will locally decrease the area of the surface. So, the index is governed by the number of holes in the surface. Increasing the number of holes results in a surface like that to the right. Continuing this process, we get a catenoid together with a plane with a disc removed. Clearly, this is not a smooth surface at the intersection of the catenoid and the plane.

Remark 8.4. It is important to ask whether we can strengthen this to guarantee that Σ has no multiplicity and is smooth. The following example suggests that, under the hypothesis of the theorem, this is not possible.

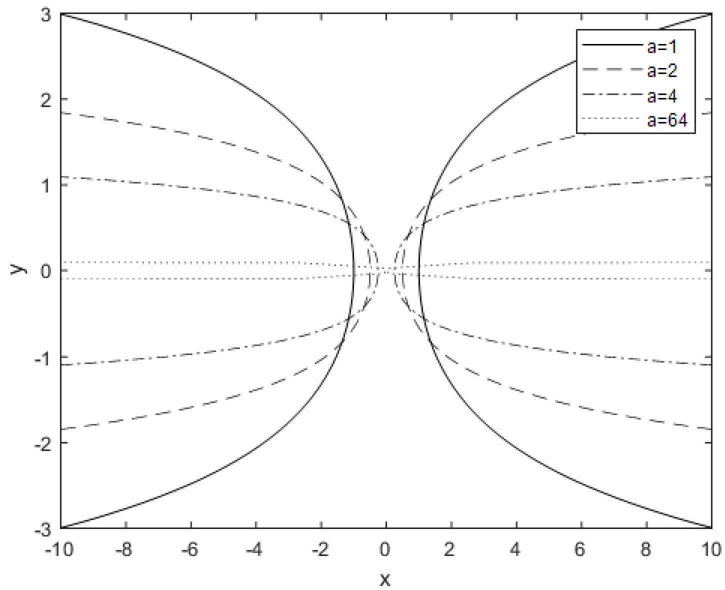

Figure 8.5. *Plots of cross sections of catenoids, in the form* $(\pm(1/a)f(at), t)$ *for various values of* a*, where* $f(t) = \cosh(t)$*.*

The above picture shows that, as $a \to \infty$, the catenoids converge to the xy plane away from the origin. Observe that both parts of the catenoid, above and below the xy plane, converge to the same part; thus, the convergence is with multiplicity two.

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