THE GALOIS ACTION ON DESSIN D'ENFANTS

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Abstract. We introduce the theory of dessin d'enfants, with emphasis on how it relates to the absolute Galois group of $\mathbb{Q}$. We prove Belyi's Theorem and show how the resulting Galois action is faithful on dessins. We also discuss the action on categories equivalent to dessins and prove its most powerful invariants for classifying orbits of the action. Minimal background in Galois theory or algebraic geometry is assumed, and we review those concepts which are necessary to this task.

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1. Introduction

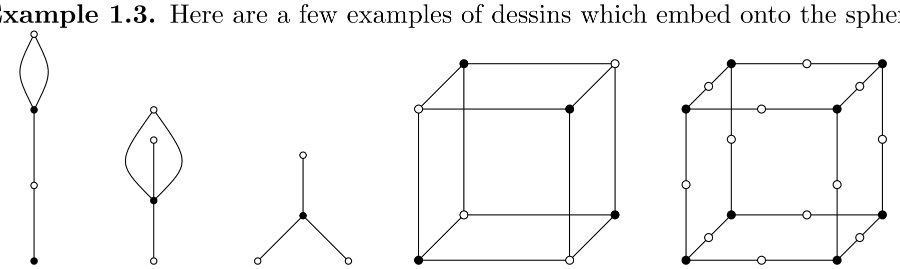
Definition 1.1. A *bigraph* is a connected bipartite graph with a fixed 2-coloring into nonempty sets of black and white vertices. We refer to the edges of a bigraph as its *darts*.

Definition 1.2. A *dessin* is a bigraph with an embedding into a topological surface such that its complement in the surface is a disjoint union of open disks, called the *faces* of the dessin.

Dessin d'enfant, often abbreviated to dessin, is French for "child's drawing." As the definition above demonstrates, these are simple combinatorial objects, yet they have subtle and deep connections to algebraic geometry and Galois theory. The main source of these connections is *Belyi's Theorem*, a result from [5] in 1979 on certain maps between algebraic curves defined over the algebraic numbers. This inspired Grothendieck to write his *Esquisse d'un Programme* in 1984; he showed then that dessins are equivalent to the maps of Belyi's Theorem, and thus may reveal combinatorial information which otherwise would be quite difficult to extract from polynomial equations.

In due time, we will explore this fascinating equivalence and the important Galois action it defines, which has been the subject of much recent study. First, however, we discuss dessins themselves and quickly go through the small amount of background necessary to understand this paper.

Example 1.3. Here are a few examples of dessins which embed onto the sphere.



(1) Trees can always be 2-colored, and always embed into the sphere with a single face.
(2) A graph which divides a surface into faces is called a *map*. We can treat the vertices of a map as black vertices and add a white vertex of degree 2 to each edge to transform a map into a dessin. A dessin with all white vertices of degree 2 is called *clean*. We might call the rightmost dessin the "clean cube," in contrast to the "2-colored cube" on its left.

Of course, degree, or number of darts meeting at a vertex, is a crucial part of the combinatorial information of a graph. We also define degree for a face: this will be half the number of darts which bound the face of a dessin, counting a dart which bounds the face on both sides twice. Note that the 2-coloring guarantees that the boundary of a face must have an even number of darts, so that degree is always a positive integer.

Though we defined bigraphs to be connected, it is good to observe that the embedding requirement would force the bigraph of a dessin to be connected anyways. Our definition leaves open the possibility that dessins may embed into nonorientable surfaces; this is common, and important for certain combinatorial descriptions of dessins. However, nonoriented dessins lack any interesting connection to algebraic curves. For the rest of this paper, we will only consider oriented dessins.

Notably, the orientation of a dessin's surface assigns a cyclic ordering of darts about each vertex of the dessin's underlying bigraph. This is an important property of the embedding, and one which we will choose to be preserved by morphisms.

Definition 1.4. A morphism between dessins is a continuous surjection between surfaces, which restricts to a continuous surjection between the underlying bigraphs and a color-preserving surjection between vertex sets. For an isomorphism, these maps are homeomorphisms.

An intuitive first approximation for isomorphism between dessins would be to consider isotopy on an embedding surface, but in fact this is slightly too narrow. The interested reader can look into *Dehn twists*, among other things; an example would be embedding a dart in a torus to wrap additionally around one of the handles, rather than go directly between vertices.

Unless otherwise noted, we will always be interested in dessins only up to isomorphism. Note that isomorphism preserves genus of the embedding surface, giving a dessin a well-defined genus. Isomorphism also preserves the cyclic order of darts about each vertex, as well as all incidence relations between faces, darts, and vertices.

The following result provides good intuition for what the geometric information of a dessin is. This example will recur throughout the paper in various different or obscured forms.

Proposition 1.5. *The category of oriented dessins is equivalent to the category of ramified covers $p : X \to S^2$, where X is a topological surface and p is ramified only in $\{0, 1, \infty\}$. A morphism $p \to q$ in this latter category is a decomposition $p = q \circ g$ for some continuous g.*

Proof. We identify S^2 with $\mathbb{CP}^1$, the complex sphere, to use notation consistent with that which will appear later.

Let p be as given. We construct the dessin D corresponding to p as $p^{-1}([0,1]) \subset X$, where black vertices are the points of $p^{-1}(0)$ and white vertices are the points of $p^{-1}(1)$. To see that this bigraph is a dessin, observe that each $x \in f^{-1}(\mathbb{CP}^1 \setminus [0,\infty])$ lies in an unramified lift of $\mathbb{CP}^1 \setminus [0,\infty]$, homeomorphic to an open disk. Therefore, the triangulation of our bigraph from adding vertices $f^{-1}(\infty)$ and edges $f^{-1}([0,\infty]), f^{-1}([1,\infty])$, satisfies the embedding requirement, and we have a dessin. Given the morphism $p = q \circ g$, we necessarily have $g(p^{-1}([0,1])) = q^{-1}([0,1])$, and thus g induces a morphism of dessins. One can easily verify that this is in fact a functor.

Conversely, let D be a dessin. Choose a designated point (labeled $*$) in the interior of each face of D, and draw edges to each vertex incident to the face. This gives a triangulation of D, and it is standard from algebraic topology that a triangulation of an orientable surface can be 2-colored so that no two adjacent triangles share the same color. We can collect these into oppositely colored pairs with the relation $T_1 \sim T_2$ if T_1 borders T_2 along a $* - - - \bullet$ edge. For each pair, glue white vertices and pairs of edges to give a sphere. Now we have a union of spheres, glued together only at black, white, and $*$ vertices. This defines a ramified cover from X, the underlying surface of D, onto the sphere, with corresponding ramification at $0, 1$, and ∞. A morphism between dessins induces a continuous map between surfaces which gives a morphism of ramified covers, so this defines another functor.

We claim that both functors are categorical equivalences. Going from dessin to ramified cover to dessin is the identity by construction. In the other direction, ramified covers which differ by the monodromy of a path in $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$ are taken to the same dessin. Therefore, by choosing a basepoint, we can define a natural transformation in terms of this monodromy to take the composite functor to the identity, so we have a categorical equivalence. □

Though to this point we have used exclusively real topological surfaces, much of our work will more closely resemble embeddings of dessins onto complex surfaces.

Definition 1.6. A *Riemann surface* is a 1-dimensional connected compact complex manifold. A morphism between Riemann surfaces is a meromorphic function: a function which is complex differentiable except at finitely many points.

A Riemann surface can be understood as a topological surface (sphere, torus, etc) with an additional complex structure, which provably exists only for oriented topological surfaces. Unlike topological surfaces, Riemann surfaces are not classified by genus. For example, there is only one complex structure on the sphere, but there are infinitely many nonequivalent complex structures on the torus that give distinct Riemann surfaces.

As it turns out, a meromorphic function between Riemann surfaces acts as a ramified cover on the underlying topological surfaces (in particular, there are only morphisms from same or higher genus to same or lower genus). Later on, we will find a natural way to assign a Riemann surface to a dessin (though we will phrase it in terms of algebraic curves), and thus Proposition 1.5 has an interpretation in this category as well.

2. Background: The Absolute Galois Group

Henceforth, we let Γ denote $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. This is the group of automorphisms of the algebraic numbers which fix the rational numbers. An element $\sigma \in \Gamma$ is a bijection $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}$ such that if p is any polynomial over the rationals and $x \in \overline{\mathbb{Q}}$, $\sigma(p(x)) = p(\sigma(x))$. As a group, Γ is fascinating and mysterious, and has been the subject of much research. Here are a few facts about Γ; for proofs and more information, the reader is advised to consult a work treating Galois Theory.

Fact 2.1. *A finite extension of $\mathbb{Q}$ (such as $\mathbb{Q}[i]$ or $\mathbb{Q}[\sqrt{2}]$) is called a number field. Any automorphism of a number field fixing $\mathbb{Q}$ extends to an element of Γ, albeit not uniquely.*

Nonetheless, the only elements of Γ which can be explicitly described are the identity, and the automorphism produced by complex conjugation. All other automorphisms are highly nontrivial. For example, the previous fact gives that we can extend an automorphism of $\mathbb{Q}[\sqrt{2}]$ sending $\sqrt{2}$ to $-\sqrt{2}$ to an element of Γ. But that element of Γ must now decide where to send numbers such as $\sqrt{3}+\sqrt{2}$, $\sqrt{6}$, $\sqrt{1+\sqrt{2}}$, and so on, to preserve algebraic relations amidst an ever increasing number of choices.

3. Background: Algebraic Curves

An extensive background in algebraic geometry is not necessary to appreciate or work with dessins. Here we summarize the famous equivalence between Riemann surfaces and algebraic curves, which should help to geometrically motivate the connection we will later encounter between curves and dessins.

Definition 3.1. An *algebraic variety* over $\mathbb{C}$ is the zero set of a collection of polynomials with coefficients in $\mathbb{C}$ defined over finitely many variables. The variety is projective if its polynomials are homogenous and we consider values in projective space.

Definition 3.2. An *algebraic curve* over $\mathbb{C}$ is an algebraic variety over $\mathbb{C}$ of dimension 1; in practice, this is a variety with n variables in $n-1$ equations. A *projective* curve corresponds to a projective variety. A *smooth* curve has no singular points.

Fact 3.3. *A smooth projective curve endowed with the subspace topology in $\mathbb{C}^n$ is a Riemann surface.*

The use of projective coordinates ensures compactness, while the absence of singular points in a curve of dimension 1 is the condition that it should be locally holomorphic to $\mathbb{C}$.

Fact 3.4. *Every projective curve is birationally equivalent to exactly one smooth projective curve. In fact, projective curves over* $\mathbb{C}$ *up to birational equivalence are equivalent to Riemann surfaces up to biholomorphism.*

The reverse direction of the equivalence is especially difficult, and is worthy of a much more thorough treatment than we provide here. Instead, our main interest in introducing algebraic curves is for the following result:

Fact 3.5. *There is a Galois action on algebraic curves defined over* $\overline{\mathbb{Q}}$*. Namely,* $\sigma \in \Gamma$ *acts on a curve by acting on the coefficients of its defining polynomials. This action preserves genus.*

This will be a piece of the Galois action on dessins, which we ultimately mean to introduce.

4. Belyi's Theorem

Definition 4.1. Let X be a smooth projective curve defined over $\overline{\mathbb{Q}}$. A *Belyi map* of X is a morphism $f : X \to \mathbb{CP}^1$ with critical values only in $\{0, 1, \infty\}$.

The name "Belyi map" comes as a consequence of these being the main subject of Belyi's Theorem, of course. We will prove that these maps are closely linked to $\overline{\mathbb{Q}}$, giving an immediate Galois action.

Theorem 4.2 (Belyi's Theorem)**.** *Let* X *be a smooth projective curve defined over* $\mathbb{C}$*. Then* X *is defined over* $\overline{\mathbb{Q}}$ *if and only if* X *has a Belyi map.*

Historically, the direction showing that X is defined over $\overline{\mathbb{Q}}$ was discovered much earlier, despite having a significantly harder proof. A detailed proof can be found in [7], which corrects certain errors which persist in older sources.

We will show the second direction, which was found by Belyi. Our proof follows the original in [5], or equivalently its presentation in [2].

Proof. Let X be defined over $\overline{\mathbb{Q}}$, so we can take g to be a rational function with coefficients in $\overline{\mathbb{Q}}$.

Let h_1 be the minimal polynomial over $\mathbb{Q}$ for the critical values of g, and let h_{i+1} be the minimal polynomial for the critical values of h_i (that is, for the set $\{h_i(z) \mid h_i'(z) = 0\}$). Observe that $\deg h_{i+1} \leq \deg h_i' = \deg h_i - 1$, so we obtain a degree 1 h_n in finitely many steps. We define $\tilde{g} = h_n \circ \cdots \circ h_1 \circ g$.

By chain rule, a critical point z of $\tilde{g}$ satisfies $g'(z) = 0$ or $h_1'(g(z)) = 0$ or ... or $h_{n-1}'(\dots h_1(g(z)) \dots) = 0$; by construction, these correspond to the critical values $h_n(\dots(h_2(0))\dots)$ or $h_n(\dots(h_3(0))\dots)$ or ... or $h_n(0)$ or 0. These are all rational, so $\tilde{g}$ has only rational critical values.

Now, by composing $\tilde{g}$ with a Möbius transformation (a rational function $\frac{az+b}{cz+d}$), we can take a triple of distinct points to any other triple. Therefore, if $\tilde{g}$ has 3 or fewer critical values, we are done. Otherwise, pick 3 and map these to $0, \frac{m}{m+n}, 1$ with m, n integers. Then, the carefully chosen transformation

$$z \to \frac{(m+n)^{m+n}}{m^m n^n} z^n (1-z)^n$$

maps 0 and 1 to 0, and $\frac{m}{m+n}$ to 1. Therefore, we can progressively reduce the number of ramification values until only 3 distinct values remain, and thus get the desired Belyi map.

□

We will see in the next section that Belyi's Theorem allows us to extend the Galois action on curves to a Galois action on Belyi maps, and obtain a much richer object of study. For now, though, we close out by observing the connection between dessins and Belyi maps.

Proposition 4.3. *Let f be a Belyi map for some X. The bigraph with black vertices $f^{-1}(0)$, white vertices $f^{-1}(1)$, and darts $f^{-1}([0,1])$ is a dessin embedded in X.*

Proof. Topologically, a Belyi map is a ramified cover with ramification at 3 values, so this is just a corollary of Proposition 1.5. □

Definition 4.4. A morphism of Belyi maps from f to g is a rational function h over $\overline{\mathbb{Q}}$ such that $f = g \circ h$.

Remark 4.5. In fact, dessins and Belyi maps are equivalent, by methods similar to Proposition 1.5.

An alternate way to reach this statement is shown in [1].

We have shown how to pass from a Belyi map to a dessin; conversely, for any dessin, we can form a system of equations using its combinatorial data which yield (uniquely) the Belyi map which passes to that dessin, up to placement of 3 points. We will give more details on this later in the section on computations.

The correspondence between isomorphism classes of Belyi maps and isomorphism classes of dessins is key to modern research, using dessins to study algebraic questions.

5. The Galois Action on Dessins

In this paper, we let $\Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Proposition 5.1. *Let $\sigma \in \Gamma$. Let X be an a smooth projective curve defined over $\overline{\mathbb{Q}}$ equipped with Belyi map f.*

Then Γ has a group action on Belyi maps, by applying σ to each coefficient of f to get a Belyi map f^σ, with domain X^σ.

This extends to a group action on dessins, by sending the dessin corresponding to f to the dessin corresponding to f^σ.

This group action on dessins may be somewhat hard to interpret at first; it admits no easy description without passing to Belyi maps. One useful property to observe is that the group action on Belyi maps extends to an action on morphisms between Belyi maps (again, by acting on the coefficients of a rational function). It is not hard to see that this action respects composition, using the properties of Γ, and as a result the action on Belyi maps is functorial. Thus, the action on dessins is also functorial.

In a different direction, we can also notice that several pieces of a dessin are trivially preserved by the action.

Proposition 5.2. *If C and D are dessins in the same orbit under Γ, they have the same number of vertices of each color-degree, the same number of darts, and the* *same number of faces of each degree. Also, their embedding surfaces have the same genus.*

Proof. A black vertex of degree k corresponds to a factor $(z-\alpha)^k$ in the Belyi map of C, and the action of $\sigma \in \Gamma$ takes this to $(z-\sigma(\alpha))^k$ (as an automorphism, σ will not map two different roots to the same point). The result for white vertices and for faces is similar.

An edge corresponds to a sheet in $f^{-1}((0,1))$, where f is the Belyi map of C. Therefore, the number of edges is simply the size of the fiber of any point in this set, or the degree of f as a rational function. Certainly, when σ changes the coefficients of f as a rational function, it preserves degree. □

Fact 5.3. *There are finitely many bigraphs for any instance of the parameters of Proposition 5.2.*

Corollary 5.4. *The orbit of a dessin is finite. We can enumerate all dessins with the same information as in Proposition 5.2, and get a union of finitely many orbits under* Γ.

These last two statements follow from basic graph theory. Information on performing such enumerations can be found in [4].

So, even though the action of Γ seems rather inscrutable, just from the dessin perspective, we can easily narrow down to *almost* the orbit of a dessin. With further invariants, we will strengthen our ability to separate orbits. Admittedly, however, presently known invariants are incapable of separating all orbits.

The following terminology for Belyi maps extends to dessins as well.

Definition 5.5. Let f be a Belyi map. A *field of definition* of f is a field containing its coefficients. The *field of moduli* of f is the field corresponding to the stabilizer of f in Γ, per the Galois correspondence.

The field of moduli K is the minimal field extension of $\mathbb{Q}$ such that $\sigma \in \Gamma$ acts nontrivially on f if and only if it acts nontrivially on K. This is a well-defined number field which is contained in every field of definition of f, though there certainly exist f which cannot be defined over the field of moduli.

These fields are obviously also invariant under the Galois action, but are very inconvenient to compute for dessins. We will examine some of the considerations that arise in moving from dessins to Belyi maps, but ultimately, the most productive way to study the Galois action will be from a more combinatorial viewpoint.

6. Computing the Belyi Map of a Spherical Dessin

Passing from dessins to Belyi maps requires solving a system of polynomial equations given by the combinatorial data of the dessin. We will outline how to do this in general, particularly for small dessins, and give a reference for automating such computations.

The following states an observation which should already be obvious from the correspondence between dessins and Belyi maps.

Proposition 6.1. *Let* D *be a dessin on the sphere, with corresponding Belyi map* f.

Let b_i *be the (unknown) coordinates of the black vertices of* D*, or equivalently the points in* $f^{-1}(0)$*. Let* n_i *be the corresponding degrees of the vertices.*

Similarly, let w_j be the (unknown) coordinates of the white vertices, or equivalently the points in $f^{-1}(1)$, and let m_j be the corresponding degrees.

Let f_k be the (unknown) coordinates of the face centers, or equivalently $f^{-1}(\infty)$. Take r_k to be the degrees of the corresponding faces.

Then

$$f(z) = c_1 \frac{\prod (z-b_i)^{n_i}}{\prod (z-f_k)^{r_k}}$$

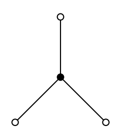
and

$$f'(z) = \frac{c_2}{p(z)} \prod (z-b_i)^{n_i-1} \prod (z-w_j)^{m_j-1}$$

for constants c_1, c_2 and a polynomial $p(z)$. We exclude any infinite b_i, w_j, f_k in these formulas.

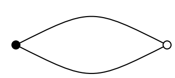
The data above is often enough to determine a Belyi map, up to an automorphism of the sphere (i.e. the placement of three points). The most common choice of normalization is to place a black vertex at 0, a white vertex at 1, and a pole at ∞. Often, however, one can find a more convenient normalization by carefully considering the dessin at hand.

Example 6.2.



We begin with a very simple example. This dessin corresponds to a Belyi map with a critical point and a pole of degree 3 mapping to 0 and ∞, and 3 unramified points in the fiber of 1. Therefore, taking the standard normalization, we have $f(z) = c_1 z^3$. To put a white vertex at 1, we take $c_1 = 1$, so in fact $f(z) = z^3$.

Example 6.3.



Here is a slightly more complicated dessin, with two poles of degree 1 and two critical points of degree 2 mapping to 0 and 1. Thus, taking the standard normalization, $f(z) = c_1 \frac{z^2}{z-f_1}$ and $f'(z) = \frac{z(z-2f_1)}{(z-f_1)^2}$. To put the white vertex at 1, we have $f_1 = \frac{1}{2}$ and $c_1 = \frac{1}{2}$, so we get that $f(z) = \frac{z^2}{2z-1}$.

Despite what these examples may suggest, there is not always a unique solution to the coefficients of a Belyi map. This is because the proposition above only made use of local data, rather than the entire data of the dessin. Fortunately, there can only be finitely many solutions. Therefore, it is feasible to enumerate all solutions via polynomial solving techniques (i.e. Groebner bases) and then plot these maps to determine which correspond to the original dessin. This process is lengthy, even on modern computers, but quite doable. A good reference for computational techniques is [1].

Multiple solutions arise from the following:

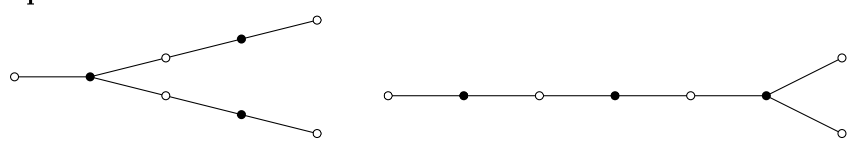
- Other Belyi maps with the same local combinatorial data.
- Degenerate solutions, which identify any pair from the b_i, w_j, f_k.

- Isomorphic solutions, resulting from symmetry in the dessin causing ambiguity after normalization.

[4] has examples of each type of failure.

The dessins we have examined so far have trivial Galois orbits. Heuristically, highly symmetric dessins are less likely to have nontrivial Galois conjugates. This can be partially formalized by applying the Galois invariants we will subsequently study to special dessins called *regular dessins*.

Example 6.4.



Here is one of the simplest instances of a nontrivial Galois action on dessins, due to [4]. The respective Belyi maps of these dessins can be computed using a nonstandard normalization as

$$f(z) = z^3(z^2 - 2z + \frac{34}{7} \pm \frac{6\sqrt{21}}{7})^2$$

An element of Γ which sends $\sqrt{21}$ to $-\sqrt{21}$ sends the dessin on the left to the dessin on the right.

It is worth noting that even though these dessin are only slightly more complex than those we explored previously, their Belyi maps are already considerably nastier. This is one of the benefits to preferring the category of dessins; it is not uncommon to see computations result in coefficients that take several lines to describe in closed form.

7. Faithfulness of the Galois Action

After seeing a few examples of the Galois action, it is only fair to wonder what this action actually tells us about Γ. We know all orbits are finite, and many of the dessins we might think to examine have trivial orbits. Nevertheless, one of the most surprising characteristics of this group action is its faithfulness, even when restricted to many different subclasses of dessins.

Theorem 7.1. *The Galois action is faithful on tree dessins.*

Proof. We make use of a result in our proof of Belyi's theorem, that a polynomial with algebraic critical points can be composed with a polynomial with rational coefficients to give a Belyi map. We also use a cancellation lemma for polynomials: if $g \circ h = \tilde{g} \circ \tilde{h}$ and $\deg h = \deg \tilde{h}$, then h and $\tilde{h}$ are equal up to left composition by a degree 1 polynomial. A proof of this lemma using elementary methods is supplied in [2].

Let $\sigma \in \Gamma$ be nontrivial, and let α be a primitive element of a number field such that σ acts nontrivially on α, taking it to β. The tree dessins correspond exactly to the Belyi maps with a single pole, and up to a change of coordinates, the polynomial Belyi maps. Define the polynomial $f_\alpha(z)$ so as to satisfy

$$f_\alpha(z) = z^3(z-1)^2(z-\alpha)$$

and likewise define f_β. The key aspect of this construction will be to guarantee three distinct critical points of different orders.

By the proof of Belyi's Theorem, there is a polynomial f over $\mathbb{Q}$ so that $g_\alpha := f \circ f_\alpha$ and $g_\beta := f \circ f_\beta$ are Belyi maps (this polynomial f is the same since it is a composition of minimal polynomials of critical points, and α and β will share a common minimal polynomial).

Suppose that g_α and g_β define isomorphic tree dessins. This implies they are equal up to a Möbius transformation; fixing the pole at ∞, we have $g_\alpha(az+b) = g_\beta(z)$ for all z. By the cancellation lemma, it follows that $f_\alpha(az+b) = cf_\beta(z) + d$ for all z.

Since $cf_\beta(z)+d$ has the same critical points as f_β, we conclude that $az+b$ maps the critical points of f_α, $(0,1,\alpha)$, to the critical points of f_β, $(0,1,\beta)$, preserving order. Solving gives $a = 1, b = 0, \alpha = \beta$, which contradicts the assumption that σ acts nontrivially on α. Therefore, the assumption that g_α and g_β were isomorphic was invalid, and σ acts nontrivially on the set of tree dessins. □

In fact, one can show that the Galois action is faithful restricted to dessins of genus g for every choice of g (trees being a special case of genus 0). A proof can be found in [3], using more machinery of algebraic curves than would be suitable here. Later on, we will define and discuss regular dessins, which are the Galois objects in the category of dessins. The Galois action is *also* faithful restricted to this class of dessins, as a consequence of a relatively involved construction in [1].

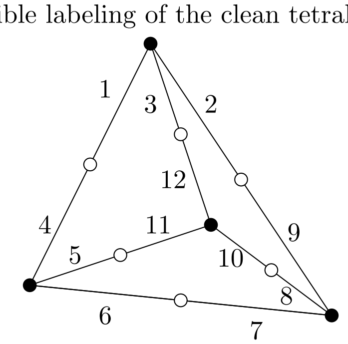
8. Cartographic and Automorphism Groups of a Dessin

Definition 8.1. Let D be a dessin, and label its n darts with $\{1, 2, \ldots, n\}$.

As a bigraph embedded into an oriented surface, we have a notion of how to rotate positively around each vertex. Let σ be the permutation in S_n obtained by rotating darts positively about each black vertex, and α be the permutation for rotating positively about each white vertex. These are well-defined permutations precisely because of the 2-coloring.

The *cartographic group* of D is the group $\langle \sigma, \alpha \rangle \subset S_n$.

Example 8.2. One possible labeling of the clean tetrahedron is given by:

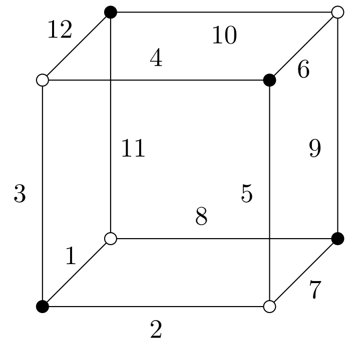


This choice of labels gives

$$\sigma = (1,2,3)(4,5,6)(7,8,9)(10,11,12)$$
$$\alpha = (1,4)(2,9)(3,12)(5,11)(6,7)(8,10)$$

This generates a subgroup of S_{12} isomorphic to A_4, with σ mapping to $(2,3,4)$ and α mapping to $(1,2)(3,4)$.

Another dessin with cartographic group A_4 is the 2-colored cube:



Here, our choice of labels gives

$$\sigma = (1, 2, 3)(4, 5, 6)(7, 8, 9)(10, 11, 12)$$

$$\alpha = (1, 11, 8)(2, 7, 5)(3, 4, 12)(6, 9, 10)$$

An isomorphism onto A_4 sends these to $(2, 3, 4)$ and $(1, 2, 3)$, respectively.

The definition of the cartographic group at first may seem rather abstract or unmotivated, but in fact this is one of the most natural constructions to associate to a dessin. One of the most important results of this section will be that the cartographic group embedded in S_n with designated generators σ and α contains all of the information of its dessin.

Theorem 8.3. *Let D be a finite set of size n equipped with permutations $\sigma, \alpha \in S_n$ that act transitively on D. Then we can associate a unique dessin D to D up to unique isomorphism, so that D is the set of darts of D and σ, α are the generators of its cartographic group.*

Proof. Using the interpretation of the cartographic group action, σ should decompose into cycles, where each cycle is rotation about a black vertex of D. In other words, given σ, we reconstruct the black vertices as the cycles in its cycle decomposition, including 1-cycles. Likewise, we get white vertices from the cycle decomposition of α, and faces from the cycle decomposition of $\phi = \alpha^{-1}\sigma^{-1}$. Furthermore, these decompositions tell us incidence relations, and we have that every dart appears in exactly one cycle of σ and α and ϕ (resp. has exactly one black and white vertex, and one "oriented" face). Therefore, we have a bigraph, and the condition that the permutations act transitively gives that the bigraph is connected. Together with the assignment of faces, this is a dessin.

Given two dessins with the given property, each has a bijection from darts onto D which is equivariant with respect to the cartographic action. Therefore, we can compose to get a bijection between the dart sets of each dessin, which is equivariant relative to the (shared) cartographic action. In particular, the cycle decompositions from the previous paragraph now induce bijections on vertices and faces, and thus a unique isomorphism of dessins. □

Corollary 8.4. *Consider the category of finite sets equipped with two permutations that act transitively and morphisms which are equivariant maps under the respective actions. This category is equivalent to the category of dessins, using the correspondence above.*

In graph theory, this would be a specialization of the more general notion of a k-constellation: a sequence of permutations $[g_1, \dots g_k]$ in S_n such that $g_1 \dots g_k = 1$ and $\langle g_1, \dots, g_k \rangle$ acts transitively on the set of n points. The constellation then is said to have cartographic group $\langle g_1, \dots, d_k \rangle \subset S_n$. A 3-constellation is sometimes instead called a (combinatorial) hypermap or dessin. We will use the term *hypermap*. The results we prove for hypermaps have immediate analogs for dessins by the above correspondence, and we will provide examples to assist in moving between the two.

Throughout, let D be an n-element hypermap with cartographic group $G = \langle \sigma, \alpha \rangle$, corresponding to the equivalent dessin D.

Proposition 8.5. *The group of automorphisms of D, $Aut(D)$, is the centralizer of G in S_n.*

Proof. This is exactly the requirement that an automorphism be a G-equivariant bijection from D to D. □

Note how much simpler the proof is for hypermaps, instead of dessins, where we would have to demonstrate that every element of the centralizer gives rise to a valid automorphism.

Remark 8.6. We define the automorphism group to act on D from the left, and the cartographic group to act on D from the right. For those familiar, this is analogous to the convention in algebraic topology that the automorphism group acts on a cover from the left and the monodromy group acts on a cover from the right. By the previous proposition, these actions commute: $(f \cdot d) \cdot \psi = f \cdot (d \cdot \psi)$ for all $f \in Aut(D), d \in D, \psi \in G$.

With this convention, we have access to a more interesting (albeit noncanonical) description of a hypermap.

Proposition 8.7. *Let D be a finite set with cartographic group $G = \langle \sigma, \phi \rangle$. Choose a basepoint $x \in D$, and let H be the stabilizer of x under the cartographic action.*

Then D is isomorphic to the right coset space $H \backslash G$ as a hypermap with cartographic group G endowed with its natural right action. The automorphism group of D acts freely and is isomorphic to $N(H)/H$ acting on $H \backslash G$ by left multiplication, where $N(H)$ is the normalizer of H in G.

Proof. If $g, g' \in G$ each take the dart x to y, then $g'g^{-1}$ stabilizes x, and thus g, g' lie in the same coset of H, which we identify with y. Then if h takes y to z, $(Hg)h = H(gh)$, which this bijection identifies to z, so the bijection is equivariant and we have an isomorphism.

Consider the map $H \backslash G \to H \backslash G$ sending $[k]$ to $[ak]$, where $a \in N(H), k \in G$, and square brackets denote a coset. This map is well-defined, since for any $h \in H$, $[ahk] = [aha^{-1}ak] = [ak]$. Then for any $g \in G$, the cartographic action gives $[ak]g = [a(kg)]$, so the left action of a defines a permutation of $H \backslash G$ which commutes with the cartographic action, i.e. an automorphism.

Thus, we have a homomorphism $N(H) \to Aut(D)$. On the other hand, an automorphism is determined by the image of x: if $f([1]) = [g]$, then $f([k]) = f([1]k) = [g]k = [gk]$ (the automorphism group acts freely). Thus, this map into the automorphism group is surjective, with kernel corresponding to H, since $[1]h = [1] \iff h \in H$, so $N(H)/H \cong Aut(D)$. □

Remark 8.8. Observe that the first result does not depend on G being exactly the cartographic group. If A is a group with a right action on D and an epimorphism $\phi : A \to G$ such that $\phi(a)d = ad$ for all $a \in A$, then D is isomorphic to $H\backslash A$, where H is the stabilizer of a point under the action of A (though we can no longer identify A to the cartographic group).

The cartographic group and the automorphism group are powerful Galois invariants, but we defer the proof for now; it will be much more straightforward with the insights from the next section.

9. Regular Dessins: A Galois Correspondence

We introduce regular dessins, which are dessins that satisfy stringent symmetry requirements that lead to very nice behavior of the cartographic and automorphism groups, as well as a Galois correspondence.

Definition 9.1. A *regular dessin* or *regular hypermap* satisfies that its automorphism group acts transitively on its dart set.

Example 9.2. Both the clean tetrahedron and the 2-colored cube are examples of regular dessins, with automorphism group A_4. These correspond to regular hypermaps with the same cartographic group but nonequivalent generators.

There are many equivalent formulations of regularity, stemming from the following observation.

Proposition 9.3. *The automorphism group acts transitively if and only if it is isomorphic to the cartographic group.*

Proof. If the two groups are isomorphic, then the result is immediate since the cartographic group acts transitively. Conversely, if the automorphism group acts freely and transitively, it has order n. Since $Aut(D) \cong N(H)/H \subset G$ and $|G| = n$, we get that $Aut(D) \cong G$. □

Corollary 9.4. *A dessin or hypermap is regular if and only if we can identify its dart set with the group G, so that the automorphism action and cartographic action are the left and right multiplication actions of G on itself.*

Referring to Proposition 8.7, perhaps it is not so surprising that we should be especially interested in hypermaps identified with $H\backslash G$, where H, the stabilizer of a point, is trivial. In fact, that notation is appropriately suggestive, as we will see that a hypermap $H\backslash G$ is obtained as a "quotient" of the regular hypermap G by automorphism group H. First, though, we see:

Definition 9.5. An *intermediate dessin* D of C satisfies that there is a morphism (continuous skeleton-preserving, color-preserving surjection) $\mathsf{C} \to \mathsf{D}$. An *intermediate hypermap* D of C satisfies that there is a morphism (G-equivariant, generator-preserving surjection) $C \to D$.

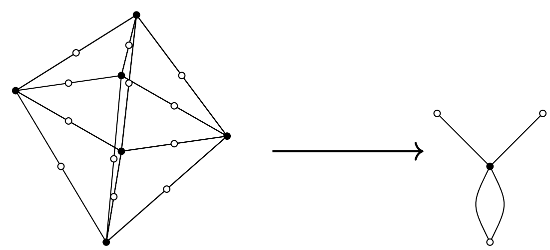
The intermediate objects of a given object are, intuitively, those that it covers. Every dessin or hypermap is covered by a regular object, of course, by Corollary 9.4 and Proposition 8.7. We will show how to construct this object, and that it must be uniquely minimal.

Theorem 9.6. *Let* D *be a dessin with cartographic group* G*. There exists a regular dessin* $\hat{\mathsf{D}}$ *with a morphism* $\hat{\mathsf{D}} \to \mathsf{D}$*. Moreover,* $\hat{\mathsf{D}}$ *can be chosen to be uniquely minimal in the sense that any morphism* $\mathsf{C} \to \mathsf{D}$ *factors through* $\hat{\mathsf{D}}$ *as* $\mathsf{C} \to \hat{\mathsf{D}} \to \mathsf{D}$*. We call this object the regular closure of* D*.*

Proof. Suppose D is a dessin corresponding to hypermap D with cartographic group G generated by σ, α. Immediately, we have that D is covered by G as a regular hypermap, as obtained in Corollary 9.4, with the natural map $G \to H\backslash G$. This induces a cover from the regular dessin corresponding to G onto D.

Suppose now that C is a regular hypermap with morphism $C \to D$. Then by Proposition 8.7 this is a C-equivariant map $C \to H\backslash G$, so we recover the action of G in the action of C, and G is a quotient group of C. This gives a natural morphism of hypermaps $C \to G$, as desired. □

Example 9.7. The clean octohedron is the regular closure of the dessin on the right, which has been sometimes named the "rabbit."



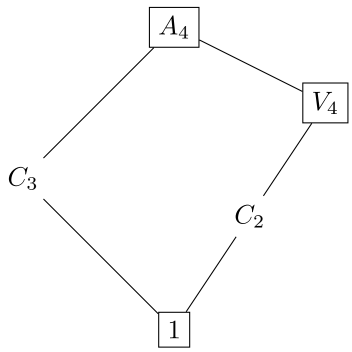
Remark 9.8. The regular closure is closely related to the Cayley graph C generated by σ, α. Specifically, its underlying bigraph is given by darts corresponding to vertices of C, and black (white) vertices corresponding to cycles of σ-labeled (α-labeled) edges.

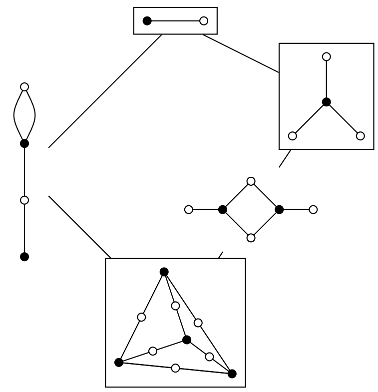
Theorem 9.9 (Galois Correspondence for Dessins)**.** *Let* D *be a regular dessin. Isomorphism classes of intermediate dessins of* D *are in bijection with conjugacy classes of subgroups of* $Aut(\mathsf{D})$*, mapping regular intermediate dessins to normal subgroups.*

Proof. This essentially restates what we have already seen. By Corollary 9.4, a regular hypermap can be understood as a group G acting on itself by left and right multiplication. If $[H]$ is the conjugacy class of a subgroup of G, then Proposition 8.7 gives that $H\backslash G$ is a well-defined intermediate hypermap of G. On the other hand, an intermediate hypermap D of G has an equivariant surjection $G \to D$, and thus corresponds to $H\backslash G$ for some H by the remark to Proposition 8.7. □

It is common to say a hypermap $H\backslash G$ is the quotient of the hypermap G by the automorphism subgroup H. By the correspondence above, we see that quotients are always well-defined.

Example 9.10. An illustrative example is given by the Galois correspondence for A_4 generated by $\langle \sigma, \alpha \rangle = \langle (2,3,4), (1,2)(3,4) \rangle$. Of course, as we saw previously, this is the correspondence for the regular dessin obtained from the tetrahedron.





On the left is the lattice of A_4 up to conjugacy, where lines indicate subgroup inclusion and boxes indicate normal subgroups of A_4. On the right is the Galois correspondence of the clean tetrahedron, where lines indicate existence of a morphism and boxes indicate regular dessins.

Theorem 9.11. *The Galois action on dessins preserves the automorphism group, regularity, regular closures, and the cartographic group.*

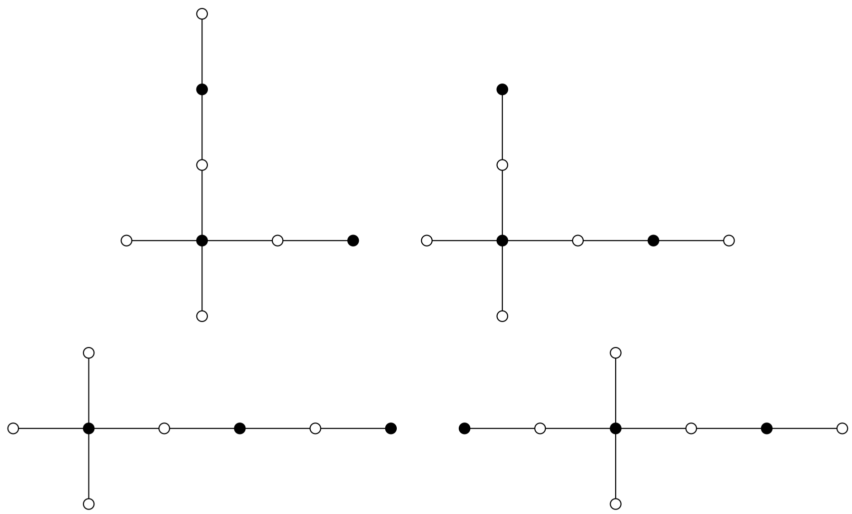
Proof. The Galois action on Belyi maps is functorial: it has an induced action on morphisms which preserves composition, identity, and inverses. Thus, $\lambda \in \Gamma$ induces a homomorphism of automorphism groups, which is an isomorphism since its inverse is induced by λ^{-1}.

Now, the Galois action preserves number of edges and automorphism group, so it maps a regular object (where the automorphism group acts transitively) to another regular object. Thus, it preserves when a morphism maps from a regular object to another object, and so we get that it preserves regular closures. Since the cartographic group of a dessin is the automorphism group of its regular closure, the Galois action preserves cartographic groups. □

Remark 9.12. The Galois action of λ takes a hypermap with underlying set D and generators σ, α to a hypermap with underlying set D and new generators $\sigma_\lambda, \alpha_\lambda$, which generate the same cartographic group. The action is nontrivial if the new generators cannot be obtained by inner automorphism of the old generators in the cartographic group. However, generating pairs which are not related by inner automorphism may lie in disjoint Galois orbits.

We finish with an example of using the cartographic group to separate orbits, from [1].

Example 9.13. Consider the tree dessins with black vertices of degrees 1, 2, and 4 and white vertices of degrees 1, 1, 1, 2, and 2. There are four such dessin, listed below:



Using only vertex degrees, these trees are all indistinguishable. However, both trees in the top row have cartographic group $PSL_2(7)$, while the trees in the second row have cartographic group A_7. Therefore, we cannot have one orbit, and as it happens, they split into two orbits as the pairs shown.

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