ENUMERATIVE GEOMETRY THROUGH INTERSECTION THEORY

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Abstract. Given two projective curves of degrees d and e, at how many points do they intersect? How many circles are tangent to three given general position circles? Given r curves $c_1, \ldots, c_r$ in $\mathbb{P}^3$, how many lines meet all r curves? This paper will answer the above questions and more importantly will introduce a general framework for tackling enumerative geometry problems using intersection theory.

Contents

1. Introduction

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Given r curves $c_1, \ldots, c_r$ in $\mathbb{P}^3$, how many lines meet all r curves? This paper is dedicated to solving enumerative geometry problems using intersection theory.

To do so, we will turn to intersection theory, specifically the Chow ring and its computation. The Chow ring is important in intersection theory since its multiplicative structure will encode the intersections of subvarieties of a given variety. With this tool, the enumerative geometry problems stated earlier can be solved using the following outline. First, we formulate the enumerative geometry problem as a problem about intersections in a parameter space. For this paper, the two parameter spaces of interest are projective space $\mathbb{P}^n$ and the Grassmanian $\mathbb{G}(1,3)$ parameterizing lines inside of $\mathbb{P}^3$. The second step is to compute the structure of the Chow rings of the parameter spaces. Finally, we will see that the enumerative geometry problems will reduce to computations inside the Chow ring. This will be a simple computation after the work of step two.

This paper will be split into four sections. The first section will briefly highlight the background on algebraic geometry that will be necessary for the remaining sections. The second section will introduce the Chow ring along with affine stratification, an important tool that will be used to compute Chow rings in nice cases. In the next section, we will compute the Chow rings of the spaces $\mathbb{P}^n$ and $\mathbb{G}(1,3)$. With this work done, we will be able to apply our results to solve enumerative geometry problems in the final section.

2. Background on Algebraic Geometry

For this section, let k be an algebraically closed field such as $\mathbb{C}$. For the purposes of this paper, it is ok to assume that $k = \mathbb{C}$, but most results will generalize to any algebraically closed field. This section will give a background on the necessary algebraic geometry needed in this paper. In particular, it will be brief in order to get to the more interesting material in the later sections. As such, some proofs will be omitted. More detailed treatments of this material can be found in [3],[4] and [6].

2.1. Affine Varieties and Zariski Topology.

An important aspect of algebraic geometry is the dictionary between geometric objects and algebraic objects. This section will explore some of these connections.

Our main geometric object of interest will be varieties which are zeros of polynomials.

Definition 2.1. Affine space over k, denoted by $\mathbb{A}^n$, is defined as the set of n-tuples with entries in k. In other words,

$$\mathbb{A}^n := k^n.$$

Definition 2.2. Let $I \subset k[x_1, \ldots, x_n]$ be an ideal. Define $V(I)$ to be the set of common zeros in $\mathbb{A}^n$ of the polynomials in I.

Definition 2.3. An affine algebraic variety is a subset $X \subset \mathbb{A}^n$ of the form $X = V(I)$ for some ideal $I \subset k[x_1, \ldots, x_n]$.

Remark 2.4. By Hilbert's Basis Theorem, $k[x_1, \ldots, x_n]$ is noetherian, i.e. every ideal is finitely generated, so we only need finitely many polynomials to define a variety.

Definition 2.5. Given a subset $X \subset \mathbb{A}^n$, the ideal of X, denoted by $I(X)$, is the set of all polynomials in $k[x_1, \dots x_n]$ which vanish on X.

The operations V and I give the first correspondence between algebra and geometry. This correspondence is given by Hilbert's Nullstellensatz.

Definition 2.6. Let I be an ideal. The radical of I is

$$\sqrt{I} := \{f \in I : f^n \in I\}.$$

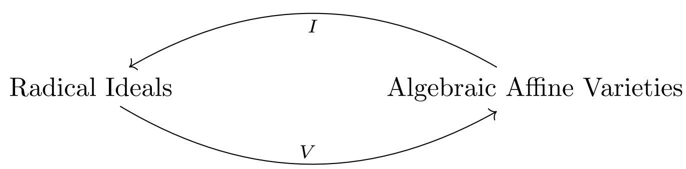
An ideal I is radical if

$$\sqrt{I} = I.$$

Theorem 2.7 (Nullstellensatz)**.** *Let J be an ideal of a commutative ring. Then,*

$$I(V(J)) = \sqrt{J}.$$

On the other hand, we always have that $V(I(X)) = X$, as long as X is an affine variety. We have the following correspondence of radical ideals on the algebraic side and affine varieties on the geometric side. The operation V takes a radical ideal and gives us a variety. Hilbert's Nullstellensatz tells us that the operation I is inverse to V in this bijective correspondence.



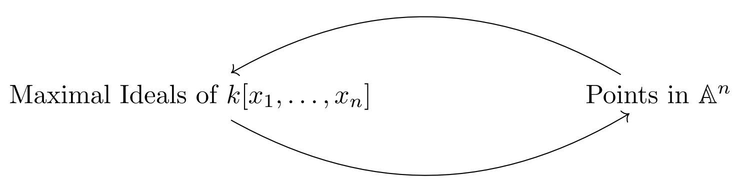
Another version of the Nullstellensatz gives us another correspondence, this time between maximal ideals and their geometric interpretation.

Theorem 2.8. *Let m be a maximal ideal of the ring $k[x_1, \dots, x_n]$. Then m is of the form*

$$m = (x_1 - a_1, \dots, x_n - a_n)$$

for some $(a_1, \dots, a_n) \subset \mathbb{A}^n$.

This version gives a bijective correspondence between maximal ideals and points.



Algebraic varieties form the closed sets of a topology on $\mathbb{A}^n$. The empty set is cut out by the polynomial $f(x) = 1$ and $\mathbb{A}^n$ is cut out by the polynomial $f(x) = 0$. Arbitrary intersections of closed sets are closed since they are given by the common zeros of the union of polynomials defining the closed sets. Finite unions of closed sets are closed by taking the common zeros of pairwise products of polynomials.

Definition 2.9. The topology given by setting the closed sets as algebraic varieties is called the Zariski topology on $\mathbb{A}^n$.

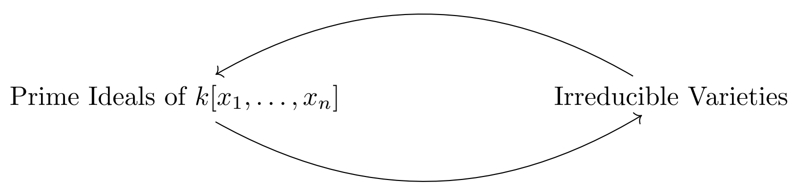
Remark 2.10. The Zariski topology is very different from the usual metric topology. In particular, the open sets in the Zariski topology are very big. For example, on $\mathbb{A}^1$, the Zariski topology is the cofinite topology.

Definition 2.11. A variety is irreducible if it cannot be written as the union of two proper zariski closed subsets.

There is another correspondence regarding irreducible varieties and their algebraic interpretation.

Proposition 2.12. *If P is a prime ideal of $k[x_1, \ldots, x_n]$, then $V(P)$ is an irreducible variety. On the other hand, if $X \subset \mathbb{A}^n$ is an irreducible variety, then $I(X)$ is a prime ideal.*

This leads to the following correspondence.



2.2. **Maps between varieties.**

Definition 2.13. Let $X \subset \mathbb{A}^n$ be an affine variety. A function f on X is regular if it is equal to some F restricted to X where $F \in k[x_1, ..., x_n]$.

Definition 2.14. The set of all regular functions on X is called the coordinate ring on X and is denoted by $k[X]$.

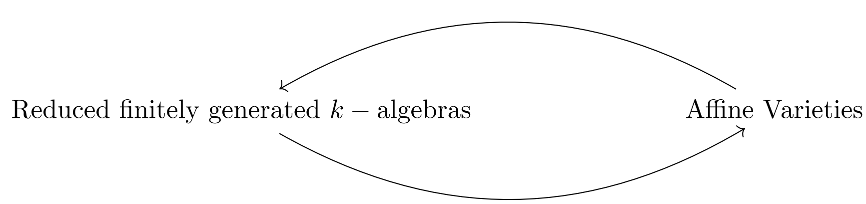
Theorem 2.15. *The coordinate ring $k[X]$ is isomorphic to $k[x_1, \ldots, x_n]/I(X)$.*

Proof. Consider the surjective homomorphism $\phi : k[x_1, \ldots, x_n] \to k[X]$ given by restricting the polynomials to X. The kernel is the ideal $I(X)$. The result is obtained by applying the first isomorphism theorem. □

Proposition 2.16. *The coordinate ring $k[x_1, \ldots, x_n]/I(X)$ has no nilpotent elements.*

Proof. Suppose that $f + I(X)$ is nilpotent. Then $f^n \in I(X)$ for some $n > 0$. By the Nullstellensatz, $I(X)$ is a radical ideal of $k[x_1, \ldots, x_n]$, so $f \in I(X)$. Then $f + I(X) = 0$. □

Thus, coordinate rings are reduced, i.e. no nonzero nilpotent elements, finitely generated k-algebras. This gives another correspondence.



Definition 2.17. Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be varieties. A regular map $f : X \to Y$ is a map such that $f(x) = (f_1(x), \ldots, f_m(x))$ where the f_i are regular functions on X.

Regular maps give us a way to compare varieties.

Definition 2.18. A map $f : X \to Y$ is an isomorphism if it is an invertible regular map. In this case, X and Y are isomorphic.

2.3. **Projective Varieties.** We will see that intersection theory in affine space is rather uninteresting. Thus, we will need projective space and projective varieties in order to analyze our proposed problems.

Definition 2.19. Projective space $\mathbb{P}^n$ is the set of points in $\mathbb{A}^{n+1} \setminus \{0\}$, under the equivalence relation that $(x_0, ..., x_n) \sim \lambda(x_0, ..., x_n)$, where $\lambda \in k^\times$. Points in $\mathbb{P}^n$ will be denoted with coordinates $(x_0 : ... : x_n)$ to emphasize that they are equivalent under scaling.

Projective space $\mathbb{P}^n$ is naturally covered by $n + 1$ affine patches, $A_0, \ldots, A_n$. Namely, take the subset of $\mathbb{P}^n$ where the first coordinate is not zero and call it A_0. Since points are equivalent under scaling, we can scale so that the first coordinate is 1. This is all the points with coordinates $(1 : x_1 : ... : x_n)$, which is isomorphic to $\mathbb{A}^n$ by mapping $(1 : x_1 : ... : x_n)$ to $(x_1, \ldots, x_n)$ in $\mathbb{A}^n$. The other A_i are defined similarly by taking subsets of $\mathbb{P}^n$ where the ith coordinate is nonzero. The affine patches will be important later on since they will be used to construct a stratification for $\mathbb{P}^n$. This stratification will be used to compute the Chow ring of $\mathbb{P}^n$.

A problem with defining varieties in projective space is that the zero sets of some polynomials are not well-defined. This is since points are equivalent under scaling. To fix this issue, we must only consider homogeneous polynomials.

Definition 2.20. A polynomial $f = \sum_k \alpha_k X_0^{i_{k,0}} \cdots X_n^{i_{k,n}}$ is homogeneous of degree d if $i_{k,0} + \cdots + i_{k,n} = d$ for all k.

Let $f \in k[x_0, \ldots, x_n]$ be a homogeneous polynomial of degree d. Then f has the desired property that

$$f(\lambda(x_0, \ldots, x_n)) = \lambda^d f(x_0, \ldots, x_n).$$

In particular, the zeros are invariant under scaling.

Definition 2.21. An ideal $I \subset k[x_0, \ldots, x_n]$ is homogeneous if it is generated by homogeneous polynomials.

With homogeneous ideals, the case of projective varieties follows in the same way as affine varieties. That is, a projective variety is of the form $V(I)$ where I is a homogeneous ideal. Similarly, we have a bijective relationship between projective varieties and radical homogeneous ideals not containing $(x_0, \ldots, x_n)$. Moreover, letting closed subsets be subvarieties $V(I) \subset \mathbb{P}^n$ defines the Zariski topology on $\mathbb{P}^n$.

Functions can be defined similarly for projective varieties. We have already defined regular functions for affine varieties. A regular projective function is simply one that is a regular affine function on every restriction to an affine patch.

Definition 2.22. Let $X \subset \mathbb{P}^n$ be a projective variety. A function $f : X \to k$ is regular if $f|_{A_i \cap X}$ is a regular function on every affine patch A_i.

Regular maps and isomorphisms can then be defined in exactly the same way as the affine case.

In many cases, we will need to use a weaker notion of a regular function. This is given by rational functions.

Definition 2.23. Let X be a variety. A rational function $f : X \dashrightarrow k$ is a regular function on a dense Zariski open subset $U \subset X$.

The rational functions on a projective variety $X \subset \mathbb{P}^n$ are as follows.

Proposition 2.24. *Let X be a projective variety. Any rational function, $f : X \dashrightarrow k$, is the restriction of a rational function $f(x) = \frac{g(x)}{h(x)}$ where $g, h \in k[x_0, \dots, x_n]$ such that g, h are homogeneous polynomials of the same degree d.*

The conditions on rational functions show that they are well defined since

$$f(\lambda x) = \frac{\lambda^d g(x)}{\lambda^d h(x)} = f(x).$$

Definition 2.25. Let $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$ be projective varieties. $f : X \dashrightarrow Y$ is a rational map if $f(x) = (f_0(x) : ... : f_m(x))$ where $f_0, \dots, f_m$ are regular functions on X.

Definition 2.26. A rational map $f : X \dashrightarrow Y$ is birational if f has an inverse that is also a rational map. In this case, X and Y are birationally equivalent.

Birational equivalence is a similar but weaker notion to that of an isomorphism. In particular, we know that there exists an open $U \subset X$ such that f restricted to U is defined everywhere and regular. Since a rational inverse exists, there exists an open $V \subset Y$ where the inverse defined everywhere and is regular. Then $f|_U : U \to V$ is a regular map that is defined everywhere and invertible. Thus two projective varieties X and Y are birationally equivalent if and only if they are generically isomorphic.

2.4. **Dimension.** In this section, we will give an algebraic formulation of dimension.

Definition 2.27. Let R be a ring and P be a prime ideal. The height of P is defined as the supremum of the length of ascending chains of unique prime ideals

$$P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n = P.$$

The dimension of R is the supremum of the heights of the prime ideals of R.

Definition 2.28. The dimension of an affine variety X is the dimension of the coordinate ring of X.

Remark 2.29. The height of a prime ideal $P \subset k[x_0, \dots, x_n]$ is the codimension of the variety $V(P)$ in $\mathbb{A}^n$.

An alternative formulation of dimension is as follows.

Proposition 2.30. *Let $X \subset \mathbb{A}^n$ be an irreducible variety and denote by $k(X)$ the field of fractions of its coordinate ring. The dimension of X is the transcendence degree of $k(X)$ over k.*

The algebraic formulations of dimension seem rather opaque. We will state a couple of examples to show that this definition does match our geometric intuition.

Example 2.31. Consider $\mathbb{A}^n$. The coordinate ring of $\mathbb{A}^n$ is $k[x_1, \ldots, x_n]$. The transcendence degree of this over k is n. Thus, $\mathbb{A}^n$ has dimension n.

Example 2.32. Suppose $X \subset \mathbb{A}^n$ is an irreducible variety of dimension 0. This means that $k(X)$ has transcendence degree 0 over k, and k is algebraically closed, so $k(X) \cong k$. The coordinate ring $k[X]$ embeds into $k(X) \cong k$ and $k[X]$ also contains a copy of k (the constant functions), thus $k[X] \cong k$. Then $k[x_1, \ldots, x_n]/I(X) \cong k$, so $I(X)$ is maximal. This shows that X is a point by the correspondence between maximal ideals and points. Thus, the irreducible dimension 0 varieties are points.

Example 2.33. In the first definition, the chains of prime ideals

$$P_0 \subsetneq \cdots \subsetneq P$$

correspond to sequences of irreducible varieties

$$V(P_0) \supsetneq \cdots \supsetneq V(P).$$

Thus, the dimension of a variety X measures the maximum number of restrictions to irreducible subvarieties to get from X to a point.

There is an important theorem regarding dimension.

Theorem 2.34 (Principal ideal theorem)**.** *Let X be a variety and $k[X]$ be its coordinate ring. If $f \in k[X]$ is not a zero divisor or constant, then the minimal prime ideal containing f has height one.*

The principal ideal theorem has an important consequence.

Proposition 2.35. *Let f be a nonconstant irreducible polynomial in $k[x_1, \ldots, x_n]$. Then $V(f) \subset \mathbb{A}^n$ is a variety of dimension $n-1$.*

Geometrically, we can think of this as cutting by a hypersurface reduces dimension by one.

We have defined dimension in the case of affine varieties, but we would also like the notion of dimension for projective varieties as well. Fortunately, the projective case follows easily from the affine case.

Definition 2.36. Let $X \subset \mathbb{P}^n$ be a projective variety and U be an affine patch of $\mathbb{P}^n$. Then $X \cap U$ is an affine variety, so $\dim(X \cap U)$ is defined. Define

$$\dim(X \cap U) := \max_U \dim(X \cap U),$$

where U is a affine patch of $\mathbb{P}^n$.

Remark 2.37. The principal ideal theorem and its consequence also hold for projective varieties since it holds for all restrictions to an affine patch.

2.5. **Riemann-Hurwitz.** In this section, we will always work with $\mathbb{C}$ as the base field. Riemann-Hurwitz gives a formula relating genus and ramification degree. We first define ramification degree.

Let $f : X \to Y$ be a finite map between smooth curves. Let B be the coordinate ring of an affine neighborhood of a point $p \in X$ and A be the coordinate ring of an affine neighborhood of $f(p) \in Y$. The point p defines a maximal ideal m_p. Define

$$\mathcal{O}_{X,p} := B_{m_p},$$

to be the localization of the ring B at m_p. We define $\mathcal{O}_{Y,f(p)}$ similarly. The map f induces a map $f^{\#}$, where

$$f^{\#} : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}.$$

The local rings $\mathcal{O}_{X,p}$ and $\mathcal{O}_{Y,f(p)}$ are discrete valuation rings, and in particular are principal ideal domains. Thus, the ideals m_p and $m_{f(p)}$ are generated by some s and t respectively.

Definition 2.38. Let v be the discrete valuation on the DVR $\mathcal{O}_{X,p}$. We define the ramification degree e_p to be

$$e_p = v(f^{\#}(t)).$$

Theorem 2.39 (Riemann-Hurwitz Formula)**.** *Let $f : X \to Y$ be a finite map of curves and $n = \deg(f)$. Define*

$$\deg R = \sum_{P \in X} (e_P - 1).$$

Then,

$$2g(X) - 2 = n(2g(Y) - 2) + \deg R.$$

3. Basics of Intersection Theory

3.1. **The Chow Ring.** We start by defining the group of cycles of an algebraic variety X. Under rational equivalence, this group forms the Chow group. An additional operation can be defined on this group giving it a ring structure. This operation will be based on intersections. As a result, the Chow ring will encode the structure of intersections of the variety X.

Definition 3.1. Let X be an algebraic variety. The group of cycles $Z(X)$ on X is the free abelian group generated by subvarieties of X. i.e. it can be viewed as finite linear combinations $\sum_i \alpha_i Y_i$, where Y_i are subvarieties of X.

Remark 3.2. $Z(X)$ is graded by codimension. That is, $Z(X) = \bigoplus Z^k(X)$, where $Z^k(X)$ is the group of cycles generated by subvarieties of codimension k.

In order to turn the group of cycles $Z(X)$ into the Chow group on X, we must first define rational equivalence.

Definition 3.3. Let $t_0, t_1 \in \mathbb{P}^1$ and Φ be a subvariety of $\mathbb{P}^1 \times X$ which is not contained in any fiber $\{t\} \times X$. Define $\mathrm{Rat}(X)$ to be the subgroup generated by

$$\langle \Phi \cap \{t_0\} \times \mathbb{P}^1 \rangle - \langle \Phi \cap \{t_1\} \times \mathbb{P}^1 \rangle.$$

Given two cycles $Z_1, Z_2 \in Z(X)$, Z_1 and Z_2 are rationally equivalent if $Z_1 - Z_2 \in \mathrm{Rat}(X)$.

The definition of rational equivalence given in Definition 3.3 is rather difficult to understand. It may be better visualized by the image below.

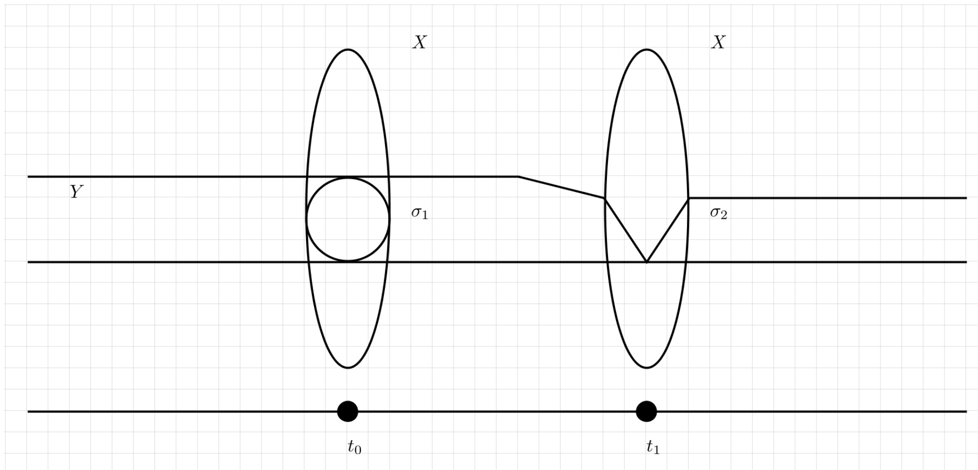


FIGURE 1. The two cycles σ_1 and σ_2 are rationally equivalent since there exists a subvariety $Y \subset \mathbb{P}^1 \times X$ such that its restriction to $\{t_0\} \times X$ is σ_1 and its restriction to $\{t_1\} \times X$ is σ_2.

The two cycles σ_1 and σ_2 are rationally equivalent if there exists a subvariety $Y \subset \mathbb{P}^1 \times X$ not contained in any fiber such that Y restricted to two fibers $\{t_0\} \times X$ and $\{t_1\} \times X$ are σ_1 and σ_2 as illustrated. In other words, two cycles are rationally equivalent if we can algebraically deform one into the other.

Definition 3.4. The classes of cycles under rational equivalence forms the Chow group on X, denoted by $A(X)$.

To give the Chow group a ring structure, we need to define a new multiplication operation which will be based on intersections, hence its importance in intersection theory. This operation will be of use when using the Chow ring to study problems in enumerative geometry. The existence and well-definedness of multiplication follows from the moving lemma and transverse intersections.

Definition 3.5. Let X be a variety and $p \in X$. The point p corresponds to a maximal ideal m_p. Define the tangent space of X at p, T_pX, to be

$$T_pX := \left(\frac{m_p^2}{m_p}\right)^{\star} = \hom\left(\frac{m_p^2}{m_p}, k\right).$$

Suppose that $p \in Z \subset X$ where Z is a subvariety of the variety X. This gives a map $k[Z] \twoheadleftarrow k[X]$ which induces a map $\frac{m_{Z,p}^2}{m_{Z,p}} \twoheadleftarrow \frac{m_{X,p}^2}{m_{X,p}}$. By taking duals, we get the inclusion,

$$T_pZ \hookrightarrow T_pX.$$

Definition 3.6. Let Y and Z be subvarieties of a variety X. Then Y and Z intersect transversely at a point p if X, Y and Z are smooth at p, and the tangent spaces of Y, Z at p span the tangent space of X at p. That is,

$$T_pY + T_pZ = T_pX.$$

Theorem 3.7. *(Moving Lemma) Let X be a smooth projective variety. Then*

1. *Given two classes, α,β, in $A(X)$, there exists representatives $\alpha = [A]$,$\beta = [B]$, such that the subvarieties in the cycle A and the subvarieties in the cycle B are transverse in all pairwise intersections.*
2. *$[A \cap B]$ does not depend on the choice of representatives A and B.*

Proof. The proof of the Moving Lemma is difficult. The proof can be found in [2]. □

The moving lemma provides the structure for multiplication in $A(X)$. Given two elements $\alpha, \beta \in A(X)$, define $\alpha\beta = [A \cap B]$, where A, B are transverse representatives of α, β as seen in the moving lemma. This result establishes the Chow ring also denoted by $A(X)$.

The Chow ring is a graded ring, inheriting the graded structure from the graded structure of the group of cycles shown earlier in remark 3.2.

Proposition 3.8. *$A(X)$ is a graded ring written as direct sum $\bigoplus A^r(X)$, where $A^r(X) = Z^r(X)/Rat(X)$.*

Proof. We need to show that $\mathrm{Rat}(X)$ is homogeneous. That is, if σ_1 and σ_2 are two rationally equivalent cycles, then they have the same dimension. Since they are rationally equivalent, there exists a subvariety $Y \subset \mathbb{P}^1 \times X$ such that $Y \cap \{t_0\} \times X = \sigma_1$ and $Y \cap \{t_1\} \times X = \sigma_2$. Notice that $Y \cap \{t_0\} \times X$ is the zero of the polynomial $t - t_0$ in Y , i.e. it is cut out by a hyperplane in Y. The principal ideal theorem then gives that $\dim(Y \cap \{t_0\} \times X) = \dim(Y) - 1$, so $\dim(\sigma_1) = \dim(Y \cap \{t_0\} \times X) = \dim(Y) - 1$. Similarly, $\dim(\sigma_2) = \dim(Y) - 1$, so these dimensions are equal. This shows that $A^r(X)$ is well-defined. We also need to check that $A^r(X) \times A^s(X) \to A^{r+s}(X)$. This follows from the fact that the transverse intersection of varieties of codimension r and s has codimension $r + s$. □

A simple example of a Chow ring that can be computed is the Chow ring of affine space $\mathbb{A}^n$.

Theorem 3.9. *$A(\mathbb{A}^n) = \mathbb{Z}[\mathbb{A}^n]$. That is, the Chow ring of $\mathbb{A}^n$ is freely generated by the class $[\mathbb{A}^n]$ over $\mathbb{Z}$.*

Proof. We will show that for every proper subvariety Y of $\mathbb{A}^n$, Y is rationally equivalent to the empty set. We can pick coordinates $z_1, ..., z_n$ on $\mathbb{A}^n$ so that Y doesn't contain the origin. Consider the set

$$W^\circ = \{(t, tz) \subset (\mathbb{A}^1 \setminus \{0\}) \times \mathbb{A}^n : z \in Y\}.$$

The fiber above t is tY. $Y \cong tY$ as varieties by the isomorphism $y \mapsto ty$. Since each fiber is Y, it follows that W° is an irreducible variety. Let W be the closure of W° in $\mathbb{P}^1 \times \mathbb{A}^n$. This is also an irreducible variety since W° is an irreducible variety. W is not contained in a single fiber. We want to show that W deforms Y into the empty set. We see that the W restricted to the fiber over 1 is Y. We want to show that W restricted to the fiber over ∞ is the empty set.

We first show that

$$W^\circ = V(\{f(z/t) : f(z) \text{ vanishes on } Y\}).$$

If $(t, tz) \in W^\circ$, then z is in Y, so it is a zero of some polynomial f which defines Y. This shows that tz is the zero of $f(z/t)$. For the other direction, if $f(z/t)$ is a zero where f vanishes on Y, then we can associate this with (t, tz) in the original definition of W°.

Let $g(z)$ be a polynomial vanishing on Y. Since Y doesn't contain the origin, we can suppose that $g(z)$ has a nonzero constant term. Consider $g(z/t)$ where $t = \infty$ (in W taking the closure of W°). This is $g(0)$ which is some non-zero constant c, which doesn't vanish, so W is empty on the fiber above ∞. Thus, W is a subvariety of $\mathbb{P}^1 \times \mathbb{A}^n$ such that its restriction over 1 is Y and its restriction over ∞ is empty, so Y is rationally equivalent to $[\emptyset]$. $\square$

In particular, $A(\mathbb{A}^n) \cong \mathbb{Z}$ since $[\mathbb{A}^n]^2 = [\mathbb{A}^n]$. This shows that the Chow ring of affine space is not very interesting. Computing the more interesting Chow rings of projective space and Grassmanians will require a further technique of affine stratification which will be described in the next section.

3.2. **Affine Stratification.** Affine stratification is an important tool that will be used to compute the Chow rings of more interesting spaces.

Definition 3.10. Let X be a variety and $U \subset X$ be a subvariety. The subvariety U is locally closed if there exists an open $V \subset X$ such that U is closed in the subspace topology on V.

Definition 3.11. Let X be a variety. A stratification of X is a finite collection of irreducible, locally closed subvarieties U_i, with the property that if $\overline{U_i} \cap U_j \neq \emptyset$, then $U_j \subset \overline{U_i}$, such that $X = \sqcup_i U_i$.

Definition 3.12. If X has a stratification $U_1, ..., U_n$, the sets U_i are called the strata and their closures, $\overline{U_i}$ are the closed strata. A stratification of X is affine if all U_i in the strata are isomorphic to an affine space $\mathbb{A}^{m_i}$ for some m_i.

Example 3.13. A simple example of an affine stratification is with $\mathbb{P}^n$. Recall that $\mathbb{P}^n$ can be covered by $n+1$ affine patches. Let U_0 be the first affine patch given by the set of $x \in \mathbb{P}^n$ where $x_0 \neq 0$. Given $(x_0 : ... : x_n) \in U_0$, we see that it can be written uniquely as $(1, x_1, \dots, x_n)$, by dividing by x_0. This gives an isomorphism between U_0 and $\mathbb{A}^n$. Now consider $\mathbb{P}^n \setminus U_0$. This is exactly the set of points with coordinates $(0 : x_1 : ... : x_n)$, where not all x_i are zero and the coordinates are again equivalent under scaling. In other words, $\mathbb{P}^n \setminus U_0$ is isomorphic to $\mathbb{P}^{n-1}$, by disregarding the first coordinate. Similarly, we can let U_1 be the set of points in $\mathbb{P}^{n-1}$, where the first coordinate is nonzero, which is the first affine patch of $\mathbb{P}^{n-1}$. This is similarly isomorphic to $\mathbb{A}^{n-1}$. Continuing this process, $\mathbb{P}^n$ can be written as a disjoint union of $\mathbb{A}^n, \mathbb{A}^{n-1}, \dots, \mathbb{A}^0$.

Each $\mathbb{A}^i$ is an irreducible, locally closed subvariety of $\mathbb{P}^n$, and $\overline{\mathbb{A}^i} = \mathbb{P}^i = \mathbb{A}^0 \cup \cdots \cup \mathbb{A}^i$, so this collection satisfies all the properties of an affine stratification.

We will see that if a variety has an affine stratification, the structure of the Chow group will be surprisingly simple. To show this, we will need the following theorem.

Theorem 3.14. *(Excision) Let X be a variety, $Y \subset X$ be a closed subvariety and $U = X \setminus Y$. Then the following sequence is right exact*

$$A(Y) \longrightarrow A(X) \longrightarrow A(U) \longrightarrow 0$$

where the maps are inclusion and restriction respectively.

Proof. First notice that the definition of the Chow group is equivalent to the following sequence being right exact

$$Z(\mathbb{P}^1 \times X) \xrightarrow{\partial_X} Z(X) \longrightarrow A(X) \longrightarrow 0$$

where the map ∂_X maps all subvarieties $\Phi \in \mathbb{P}^1 \times X$ contained in a fiber to 0 and to

$$\langle \Phi \cap \{t_0\} \times \mathbb{P}^1 \rangle - \langle \Phi \cap \{t_1\} \times \mathbb{P}^1 \rangle$$

otherwise.

This is since being right exact gives that

$$A(X) \cong \operatorname{coker}(\partial_X) \cong Z(X)/\operatorname{image}(\partial_X).$$

By construction, the image of ∂_X is exactly $\operatorname{Rat}(X)$, so this gives that $A(X)$ is the Chow group.

We have the following commutative diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z(Y \times \mathbb{P}^1) & \longrightarrow & Z(X \times \mathbb{P}^1) & \longrightarrow & Z(U \times \mathbb{P}^1) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_Y} & & \downarrow{\scriptstyle \partial_X} & & \downarrow{\scriptstyle \partial_U} & & \\
0 & \longrightarrow & Z(Y) & \xrightarrow{\alpha_1} & Z(X) & \xrightarrow{\alpha_2} & Z(U) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi_Y} & & \downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \pi_U} & & \\
 & & A(Y) & \xrightarrow{\beta_1} & A(X) & \xrightarrow{\beta_2} & A(U) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

From the new formulation of the Chow group, we have that all columns are exact. The maps along the rows are given by inclusion and restriction respectively. The first two rows are exact. For instance, it is clear that α_1 is injective and α_2 is surjective. Moreover, $\operatorname{image}(\alpha_1) = \ker(\alpha_2)$, since a cycle restricts to 0 on U if and only if it is supported on Y. Similarly, the first row is also exact. We would like to show that the third row is right exact.

We start by showing that β_2 is surjective. Take $x \in A(U)$. By surjectivity of π_U there exists $y \in Z(U)$ such that $\pi_U(y) = x$. By surjectivity of α_2, there exists $z \in Z(X)$ so that $\alpha_2(z) = y$. Consider $\pi_X(z)$. By commutativity, $\beta_2(\pi_X(z)) = \pi_U(\alpha_2(z)) = x$. This shows surjectivity. A similar diagram chase shows that $\operatorname{image}(\beta_1) = \ker(\beta_2)$. □

The following proposition gives an important tool in computing the Chow ring of certain spaces with an affine stratification. It will be used in computing the Chow ring of both $\mathbb{P}^n$ and the Grassmanian $\mathbb{G}(1,3)$.

Proposition 3.15. *Let X be a variety with an affine stratification $U_0, ..., U_n$. Then the Chow ring, $A(X)$, is generated by the classes $[\overline{U_i}]$.*

Proof. We will prove this using induction on the number of strata. If there is only one stratum, then X is isomorphic to affine space. The proposition then follows from the computation of the Chow ring of affine space.

Now suppose we have an affine stratification $U_0, ..., U_n$. Without loss of generality, let U_0 be the minimal stratum. Since the closure is a union of strata, it must be that the closure is itself, so U_0 is closed. Then, let $U = X \setminus U_0$. By the previous theorem, we have the sequence

$$A(U_0) \longrightarrow A(X) \longrightarrow A(U) \longrightarrow 0$$

which is right exact. By the inductive hypothesis, $A(U)$ is generated by the classes of the closed strata of $U_1, ..., U_n$. The Chow group $A(U_0)$ is generated by the closure of the class of U_0, since U_0 is affine. By definition of the restriction map, the map $A(X) \to A(U)$ sends $[\overline{U_i}] \mapsto [\overline{U_i}]$, the generators of $A(U)$. Thus, the sequence shows us that $A(X)$ is generated by the class $[\overline{U_0}]$ from $A(U_0)$ and the classes $[\overline{U_1}], \ldots, [\overline{U_n}]$ from $A(U)$. □

A stronger statement is given by Totaro. The proof is difficult. A source for the proof is [7].

Theorem 3.16. *(Totaro) Let X be a variety with an affine stratification. Then the classes of the strata form a basis of $A(X)$.*

4. Examples

4.1. **Chow Ring of $\mathbb{P}^n$.** The first interesting example of a Chow ring we present is of projective space $\mathbb{P}^n$.

Theorem 4.1. *Let X be a subvariety of $\mathbb{P}^n$ with codimension k and degree d. Then the class $[X]$ in the Chow ring $A(\mathbb{P}^n)$ is $d\xi^k$ where ξ is the class of a hyperplane.*

Proof. Take the affine stratification of $\mathbb{P}^n$ as shown in Example 3.13. Using Theorem 3.16, we see that $A^k(\mathbb{P}^n)$ has basis $\mathbb{P}^{n-k}$ which is a $(n-k)$ plane $L \subset \mathbb{P}^n$. The plane L is the transverse intersection of k hyperplanes so $[L] = \xi^k$. The subvariety X intersects a general k plane transversely in d points since it has degree d, so $[X]\xi^{n-k} = d[p]$ where p is a point. Then

$$[X]\xi^n = d\xi^k[p]$$

so

$$[X][p] = d\xi^k[p]$$

since an intersection of n hyperplanes is a point. Thus, we see that

$$[X] = d\xi^k.$$ □

Corollary 4.2. *The Chow ring of $\mathbb{P}^n$ is*

$$A(\mathbb{P}^n) = \mathbb{Z}[\xi]/\xi^{n+1},$$

where ξ is the class of a hyperplane.

4.2. **Chow Ring of the Grassmanian $\mathbb{G}(k, n)$.**

Definition 4.3. The Grassmanian $\mathbb{G}(k, n)$ will denote the space of $k-$dimensional subspaces of $\mathbb{P}^n$.

We first describe special cycles called Schubert cycles.

Definition 4.4. Given an n dimensional variety X, a complete flag in X is an increasing sequence of linear subvarieties, $0 \subset V_0 \subset \cdots \subset V_n = X$, where dim $V_i = i$.

Definition 4.5. Fix a complete flag, $\mathcal{V}$ on $\mathbb{P}^n$. Take any sequence $a = (a_1, \dots, a_k)$ of integers where $n - k \geq a_1 \geq a_2 \geq \cdots \geq a_k \geq 0$. Define

$$\Sigma_a(\mathcal{V}) := \{\Lambda \in \mathbb{G}(k,n) : \dim(V_{n-k+i-a_i} \cap \Lambda) \geq i \ \forall i\}.$$

These Σ_a are called Schubert cycles. Define

$$\sigma_a := [\Sigma_a].$$

These are the classes of the Schubert cycles in the Chow ring, called Schubert classes. Note that this is well-defined since the class does not depend on the flag chosen. This is since we can linearly interpolate between any two choices of flags.

Theorem 4.6. *The Schubert classes σ_a generate the Chow ring of the Grassmanian $\mathbb{G}(n,k)$.*

Theorem 4.6 together with Totaro's Theorem (Theorem 3.16) gives an explicit formulation of the composition of the Chow ring of Grassmanians. In this paper, we will only state the proof of the case $\mathbb{G}(1,3)$, the space of lines in $\mathbb{P}^3$.

Fix a flag $p \in L \subset H \subset \mathbb{P}^3$, where p is a point, L is a line and H is a plane.

The Schubert cycles are of the form

$$\Sigma_{a,b} = \Big\{\Lambda \in \mathbb{G}(1,3) : \Lambda \text{ meets } (2-a) \text{ plane at a point and } (3-b) \text{ plane at a line}\Big\}.$$

We can directly compute each case in $\mathbb{G}(1,3)$ giving the following six cycles.

(1) $\Sigma_{0,0} = \mathbb{G}(1,3)$
(2) $\Sigma_{1,0} = \{\Lambda : \Lambda \cap L \neq \emptyset\}$
(3) $\Sigma_{2,0} = \{\Lambda : p \in \Lambda\}$
(4) $\Sigma_{1,1} = \{\Lambda : \Lambda \subset H\}$
(5) $\Sigma_{2,1} = \{\Lambda : p \in \Lambda \subset H\}$
(6) $\Sigma_{2,2} = \{L\}$

Remark 4.7. If the second index in a Schubert cycle is 0, we will leave it out in the notation. For example,

$$\Sigma_1 := \Sigma_{1,0}.$$

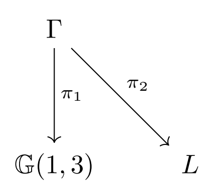
We would like to take the class of each Schubert cycle. In order to do so, we must check that the Schubert cycles are indeed cycles and thus belong in the Chow ring. We will do this by showing that each cycle is an irreducible variety. Moreover, the fact that Schubert cycles are irreducible varieties will also be important since they will be used to form an affine stratification.

Proposition 4.8. *The Schubert cycle $\Sigma_{1,0}$ is an irreducible variety.*

Proof. Consider the correspondence

$$\Gamma = \{(L', p) \in \mathbb{G}(1,3) \times L : p \in L'\}.$$

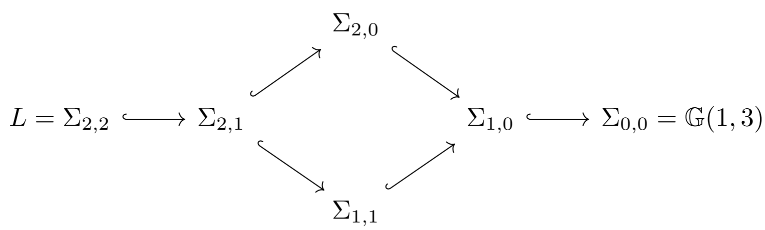
This correspondence comes with the projections



We first look at the projection π_2 to L. Each fiber $\pi_2^{-1}(p')$ is the set of lines through p' which is isomorphic to $\mathbb{P}^2$. Since all fibers are isomorphic to $\mathbb{P}^2$ and the base L is irreducible, Γ is an irreducible variety. Notice that the image of π_1 is $\Sigma_{1,0}$. The projection π_1 is a birational map to its image since given a line L' intersecting L, we can map it back to (L', p) where p is the point of intersection unless $L' = L$. Thus, the image $\Sigma_{1,0}$ is an irreducible variety since Γ is an irreducible variety. □

The proof for the other Schubert cycles follows similarly.

We would now like to find an affine stratification of $\mathbb{G}(1,3)$. The Schubert cycles have natural inclusions illustrated below.



Definition 4.9. Define $\Sigma^0_{a,b}$ to be $\Sigma_{a,b}$ removing the other Schubert cycles that come before it in the diagram of inclusions. In other words,

$$\Sigma^0_{a,b} = \Sigma_{a,b} \setminus \bigcup_{(c,d)>(a,b)} \Sigma_{c,d},$$

where $(c,d) \geq (a,b)$ if $c \geq a$ and $d \geq b$.

By construction, $\mathbb{G}(1,3)$ is the disjoint union of the $\Sigma^0_{a,b}$. Moreover, from the previous proposition, each $\Sigma^0_{a,b}$ is an irreducible subvariety. It follows that they form a stratification of $\mathbb{G}(1,3)$. To show that it is an affine stratification, we need to show that each piece is isomorphic to affine space.

Proposition 4.10. *Σ^0_1 is isomorphic to $\mathbb{A}^3$.*

Proof. Let H' be a general plane that contains p but not L. Consider the map

$$\Sigma^0_1 \to (L \setminus \{p\}) \times (H' \setminus H \cap H')$$
$$\Lambda \mapsto (\Lambda \cap L, \Lambda \cap H').$$

This is an isomorphism since every line in Σ^0_1 intersects both $L \setminus \{p\}$ and $H' \setminus H \cap H'$ uniquely at one point, and moreover, given a point in $(L \setminus \{p\}) \times (H' \setminus H \cap H')$, the line between the two is contained in Σ^0_1. The space $L \setminus \{p\}$ is a projective line minus a point, which is isomorphic to the affine line. Similarly, $H' \setminus H \cap H'$ is a projective plane minus a projective line which is the same as the affine plane. Thus, $(L \setminus \{p\}) \times (H' \setminus H \cap H') \cong \mathbb{A}^1 \times \mathbb{A}^2 \cong \mathbb{A}^3$. □

Similar arguments show that the other strata are also affine. It follows that $\mathbb{G}(1,3)$ is generated by the classes of the closures of $\Sigma^0_{a,b}$, which are the classes $\sigma_{a,b}$. Thus, we have determined the group structure of $A(\mathbb{G}(1,3))$.

Theorem 4.11. *$A(\mathbb{G}(1,3))$ is generated by the classes $\sigma_0, \sigma_1, \sigma_2, \sigma_{1,1}, \sigma_{2,1}$ and $\sigma_{2,2}$.*

Remark 4.12. A Schubert cycle $\Sigma_{a,b}$ has codimension $a+b$. Thus, the graded component $A^k(\mathbb{G}(1,3))$ is generated by the $\sigma_{a,b}$ where $a+b=k$.

We would also like to figure out the multiplicative structure of this Chow ring. The multiplicative structure is determined by the following relations.

Theorem 4.13. *The Schubert classes $\sigma_{a,b}$ in $A(\mathbb{G}(1,3))$ satisfy the following relations*

- $\sigma_1^2 = \sigma_{1,1} + \sigma_2$
- $\sigma_1\sigma_{1,1} = \sigma_1\sigma_2 = \sigma_{2,1}$
- $\sigma_1\sigma_{2,1} = \sigma_{2,2}$
- $\sigma_{1,1}^2 = \sigma_2^2 = \sigma_{2,2}$
- $\sigma_{1,1}\sigma_2 = 0.$

Moreover, all relations in $A(\mathbb{G}(1,3))$ are generated from the above five relations.

Proof. First we fix two flags $\mathcal{V}$ and $\mathcal{V}'$, with corresponding Schubert cycles $\Sigma_{a,b}$ and $\Sigma'_{a,b}$. We will compute σ_2^2. By the multiplication structure of the Chow ring, this is the class $[\Sigma_2 \cap \Sigma'_2]$. This is in the codimension four part of the Chow ring, so it is of the form $c\sigma_{2,2}$, for a constant c. The Schubert class $\sigma_{2,2}$ is the class of a point in $\mathbb{G}(1,3)$, so c is the number of points in $\Sigma_2 \cap \Sigma'_2$. The intersection $\Sigma_2 \cap \Sigma'_2$ consists of the lines that contain the point p in the flag $\mathcal{V}$ and p' in the flag $\mathcal{V}'$. There is exactly one line through p and p', so $c = 1$.

The other relations are shown in a similar manner. These five relations completely characterize multiplication in $A(\mathbb{G}(1,3))$ since the product of two cycles with codimension greater than four is sent to 0. For example, $\sigma_{2,2}\sigma_2 = 0$. □

In the next section, we will give another method of computing inside the Chow ring of Grassmanians. This method will follow a simple rule and does not require memorizing relations. Moreover, it generalizes to arbitrary Grassmanians where in general the relations are hard to determine.

4.3. **Symmetric Functions and the computation of $A(\mathbb{G}(k,n))$.** The basis of the computation of $A(\mathbb{G}(k,n))$ revolves around symmetric functions. As such, we take a quick detour to summarize the necessary definitions and results about symmetric functions and their relationship with Schubert classes. Some results are stated without proof. A good reference for the proofs and symmetric functions in general is [5], Steven Sam's notes on symmetric functions.

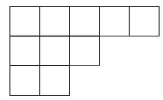
Definition 4.14. A partition of a natural number n is a weakly decreasing sequence $\lambda = (\lambda_1, ..., \lambda_k)$ where $\lambda_1 + \cdots + \lambda_k = n$.

Definition 4.15. Given a partition λ, we can define a Young's diagram, given by a left justified array of boxes with the number of boxes in the ith row corresponding to λ_i.

Definition 4.16. A semistandard Young's Tableaux, is a numbered Young's diagram, where the numbers are weakly increasing in the rows and strictly increasing in the columns.

Definition 4.17. Given a semistandard Young's Tableaux T, with numbers $1, ..., k$ occuring $n_1, ..., n_k$ times in the tableaux, we can define a polynomial x^T given by $x_1^{n_1} \cdots x_k^{n_k}$.

Example 4.18. For an example, take the partition $(5, 3, 1)$. It has the Young's diagram



An example of a semistandard Young's Tableaux of this partition is

$T =$

1	1	2	4
2	3	3	
4			

Its corresponding polynomial is

$$x^T = x_1^2 x_2^2 x_3^2 x_4^2.$$

Definition 4.19. A Schur polynomial given by a partition λ is the polynomial $s_\lambda = \Sigma_T x^T$ where T is a semistandard Young's Tableaux on λ.

Definition 4.20. A symmetric polynomial in n variables is a polynomial $p(x_1, ..., x_n)$ such that $p(\sigma(x_1), ..., \sigma(x_n)) = p(x_1, ..., x_n)$ where $\sigma \in S_n$.

Theorem 4.21. *The set $\{s_\lambda : |\lambda| = d, l(\lambda) \leq n\}$ is a basis for symmetric polynomials in n variables of degree d.*

From this, we can work out the multiplication structure of symmetric polynomials. We have that the product of two basis elements is

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu,\nu} s_\lambda.$$

The coefficients $c^\lambda_{\mu,\nu}$ are called Littlewood-Richardson coefficients. We would like to compute the Littlewood-Richardson coefficients. This is difficult in general. For special cases, Pieri's rule gives an easy way to compute these coefficients. These special cases are all we need for the computation of $A(\mathbb{G}(1, 3))$.

Theorem 4.22. *(Pieri's rule) In the case where one partition is of the form (1^k) or (k), the Littlewood-Richardson coefficients are given by the following rules*

$$c^\lambda_{(1^k),\nu} = \begin{cases} 1 & |\lambda| = |\nu| + k \text{ and } \lambda \setminus \nu \text{ is a vertical strip} \\ 0 & \text{otherwise} \end{cases}$$

That is, the Schur polynomials in the product correlate to the diagram by adding k squares to the tableaux ν where no two of the added boxes are in the same row.

The other case is

$$c^\lambda_{(k),\nu} = \begin{cases} 1 & |\lambda| = |\nu| + k \text{ and } \lambda \setminus \nu \text{ is a horizontal strip} \\ 0 & \text{otherwise} \end{cases}$$

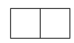
There is a correspondence between these polynomials and Schubert classes in $A(\mathbb{G}(k, n))$. The connection is that the indices in the Schubert classes are partitions which correspond to a Young's diagram. As such, we can relate a Schubert class σ_μ to the Schur polynomial s_μ. Thus, we can compute

$$\sigma_\mu \sigma_\nu = \sum_\lambda c^\lambda_{\mu,\nu} \sigma_\lambda,$$

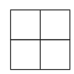
where the diagram of λ is contained in a k by $n-k$ rectangle. That is since as mentioned earlier, if the dimension of the Schubert class (the sum of the indicies of the partition) is greater than $n+k$, then the Schubert class is 0.

Pieri's rule will be able to easily compute the relations of this Chow ring. We will give an example in $A(\mathbb{G}(1,3))$.

Example 4.23. Let us compute σ_2^2 again this time using Pieri's rule. The diagram corresponding to σ_2 is



Pieri's rule tells us that the indices that show up in the product are given by adding two boxes to the above diagram so that the result is a Young's diagram, the two boxes do not lie in the same column and must fit in a 2 by 2 box. The only such diagram is



corresponding to the cycle $\sigma_{2,2}$. Thus, $\sigma_2^2 = \sigma_{2,2}$.

As we see, Pieri's rule gives a simple way to compute the product of Schubert classes. The remaining relations in Theorem 4.13 can be easily computed in this way.

5. Applications

In this section, we will be able to reap the rewards of our work in the previous sections. That is, we will apply our results in order to solve enumerative geometry problems. The following subsections are each organized by a different enumerative geometry problem.

5.1. **Bezout's Theorem.**

One application of the Chow ring of $\mathbb{P}^n$ is describing the intersection of varieties. Bezout's theorem is an example which is described in the following problem.

Problem 5.1. Given two varieties $X, Y \subset \mathbb{P}^n$ of degrees d and e with complementary dimension, i.e. $\dim X + \dim Y = n$, that intersect transversely, at how many points do they intersect?

Solution. Look at the classes of X and Y in the Chow ring $A(\mathbb{P}^n)$. By Theorem 4.1, $[X] = d\xi^{\dim Y}$ and $[Y] = e\xi^{\dim X}$. Using the multiplication structure of the Chow ring, $[X \cap Y] = de\xi^{\dim X + \dim Y} = de[p]$ where p is a point. Thus, we see that the intersection of X and Y consists of de points.

Problem 5.1 can be generalized to the following theorem.

Theorem 5.2 (Bezout)**.** *Let $X_1, ..., X_k \subset \mathbb{P}^n$ be subvarieties of codimension $c_1, ..., c_k$ and degree $d_1, ..., d_k$ which intersect transversely. If $\sum_i c_i = n$, then the intersection $X_1 \cap \cdots \cap X_k$ consists of $\prod_i d_i$ points.*

5.2. **Circles of Apollonius.**

This section will discuss Apollonius' classical problem on three circles. The solution will reduce to a simple computation using Bezout's theorem from the previous section and will illustrate how one uses parameter spaces to solve questions in enumerative geometry. In particular, we will parameterize the space of circles with the projective space $\mathbb{P}^3$.

Problem 5.3 (Apollonius)**.** Given three general position circles, how many are tangent to all three?

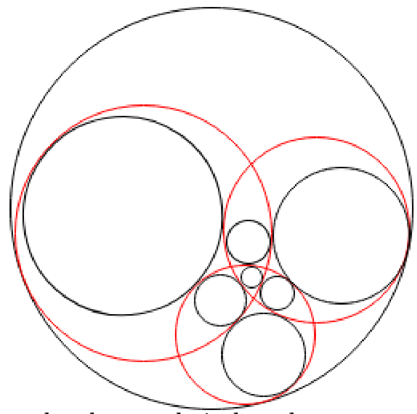


FIGURE 2. Given the three red circles, there are exactly eight circles tangent to all three illustrated by the black circles.

For this problem, we will work in complex projective space. First, we must define what a circle is in this space. In affine space, the equation for a circle centered at point (a, b) with radius r is given by

$$(x - a)^2 + (y - b)^2 = r^2.$$

To define a circle in projective space, the circle equation will be homogenized by adding the additional variable z giving the following definition.

Definition 5.4. A circle in $\mathbb{P}^2$ is given by the equation

$$(x - az)^2 + (y - bz)^2 = r^2 z^2.$$

The new variable z introduces a line at infinity, $z = 0$. By plugging into the circle equation, it is evident that all circles cross through two points at infinity, $(1 : i : 0)$ and $(1 : -i : 0)$, called circular points.

There is another useful definition of a circle.

Definition 5.5. A circle in $\mathbb{P}^2$ is a conic, given by $V(f)$, where $f \in (z, x^2 + y^2)$.

In other words, a circle satisfies

$$a(x^2 + y^2) + bzx + czy + dz^2 = 0.$$

Algebraic manipulation shows that this definition matches the first definition. More importantly, this definition gives insight into the structure of the space of circles. In particular, we see that circles are defined by four parameters, a, b, c, d that are equivalent under scaling. i.e. if a, b, c, d are scaled by a constant, the resulting circle described by the scaled equation is the same. Thus, the space of circles can be identified with $\mathbb{P}^3$.

Definition 5.5 also includes singular circles. A circle is singular there exists a point on the circle where the partial derivatives all vanish at that point. The singular circles have a simple classification.

Proposition 5.6. *There are two forms of singular circles.*

(1) *Union of line at infinity* ($z = 0$)*, with another line in* $\mathbb{P}^2$*.*
(2) *Union of two lines in* $\mathbb{P}^2$ *with one containing* $(1 : i : 0)$ *and the other containing* $(1 : -i : 0)$*.*

Example 5.7. The singular circles are the limits as $r \to 0$ and $r \to \infty$. For example, consider the case $r \to 0$, with the circle centered at zero. Over complex projective space, the resulting equation $x^2 + y^2 = 0$ is the two lines $x = iy$ and $x = -iy$. These are two lines each containing one circular point so it falls under the second case of the previous proposition.

The final part of the problem that needs to be defined is tangency. Tangency relates to the intersection of two circles. In projective space, two circles will always intersect at the circular points. By Bezout's theorem, there are two additional points of intersection. Two circles will be called tangent if the additional points of intersection coincide.

In order to solve our problem, we will show that given a smooth circle D, the circles tangent to D, denoted by Z_D, form a quadric cone in the space of circles, identified as $\mathbb{P}^3$.

We start by studying a correspondence Φ which will help understand the tangent circles of a given circle D.

Definition 5.8. Define $\Phi = \{(r, C) \in D \times \mathbb{P}^3 : C \text{ is tangent to } D \text{ at } r\}$, where D is a given smooth circle, and $\mathbb{P}^3$ is viewed as the space of circles.

This correspondence naturally comes with two projections illustrated below.

$$\begin{array}{ccc} \Phi & & \\ \Big\downarrow{\scriptstyle \pi_1} & \searrow^{\pi_2} & \\ D & & \mathbb{P}^3 \end{array}$$

where π_1 sends a point (r, C) to the point of tangency r on D. The other projection π_2 sends (r, C) to the point of $\mathbb{P}^3$ paramerterizing the circle C.

We start by analyzing the first projection π_1 to D. This will be used to show the following proposition.

Proposition 5.9. *The correspondence* Φ *is 2-dimensional and irreducible.*

Proof. Consider a fiber $\pi_1^{-1}(r)$. This is the set of all circles tangent to D at the point r. Note that tangency means that another circle has to meet D at r with multiplicity 2, so it is described by 2 linear equations inside the parameter space of circles $\mathbb{P}^3$, given by requiring the circles to intersect and have the same tangent direction at the intersection. Then $\pi_1^{-1}(r)$ is the intersection of 2 hyperplanes in $\mathbb{P}^3$ and is isomorphic to $\mathbb{P}^1$. Since, Φ is a $\mathbb{P}^1$ bundle over D, it is 2 dimensional and irreducible. □

We also want to study the other projection π_2. Its image Z_D is the object of interest.

Corollary 5.10. *The circles tangent to* D*,* Z_D*, has dimension two.*

Proof. The projection π_2 is birational from Φ to its image Z_D, since the map from Z_D to Φ taking C to (r, C) where r is the point of tangency is a well-defined map except if C is tangent at both circular points. From the previous proposition Φ is two-dimensional so Z_D is also two-dimensional. □

Now we move on to showing that Z_D is a quadric cone. For this, consider a line L inside $\mathbb{P}^3$. Viewing $\mathbb{P}^3$ again as the space of circles, L parameterizes a family of circles $\{C_t\}_{t\in\mathbb{P}^1}$. We now consider the intersection L with Z_D.

Proposition 5.11. *For a generic line L in $\mathbb{P}^3$, $L \cap Z_D$ consists of 2 points.*

Proof. Take such an L and consider f, g, where $C_0 = V(f)$ and $C_\infty = V(g)$. In general, C_0 and C_∞ intersect with D at two points disregarding the points at infinity. As a result, the rational function f/g on D has two zeros corresponding to when C_0 intersects D and two poles for when C_∞ intersects D. Thus, $f/g : D \to \mathbb{P}^1$ is degree 2. The circles on the line between C_0 and C_∞ which are tangent to D intersect D at the ramification points of f/g. The Riemann-Hurwitz formula shows that there are two such points. Then $L \cap Z_D$ is two points. □

Since a generic line intersects Z_D at 2 points, Z_D is degree 2. Since Z_D is degree 2 and dimension 2, it is a quadric surface. The following proposition will establish that Z_D is indeed a cone.

Proposition 5.12. *If C is tangent to D, then the line between C and D is contained in Z_D.*

Proof. Suppose that $C \in Z_D$ $C \neq D$. Consider the line of circles in $\mathbb{P}^3$ between C and D. The circles that lie on this line satisfy the condition of tangency at the same point as C. Intuitively, these circles are rescalings of the circle C at the point of tangency. In particular, these circles are still tangent to D at r so this line is contained in Z_D. This is since tangency is given by 2 linear equations inside $\mathbb{P}^3$. Thus, a point in the line between two points that satisfy these linear conditions will still satisfy them. □

Thus, we conclude that Z_D is a quadric cone with the vertex at D. With the hard work done, the result of Apollonius is reduced to a simple computation with Bezout's Theorem.

Solution (Apollonius)**.** Let D_1, D_2, D_3 be three general circles. The circles that are tangent to all three are exactly parameterized by the points in $Z_{D_1} \cap Z_{D_2} \cap Z_{D_3}$. We know each Z_{D_i} is of degree 2, so by Bezout, if the intersection is finite (general position), then the intersection consists of 8 points. Thus, there are 8 circles tangent to all three.

5.3. **Lines meeting four general lines in $\mathbb{P}^3$.** The Grassmanian $\mathbb{G}(1, 3)$ parameterizes lines inside $\mathbb{P}^3$. As such, the Chow ring of $\mathbb{G}(1, 3)$ can be used to answer enumerative geometry questions regarding lines.

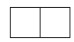
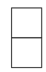
Problem 5.13. Let $L_1, ..., L_4$ be four general lines in $\mathbb{P}^3$. How many lines $L \in \mathbb{P}^3$ intersect all four?

Solution. The class of any line L_i inside $A(\mathbb{G}(1, 3))$ is the Schubert class σ_1. Since multiplication inside the Chow ring corresponds to intersection, we see that

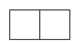
$$[L_1 \cap \cdots \cap L_4] = \sigma_1^4.$$

To answer our question, we should compute σ_1^4. We will do this using Young's diagrams and Pieri's rule as seen in section 4.3. Let us start by computing σ_1^2. The Young's diagram corresponding to σ_1 is

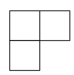
Pieri's rule tells us we need to add one box to this diagram. There are two ways to do this.

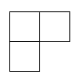
Thus, $\sigma_1^2 = \sigma_2 + \sigma_{1,1}$. Then $\sigma_1^3 = \sigma_1\sigma_2 + \sigma_1\sigma_{1,1}$. We need to compute the two cases, $\sigma_1\sigma_2$ and $\sigma_1\sigma_{1,1}$. In the first case we need to add one box to



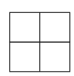
making sure that the result is a Young's diagram and we stay inside a 2 by 2 square. The only way to do this results in the following diagram



so $\sigma_1\sigma_2 = \sigma_{2,1}$. A similar calculation shows that $\sigma_1\sigma_{1,1} = \sigma_{2,1}$ as well. Thus, $\sigma_1^3 = 2\sigma_{2,1}$. Then $\sigma_1^4 = 2\sigma_1\sigma_{2,1}$. To compute $\sigma_1\sigma_{2,1}$, we need to add one box to



where the resulting diagram has to stay inside a 2 by 2 square. The only way to do this is



Thus, $\sigma_1^4 = 2\sigma_{2,2}$. The class $\sigma_{2,2}$ is the class of a line, so there are two lines that intersect $L_1, ..., L_4$.

5.4. **Lines meeting general curves in $\mathbb{P}^3$.** We can generalize the result in the previous section to curves.

Problem 5.14. Let $C_1, ..., C_r$ be general curves in $\mathbb{P}^3$ of degrees $d_1, ..., d_r$. For what r are there finitely many lines in $\mathbb{P}^3$ which intersect all of these curves? For such r, how many lines in $\mathbb{P}^3$ intersect all of the curves?

To do this, we need to compute the class of the lines intersecting a given curve in the Chow ring $A(\mathbb{G}(1,3))$.

Proposition 5.15. *Let $C \subset \mathbb{P}^3$ be a curve of degree d and*

$$\Gamma_C = \{L \in \mathbb{G}(1,3) : L \cap C \neq \emptyset\}.$$

Then,

$$[\Gamma_C] = d\sigma_1.$$

Proof. We first show Γ_C is a cycle so that we may take a class of it in the Chow ring. To do this, we show that Γ_C is a variety. We have the correspondence

$$\Sigma = \{(p, L) \in C \times \mathbb{G}(1,3) : p \in L\}.$$

It comes with the following projections

$$\begin{array}{ccc} \Sigma & & \\ \downarrow{\scriptstyle \pi_1} & \searrow^{\pi_2} & \\ C & & \mathbb{G}(1,3) \end{array}.$$

Consider a fiber of π_1. These are all the lines that go through a point p on the curve C, so it is a projective plane. Since this is true for all fibers, Σ is an irreducible variety of dimension 3. Now consider the projection π_2. Its image is Γ_C. It is birational to its image, since a line generally intersects a curve at one point. Thus, Γ_C is an irreducible variety of dimension 3. Let $\gamma_C = [\Gamma_C]$. Since Γ_C has dimension 3, we know that γ_C belongs in the codimension 1 graded part of the Chow ring. Thus,

$$\gamma_C = \alpha\sigma_1.$$

We would like to find this α. We multiply the equation by $\sigma_{2,1}$ on both sides to get

$$\gamma_C\sigma_{2,1} = \alpha\sigma_1\sigma_{2,1} = \alpha\sigma_{2,2}.$$

Since $\sigma_{2,2}$ is the class of a point in $\mathbb{G}(1,3)$, to find α, we only need to find the number of points in the intersection $\gamma_C\sigma_{2,1}$. Fix a flag $p \in L \subset H$, and choose the corresponding representative $\Sigma_{2,1}$.

$$\Gamma_C \cap \Sigma_{2,1} = \{\Lambda \in \mathbb{G}(1,3) : p \in \Lambda \subset H, L \cap C \neq \emptyset\}.$$

Since H is general plane, H will intersect C in d points none of which are colinear with p. Then the only lines in the intersection are the d lines through p and a point of intersection of C and H. Thus, $\alpha = d$. □

Solution. Returning to our problem, we see the class of lines intersecting a curve C_i, Γ_{C_i} is given by $d_i\sigma_1$. To find the lines intersecting all the curves we need to take the intersection of the Γ_{C_i} which corresponds to multiplication in the Chow ring. Thus, to see when the intersection is finite, we need to look at the powers of σ_1. For $r \leq 3$, the power σ_1^r correspond to classes that represent infinitely many lines, so there are infinite lines that intersect these curves. For $r > 4$, the corresponding power is 0, so there are no curves that intersect more than 4 general curves. We need to further analyze the case when $r = 4$. In this case we see that,

$$[\Gamma_{C_1} \cap \cdots \cap \Gamma_{C_4}] = d_1 \cdots d_4\sigma_1^4 = 2d_1 \cdots d_4\sigma_{2,2}.$$

The class $\sigma_{2,2}$ is the class of a line in $\mathbb{P}^3$ so there are $2d_1 \cdots d_4$ lines that meet the four general curves.

Remark 5.16. Lines are curves of degree one, so using the above solution, there are 2 lines that meet four general lines. This matches our solution in section 5.3.

5.5. **Chords of curves in $\mathbb{P}^3$.** Another application of the Chow ring $A(\mathbb{G}(1,3))$ is analyzing chords.

Definition 5.17. Let $C \subset \mathbb{P}^n$ be a curve of degree d and genus g. The set of chords $\Psi(C) \subset \mathbb{G}(1,3)$ is the closure of the set of lines $\overline{pq}$ where $p, q \in C$.

One such enumerative geometry question we may ask about chords is the following.

Problem 5.18. Let $C, C' \subset \mathbb{P}^3$ be general twisted cubic curves. How many chords do they have in common?

To solve this problem, we need to first compute the class of chords of a given curve in the Chow ring.

In this computation, we will require a genus formula for singular curves.

Lemma 5.19. *Suppose $C \subset S$ is a curve with singular points, $p_1, \dots, p_\delta$ of multiplicity $m_1, \dots, m_\delta$, where all singular points are nodes. Then*

$$g = \frac{C^2 + K_S \cdot C}{2} + 1 - \sum \binom{m_i}{2}.$$

Proposition 5.20. *Let $C \in \mathbb{P}^3$ be a smooth curve of genus g and degree d. Then,*

$$[\Psi(C)] = (\binom{d-1}{2} - g)\sigma_2 + \binom{d}{2}\sigma_{1,1}.$$

Proof. We have a map $\tau : C \times C \dashrightarrow \mathbb{G}(1,3)$ sending (p,q) to the line containing p, q. This is a rational map since it is defined everywhere except for points of the form (p,p). The image of this map is $\Psi(C)$ so $\Psi(C)$ is dimension 2.

The codimension 2 graded component of $A(\mathbb{G}(1,3))$ is spanned by $\sigma_{1,1}$ and σ_2, so

$$[\Psi(C)] = \alpha\sigma_2 + \beta\sigma_{1,1}.$$

We first solve for β. Fix a flag $p \in L \subset H$. Let $\Sigma_{1,1}$ be the Schubert cycle corresponding to this flag. The coefficient β is the number of points in the intersection of $\Psi(C)$ and $\Sigma_{1,1}$. These are the lines contained in H that are chords of the curve C. H will intersect C at d points $p_1, \dots, p_d$, where no three are colinear. The chords are formed by the lines through pairs of these points. There are $\binom{d}{2}$ possible options so

$$\beta = \binom{d}{2}.$$

Now we solve for α. Similarly fix a flag and take Σ_2 to be the Schubert cycle associated with this flag. We want to find the number of points in $\Sigma_2 \cap \Psi(C)$, which is the set of chords that goes through the point p in the flag. Consider the projection

$$\pi_p : C \to \mathbb{P}^2.$$

Fix a general plane H' not containing p. The map π_p takes a point q in C and maps it to the intersection of the line $\overline{pq}$ and the plane H'. Let $\overline{C}$ be the image. The preimage of any point in $\overline{C}$ is either two points if it corresponds to a secant line through p and consists of one point otherwise. Thus, we have that π_p is birational to its image since its inverse is defined for all points not corresponding to a secant line through p. Moreover, by inverse function theorem, the points of the image corresponding to secant lines are singular on $\pi_p(C)$ while the other points are not. The generic choice of p and H' allow us to assume that these singularities are nodes. Thus, $\Sigma_2 \cap \Psi(C)$ corresponds to singular nodes on the curve π_p. Computing with the genus formula of lemma 5.19, we get

$$g = \frac{\overline{C}^2 + K_{\mathbb{P}}^2 \cdot \overline{C}}{2} + 1 - \sum \binom{m_i}{2} = \frac{1}{2}(d^2 - 3d) + 1 - \alpha = \frac{d^2 - 3d + 2}{2} - \alpha = \binom{d-1}{2} - \alpha.$$

Thus, $\alpha = \binom{d-1}{2} - g$ completing our computation. □

Solution. Suppose C, C' are twisted cubic curves. By Proposition 5.20, the class of chords in the Chow ring of a twisted cubic curve is $\sigma_2 + 3\sigma_{1,1}$. We want to compute $(\sigma_2 + 3\sigma_{1,1})^2$.

$$(\sigma_2 + 3\sigma_{1,1})^2 = \sigma_2^2 + 6\sigma_2\sigma_{1,1} + 9\sigma_{1,1}^2 = \sigma_{2,2} + 0 + 9\sigma_{2,2} = 10\sigma_{2,2}.$$

The class $\sigma_{2,2}$ is the class of a line so there are exactly 10 chords in common.

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