# ELEMENTARY DIFFERENTIAL GEOMETRY AND THE GAUSS-BONNET THEOREM

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Abstract. In this paper we discuss examples of the classical Gauss-Bonnet theorem under constant positive Gaussian curvature and zero Gaussian curvature. We then develop the necessary geometric preliminaries with example calculations. We then use our newly developed tools to prove the local Gauss-Bonnet theorem. After a foray into triangulation and the Euler characteristic, we will use the local Gauss-Bonnet theorem to prove the global Gauss-Bonnet theorem for compact surfaces.

Contents



## 1. Introduction

The Gauss-Bonnet theorem serves as a fundamental connection between topology and geometry. It relates an inherently topological quanitity of a surface, the Euler characteristic, with an intrinsic geometric property, the total Gaussian curvature. Gaussian curvature is, roughly speaking, how much a surface deviates from a tangent plane at a point. The flatter the surface, the closer a tangent plane approximates it, and the lower the Gaussian curvature. The Gauss-Bonnet theorem remarkably states that under homeomorphism, the total Gaussian curvature remains invariant. This seems quite odd upon first glance as twisting, squishing, or stretching an object seems like it would radically alter many of its geometric properties. Yet, since the Euler characteric remains invaraiant under homeomorphism, the Gauss-Bonnet theorem predicts that total curvature should remain unchanged. Our goal is to build the necessary machinery to understand the theorem and then prove it. It formally states:

**Theorem 1.1.** *Let $R \subset S$ be a regular region of an oriented surface and let $C_1 \cdots C_n$ be the closed, simple, piecewise regular curves which form the boundary $\partial R$ of R. Suppose that each $C_i$ is positively oriented and let $\theta_1 \cdots \theta_p$ be the set of all external angles of the curve $C_1 \cdots C_n$ . Then*

$$\sum_{i=1}^{n} \int_{c_i} k_g(s)ds + \iint_R K d\sigma + \sum_{l=1}^{p} \theta_l = 2\pi\chi(R),$$

*where $s$ denotes the arc length of $C_i$, and the integral over $C_i$ means the sum of integrals in every regular arc of $C_i$.*

$\chi$ refers to the Euler characteristic and K refer to the total Gaussian curvature. The importance of Gauss-Bonnet is the connection it makes between these seemingly unrelated values. We will motivate both of these notions later in the paper and supply the requisite definitions.

## 2. Gauss-Bonnet for particular triangles

The first version of the Gauss-Bonnet theorem that we will discuss concerns itself with geodesic triangles on a surface. It states that the difference between the angles of the triangle and $\pi$ equals to the total Gaussian curvature of the triangle or

$$\sum_{i=1}^{3} \phi_i - \pi = \iint_T K dA. \tag{2.1}$$

Geodesics, roughly speaking, are the shortest path along a surface. On a plane they are lines, on a sphere they are great circles, etc. We can think of curvature as how fast a tangent plane to a given point deviates from a surface. This formula enables us to compute the total Gaussian curvature along the triangle in a relatively simple manner. As opposed to complex computation with integrals, we calculate the angles in a given geodesic triangle.

2.1. **Zero Gaussian Curvature.** In a plane, the value K, or the Gaussian curvature, is equal to 0. This implies that the value on the right side of (2.1) evaluates to 0. Additionally, geodesics along the plane are merely straight lines. This implies that a geodesic triangle in a surface of flat curvature is a planar triangle. Rearranging the terms yields

$$\sum_{i=1}^{3} \phi_i = \pi.$$

This reduces to the theorem that the sum of the angles in a planar triangle add up to $\pi$ radians.

2.2. **Constant Positive Gaussian Curvature.** Our next example is the unit sphere, which has constant positive curvature. The Gaussian curvature of a unit sphere is 1, meaning the right side of (2.1) is

$$\iint_T 1dA, \tag{2.2}$$

but this is the area of the region we are integrating, which in our case is the geodesic triangle. Hence it is sufficient to prove

$$\sum_{i=1}^{3} \phi_i - \pi = \triangle, \tag{2.3}$$

where $\triangle$ refers to the area of the geodesic triangle.

*Proof.* Suppose there exists a geodesic triangle with angles $\alpha$, $\beta$, $\gamma$. For each pair of edges of the triangle, their extensions along great circles form a lune, the 3 dimensional analogue of a sector of a circle. The surface area of a lune is proportional to the angle it makes divided by $2\pi$. In this case, the surface area of a lune with angle $\alpha$ is $\frac{\alpha}{2\pi}$S, where S is the surface area of the sphere. Now, for each angle in the geodesic triangle, there exists two corresponding lunes. Hence, there are 6 lunes total. Their union is the surface area of the sphere. However, the geodesic triangle and its antipodal image are overcounted an additional 2 times by each lune, thus the geodesic triangle is overcounted 4 times. We then obtain the following equation:

$$2(\frac{\alpha}{2\pi} + \frac{\beta}{2\pi} + \frac{\gamma}{2\pi})S = S + 4\Delta.$$

However, S $= 4\pi$, the surface area of a unit sphere. Therefore, substitution and simplification yields:

$$\alpha + \beta + \gamma - \pi = \Delta.$$

$\square$

## 3. Geometric Preliminaries

3.1. **Curves.** Before we prove the local and global Gauss-Bonnet theorems, we need to construct some objects and the operations we wish to perform on them.

**Definition 3.1.** A parametrized differentiable curve *is a differentiable curve* $\alpha$: I $\to \mathbb{R}^3$ *of an open interval* I = (a,b) *of the real line* $\mathbb{R}$ into $\mathbb{R}^3$.

*Remark* 3.2. For each t $\in I$, we denote $\alpha$ at $t$ by $\alpha(t) = (x(t), y(t), z(t))$ for some parametrization of the curve.

**Definition 3.3.** The tangent vector $\alpha$'(t) is the vector $(x'(t), y'(t), z'(t))$

**Example 3.4.** The map $\alpha$(t) = $(\cos 2t, \sin 4t, t^2)$, where $t \in \mathbb{R}$ is a parametrized differentiable curve. Each component is differentiable and $\alpha$ is a map from $\mathbb{R}$ into $\mathbb{R}^3$.

The tangent vector at a point $t$ is $\alpha'(t) = (-2\sin(2t), 4\cos(4t), 2t)$

**Definition 3.5.** *A parametrized differentiable curve* $\alpha$: I $\to \mathbb{R}^3$ *is said to be regular if* $\alpha$'(t) $\neq 0$ *for all* t $\in$ I.

A variety of analytic tools can be used to explore the geometry of curves:

(1) The first derivative is the *tangent vector*.
(2) The magnitude of the second derivative is defined to be the *curvature*. It measures the rate of the change of the tangent line.
(3) The plane determined by the unit vectors in the direction of the first and second derivatives is the *osculating plane* at a point.
(4) The measure of how quickly neighboring osculating planes differ from a given one is *torsion*.

Curvature and torsion roughly correspond to the **bending** and **twisting** of a curve. Therefore we can think of any curve as bending, twisting, and stretching an interval. These serve as examples of how we can use analysis to explore geometry. Corresponding notions of bending and twisting as well as geometric properites such as area and length will be constructed for regular surfaces using the same idea.

3.2. **Surfaces.** We will now examine the notion of a surface. Surfaces are defined to be subsets rather than mappings.

**Definition 3.6.** *A subset* $\mathrm{S} \subset \mathbb{R}^3$ *is* a regular surface *if, for each* $\mathrm{p} \in \mathrm{S}$, *there exists a neighborhood* V *in* $\mathbb{R}^3$ *and a map* $\mathbf{x}$: $\mathrm{U} \to \mathrm{V} \cap \mathrm{S}$ *of an open set* $\mathrm{U} \subset \mathbb{R}^2$ *onto* V $\cap$ S such that:

(1) $\mathbf{x}$ is smooth.
(2) $\mathbf{x}$ is a homeomorphism .
(3) *For each* $\mathrm{q} \in \mathrm{U}$, *the differential* $\mathrm{d}\mathbf{x}_q$: $\mathbb{R}^2 \to \mathbb{R}^3$ is injective.

**Definition 3.7.** The mapping $\mathbf{x}$ is called a parametrization of $p$.

**Definition 3.8.** The neighborhood $\mathrm{V} \cap \mathrm{S}$ of $p$ in S is called a *coordinate neighborhood.*

In order to do differential geometry, we need all derivatives of $\mathbf{x}$ to be well-defined. This is guaranteed by Condition 1. Condition 2 enables our surface to be locally Euclidean, meaning it will inherit many of the same tools present in Euclidean space when viewed in a sufficently small neighborhood. Condition 3 takes on a more familiar form if we compute its matrix in the canonical basis of $\mathbb{R}^2$ and $\mathbb{R}^3$.

Let $\mathrm{d}\mathbf{x}_q$ have coordinates (u,v) in $\mathbb{R}^2$ and $(x, y, z)$ in $\mathbb{R}^3$. The vector $e_1$ is tangent to the curve $u \to$ (u, $v_0$). It's image under $\mathbf{x}$ is $(x(u, v_0), y(u, v_0), z(u, v_0))$. The tangent vector to the curve at $\mathbf{x}$(q) is

$$\frac{\partial \mathbf{x}}{\partial u} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \end{pmatrix}.$$

Thus, by the defintion of differential:

$$d\mathbf{x}_q(e_1) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \end{pmatrix} = \frac{\partial \mathbf{x}}{\partial u},$$

hence

$$d\mathbf{x}_q = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}.$$

Condition 3 states that the two columns of the matrix of $d\mathbf{x}_q$ are linearly independent. This will prove useful when creating a coordinate system for the space of all tangent vectors at a point.

**Example 3.9.** We can demonstrate that the unit sphere is a regular surface using Definition (3.4). Let

$$\mathbf{x}_{1,2}(x,y) = (x, y, \pm\sqrt{1-(x^2+y^2)})$$
$$\mathbf{x}_{3,4}(y,z) = (\pm\sqrt{1-(y^2+z^2)}, y, z)$$
$$\mathbf{x}_{5,6}(x,z) = (x, \pm\sqrt{1-(x^2+z^2)}, z)$$

and

$$U_{1,2} = (x,y) \in \mathbb{R}^2; x^2+y^2 < 1$$
$$U_{3,4} = (y,z) \in \mathbb{R}^2; y^2+z^2 < 1$$
$$U_{5,6} = (x,z) \in \mathbb{R}^2; x^2+z^2 < 1$$

Since $x^2+y^2 < 1$, $y^2+z^2 < 1$, and $x^2+z^2 < 1$, all 6 parametrizations are continously differentiable, hence Condition 1 is satisfied. The inverse of each parametrization is a projection map, which is continuous, thus Condition 2 is satisfied. Condition 3 can be reduced to a computation. Additionally, each parametrization $\mathbf{x}$ maps a point on the sphere from a point on a 2-dimensional open subset.

3.3. **The Tangent Plane.**

**Definition 3.10.** A *tangent vector* to $S$ at a point $p \in S$, is the tangent vector $\alpha'(0)$ of a differentiable parametrized curve $\alpha$: (-$\epsilon$, $\epsilon$) $\to S$ with $\alpha(0) = p$.

**Proposition 3.11.** *Let* $\mathbf{x}$*:* $U \subset \mathbb{R}^2$ *be a parametrization of a regular surface* $S$ *and let* $q \in U$*. The 2-dimensional vector space,*

$$d\mathbf{x}_q(\mathbb{R}^2) \subset \mathbb{R}^3,$$

*coincides with the set of tangent vectors to* $S$ *at* $\mathbf{x}$(q) and is spanned by $(\frac{\partial \mathbf{x}}{\partial u}, \frac{\partial \mathbf{x}}{\partial v})$.

*Proof.* Let $\alpha$(t) = $\mathbf{x}(u(t), v(t))$ be a curve in $S$. Then $\alpha'$(t) = $u'\mathbf{x}_u + v'\mathbf{x}_v$. Additionally, if $\mathbf{v} = a_1\mathbf{x}_u + a_2\mathbf{x}_v$, then $\alpha$(t) = $\mathbf{x}(u_0 + a_1$t, $v_0 + a_2$t) implies $\alpha' = \mathbf{v}$. □

**Definition 3.12.** The set of all tangent vectors to S at a point $\mathbf{x}$(q) is called the *tangent plane* to $S$ at $p$ and is denoted $T_p$(S).

*Remark* 3.13. The tangent plane $T_p$(S) does not depend on the choice of parametrization. However the basis of $T_p$(S) does depend on the choice of parametrization. The vectors $(\frac{\partial \mathbf{x}}{\partial u}, \frac{\partial \mathbf{x}}{\partial v})$ form a basis for $T_p$(S) as they are two linearly independent vectors in a 2-dimensional subspace. We will use $(\mathbf{x}_u, \mathbf{x}_v)$ for short.

3.4. **The First Fundamental Form.**

*Remark* 3.14. In order to define length, angles, and area on our surface without going back to $\mathbb{R}^3$, we will need to induce an inner product on our tangent plane, $T_p$(S). We will denote this inner product $\langle\ ,\ \rangle_p$.

**Definition 3.15.** The quadratic form $I_p$ (w) = $\langle w, w\rangle_p$ for $w \in T_p(\mathrm{S}) \subset \mathbb{R}^3$ is the *first fundamental form* of the regular surface $S$ at $p \in S$.

*Remark* 3.16. Since we have a basis of $T_p(\mathrm{S})$ given by a parametrization $\mathbf{x}$ at a point p $\in$ S, we want to write down the *first fundamental form* in that basis. Let w $\in T_p(\mathrm{S})$, then, w $= \alpha'(0)$ for some parametrized curve $\alpha(\mathrm{t}) = \mathbf{x}\ (u(t),\, v(t))$ with p $= \alpha(0) = \mathbf{x}(u_0,\, v_0)$. The first fundamental form in the basis $(\mathbf{x}_u, \mathbf{x}_v)$ is

$$\begin{aligned}
I_p(\alpha'(0)) &= \langle \alpha'(0), \alpha'(0)\rangle_p \\
&= \langle \mathbf{x}_u u' + \mathbf{x}_v v', \mathbf{x}_u u' + \mathbf{x}_v v'\rangle_p \\
&= \langle \mathbf{x}_u, \mathbf{x}_u\rangle_p (u')^2 + 2\langle \mathbf{x}_u, \mathbf{x}_v\rangle_p u'v' + \langle \mathbf{x}_v, \mathbf{x}_v\rangle_p (v')^2 \\
&= E(u')^2 + 2Fu'v' + G(v')^2.
\end{aligned}$$

**Definition 3.17.** The following values are the *coefficients of the first fundamental form* in the basis $(\mathbf{x}_u,\, \mathbf{x}_v)$ of $T_p(\mathrm{S})$ for $t = 0$.

$$\begin{aligned}
E(u_0, v_0) &= \langle \mathbf{x}_u, \mathbf{x}_u\rangle_p \\
F(u_0, v_0) &= \langle \mathbf{x}_u, \mathbf{x}_v\rangle_p \\
G(u_0, v_0) &= \langle \mathbf{x}_v, \mathbf{x}_v\rangle_p.
\end{aligned}$$

**Example 3.18.** We can compute the first fundamental form of a cylinder with the parametrization $\mathbf{x}$: U $\to \mathbb{R}^3$ given by $\mathbf{x}$(u,v) = (cos($u$), sin($v$)) where (u,v) $\in \mathbb{R}^2$, 0< $u < 2\pi$, -$\infty <$ v $< \infty$

$$\begin{aligned}
\mathbf{x}_u &= (-\sin(u), \cos(u), 0) \\
\mathbf{x}_v &= (0, 0, 1),
\end{aligned}$$

Hence,

$$\begin{aligned}
E &= \sin^2(u) + cos^2(u) = 1 \\
F &= 0 \\
G &= 1.
\end{aligned}$$

Therefore, given any $w \in T_p(\mathrm{S})$ with coordinates (a,b), then the first fundamental form is

$$I_p(w) = Ea^2 + 2Fab + Gb^2 = a^2 + b^2.$$

which is a confirmation of the Pythagorean theorem.

**Definition 3.19.** Let R $\subset$ S be a bounded region of a regular surface contained in the coordinate neighborhood of a parametrization $\mathbf{x}$: U $\subset \mathbb{R}^2 \to$ S, then we define

$$\iint_Q \|\mathbf{x}_u \wedge \mathbf{x}_v\| \mathrm{dudv}$$

as the area of R.

*Remark* 3.20. Note that $\|\mathbf{x}_u \wedge \mathbf{x}_v\|^2 + \langle\ \mathbf{x}_u\ ,\ \mathbf{x}_v\ \rangle^2 = \|\mathbf{x}_u\|^2 \cdot \|\mathbf{x}_v\|^2$, hence, by plugging in the coefficients of the first fundamental form, we obtain $\|\mathbf{x}_u \wedge \mathbf{x}_v\| = \sqrt{EG - F^2}$.

3.5. **The Gauss Map and the Second Fundamental Form.** Earlier we examined the notion of curvarture for a curve, namely how much a curve pulls away from the tangent vector at a point. We would like to extend this notion to a regular surface, seeing how quickly a surface $S$ pulls away from a tangent plane, $T_p(S)$ at a point p. One way to do this is by assigning a vector field to a given neighborhood of $S$ and measuring its rate of change. Since a plane is determined by a unit vector, each of these vectors will correspond to the tangent plane at a point. We will need to build some further machinery to define different kinds of curvature, namely the Gauss Map and the Second Fundamental Form.

Given a parametrization $\mathbf{x} : U \subset \mathbb{R}^2 \to S$ of a regular surface, we can assign a unit normal vector to each point of our surface in $U$ by

$$N(q) = \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|}(q), \qquad q \in \mathbf{x}(U) \tag{3.21}$$

Since both $\mathbf{x}_u$ and $\mathbf{x}_v$ are differentiable, $N(q)$ is a differentiable map as well. It assigns a unit normal vector to each point in $\mathbf{x}$(U).

**Definition 3.22.** Let S be a regular surface, V $\subset$ S be an open set in S, and $N : V \to \mathbb{R}^3$ be a differentiable map which assigns a unit normal vector to each $p \in S$. Then $N$ *is a differentiable field of unit normal vectors on* $V$*.*

*Remark* 3.23. If a differentiable field of unit normal vectors can be defined on an entire surface, the surface is *orientable*. The Mobius strip is not orientable for if we go around the surface once, the vectors in our vector field flip sign.

**Definition 3.24.** Let S $\subset \mathbb{R}^3$ be a surface with orientation $\mathbf{N}$. The map N: $S \to S^2$ defined in (3.21) is called the *Gauss Map* of $S$ where $S^2$ is the unit sphere in $\mathbb{R}^3$.

The Gauss map is differentiable and its differential $dN_p$ at $p \in S$ is a linear map from $T_p(S)$ to $T_{N(p)}(S^2)$. Since $T_p(S)$ and $T_p(S^2)$ are the same vector space, we can consider $dN_p$ as a linear map from the same vector space to itself.

We will now verify that $dN_p$ is a self-adjoint linear map so that we can associate a quadratic form $\langle dN_p(v), v\rangle$ and compute it in the local coordinates of $T_p(S)$.

**Proposition 3.25.** *The differential* $dN_p : T_p(S) \to T_p(S)$ *of the Gauss map is a self-adjoint linear map.*

*Proof.* We know that $dN_p$ is already linear, hence it sufficient to show that $\langle \mathbf{x}_u, dN_p(\mathbf{x}_v)\rangle$ $= \langle dN_p(\mathbf{x}_u), \mathbf{x}_v\rangle$ for a given parametrization $\mathbf{x}$(u,v) and a corresponding basis ($\mathbf{x}_u$, $\mathbf{x}_v$) of $T_p(S)$. Let $\alpha(t) = \mathbf{x}(u(t), v(t))$ be a parametrized curve in S with $\alpha(0) = p$, then

$$\begin{aligned} dN_p(\alpha'(0)) &= dN_p(\mathbf{x}_u u'(0) + \mathbf{x}_v v'(0)) \\ &= \frac{d}{dt} N(u(t), v(t))\Big|_{t=0} \\ &= N_u u'(0) + N_v v'(0) \end{aligned}$$

Plugging $\mathbf{x}_u$ and $\mathbf{x}_v$ into $dN_p$, we get $dN_p(\mathbf{x}_u) = N_u$ and $dN_p(\mathbf{x}_v) = N_v$. Thus we must prove $\langle N_u, \mathbf{x}_v\rangle = \langle N_v, \mathbf{x}_u\rangle$.

We can do this by differentiating $\langle N, \mathbf{x}_u \rangle = 0$ $\langle N, \mathbf{x}_v \rangle = 0$ with respect to u and v, yielding:

$$\langle N_v, \mathbf{x}_u \rangle + \langle N, \mathbf{x}_{uv} \rangle = 0$$
$$\langle N_u, \mathbf{x}_v \rangle + \langle N, \mathbf{x}_{vu} \rangle = 0$$

Combining these two equations, we yield the desired result,

$$\langle N_u, \mathbf{x}_v \rangle = \langle N_v, \mathbf{x}_u \rangle$$

□

**Definition 3.26.** The quadratic form, $II_p(v) = -\langle dN_p(v), v \rangle$ is called the *second fundamental form* of $S$ at $p$.

3.6. **Curvature(s).**

**Definition 3.27.** Let $C$ be a regular curve in $S$ going through $p \in S$ with curvature $k$ at $p$. Let $n$ be the normal vector to $C$ at $p$, $N$ be the normal vector to $S$ at $p$, and $\cos\theta = \langle n, N \rangle$. We define $k_n = k\cos\theta$ as the *normal curvature* of $C$ at$p$.

We now provide geometric intution for the second fundamental form. Let $C = \alpha(s) \subset S$ be a regular curve parametrized by arc length and we restrict $N(s)$ to $\alpha(S)$. Since the normal vector is perpendicular to the tangent plane, we have $\langle N(s), \alpha'(s) \rangle = 0$. Differentiating yields,

$$\langle N(s), \alpha''(s) \rangle = -\langle N'(s), \alpha'(s) \rangle$$

Hence,

$$\begin{aligned} II_p(\alpha'(0)) &= -\langle dN_p(\alpha'(0), \alpha'(0) \rangle \\ &= -\langle N'(0), \alpha'(0) \rangle \\ &= \langle N(0), \alpha''(0) \rangle \\ &= \langle N, kn \rangle(p) = k_n(p) \end{aligned} \tag{3.28}$$

Therefore the second fundamental form of a tangent vector $v \in T_p(S)$ is the normal curvature of a regular curve through $p$ and tangent to $v$.

**Proposition 3.29.** *All curves on a surface $S$ at a point $p \in S$ with the same tangent line have the same normal curvature at $p$.*

*Remark* 3.30. We won't prove this statement but it enables us to talk about normal curvature in a given direction since it is the tangent line at a point and not the curve that goes through it that determines the normal curvature.

**Definition 3.31.** Let $p$ be a point in a regular surface $S$ and $v$ be a unit vector in $T_p(S)$ and $N(p)$ be the normal vector at point $p$. Then the *normal section* of $S$ at $p$ along $v$ is the intersection of $S$ with the plane determined by the normal vector and tangent vector.

If we restrict the second fundamental form to a unit circle along $T_p(S)$, we can denote the maximum and minimum normal curvature by $k_1$ and $k_2$ respectively. By a property of self-adjoint operators, there exists an orthonormal basis $(e_1, e_2)$ of $T_p(S)$ such that

$$\begin{aligned} dN_p(e_1) &= -k_1 e_1 \\ dN_p(e_2) &= -k_2 e_2 \end{aligned} \tag{3.32}$$

**Definition 3.33.** The *maximum normal curvature* $k_1$ and the *minimum normal curvature* $k_2$ are called the *principal curvatures* at p. Their corresponding directions given by the eigenvectors $e_1$ and $e_2$ are called the *principal directions* at p.

Since the determinant and trace of a linear operator do not depend on the choice of basis, by (3.31) the determinant of $dN$ is $k_1k_2$ and the trace of $dN$ is $k_1 + k_2$.

**Definition 3.34.** The determinant of $dN_p$ is the *Gaussian curvature* K of a regular surface S at p. The negative half of the trace is the *mean curvature* H of S at p.

**Example 3.35.** In a unit sphere $S^2$, the normal sections through a point $p \in S^2$ are unit circles of radius one, but the curvature of a unit circle is identically one everywhere. Therefore the normal curvatures are equal to one around any point p in $S^2$. This implies that the Gaussian curvature $K$ is 1 everywhere as well.

For more difficult examples and calculations, we will need to first compute the second fundamental form and the differential of the Gauss map in the local coordinates of our tangent plane.

Let $\mathbf{x}(u, v)$ be a parametrization of regular surface $S$ at a point $p$ and $\alpha(t) = (u(t), v(t))$ be a parametrized curve on $S$ with $\alpha(0) = p$ given by $\alpha' = \mathbf{x}_u u' + \mathbf{x}_v v'$. Then

$$dN(\alpha') = N'(u(t), v(t)) = N_u u' + N_v v' \tag{3.36}$$

but $N_u, N_v \in T_p(S)$, hence

$$\begin{aligned} N_u &= a_{11}\mathbf{x}_u + a_{21}\mathbf{x}_v \\ N_v &= a_{12}\mathbf{x}_u + a_{22}\mathbf{x}_u \end{aligned} \tag{3.37}$$

plugging back into (3.36) yields,

$$dN(\alpha') = (a_{11}u' + a_{12}v')\mathbf{x}_u + (a_{21}u' + a_{22}v')\mathbf{x}_v \tag{3.38}$$

which can be written compactly as,

$$dN \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} \tag{3.39}$$

If we can compute each $a_{ij}$ in terms of the coordinates of the basis $\mathbf{x}_u$, $\mathbf{x}_v$, we will have a matrix for the differential of the Gauss map. We will first compute the second fundamental form in this basis and develop shorthand much like we did with the first fundamental form.

We first note that $\langle N, \mathbf{x}_u\rangle = \langle N, \mathbf{x}_v\rangle = 0$ since the normal vector is by definition perpendicular to the tangent plane.

$$\begin{aligned} II_p(\alpha') &= -\langle dN(\alpha'), \alpha'\rangle \\ &= -\langle N_u u' + N_v v', \mathbf{x}_u u' + \mathbf{x}_v v'\rangle \\ &= -(\langle N_u, \mathbf{x}_u\rangle(u')^2 + \langle N_u, \mathbf{x}_v\rangle u'v' + \langle N_v, \mathbf{x}_u\rangle u'v' + \langle N_v, \mathbf{x}_v\rangle(v')^2) \end{aligned}$$

We will express the inner-products using shorthand,

$$\begin{aligned} e &= -\langle N_u, \mathbf{x}_u \rangle \\ f &= \langle N_v, \mathbf{x}_u \rangle = -\langle N_u, \mathbf{x}_v \rangle \\ g &= -\langle N_v, \mathbf{x}_v \rangle \end{aligned} \tag{3.40}$$

**Definition 3.41.** The values expressed in (3.40) are known as the *coefficients of the second fundamental form* in the basis ($\mathbf{x}_u$, $\mathbf{x}_v$) of $T_p(\mathrm{S})$.

We can now express the terms $a_{ij}$ in terms of $e, f, g$. Using the equations from (3.37) and bilinearity, we obtain

$$\begin{aligned} e = -\langle N_u, \mathbf{x}_u \rangle &= a_{11}E + a_{21}F \\ f = -\langle N_v, \mathbf{x}_u \rangle &= a_{12}E + a_{22}F \\ f = -\langle N_u, \mathbf{x}_v \rangle &= a_{11}F + a_{21}G \\ g = -\langle N_v, \mathbf{x}_v \rangle &= a_{12}F + a_{22}G \end{aligned}$$

which can be summarized neatly with matrices as

$$-\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

solving for the matrix with terms $a_{ij}$, we obtain

$$\begin{aligned} \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} &= -\begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \\ &= -\frac{1}{EG - F^2} \begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \end{aligned} \tag{3.42}$$

Hence,

$$\begin{aligned} a_{11} &= \frac{fF - eG}{EG - F^2} \\ a_{12} &= \frac{gF - fG}{EG - F^2} \\ a_{21} &= \frac{eF - fE}{EG - F^2} \\ a_{22} &= \frac{fF - gE}{EG - F^2} \end{aligned}$$

The result from (3.39) gives us the Gaussian curvature explicitly in terms of the coefficients of the first and second fundamental form. Namely, if we take the determinant of the matrix on the left side of (3.42), we obtain

$$K = \frac{eg - f^2}{EG - F^2} \tag{3.43}$$

**Example 3.44.** We previously expressed the Gaussian curvature of a sphere, however we exploited its constant normal curvature at every point. Using our new machinery, we can examine more complicated surfaces. We can compute the Gaussian curvature of torus with the following parametrization

$$\mathbf{x}(u, v) = ((a + r\cos)\cos u, (a + r\cos u)\sin(v), r\sin u)$$

Computing the partial derivatives yields the coefficients of each fundamental form:

$$\begin{aligned} E &= \langle \mathbf{x}_u, \mathbf{x}_u \rangle = r^2 \\ F &= \langle \mathbf{x}_u, \mathbf{x}_v \rangle = 0 \\ G &= \langle \mathbf{x}_v, \mathbf{x}_v \rangle = (a + r\cos u)^2 \end{aligned}$$

similar computation following from the definitions yields,

$$\begin{aligned} e &= \langle N, \mathbf{x}_{uu} \rangle = \langle \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|}, \mathbf{x}_{uu} \rangle = \frac{r^2(a + r\cos u)}{r(a + r\cos u)} = r \\ f &= 0 \\ g &= \cos u(a + r\cos u) \end{aligned}$$

hence, by (3.43), we obtain

$$K = \frac{eg - f^2}{EG - F^2} = \frac{r\cos u(a + r\cos u)}{r^2(a + r\cos u)^2} = \frac{\cos u}{r(a + r\cos u)} \tag{3.45}$$

3.7. **The Covariant Derivative and Geodesics.** We will now build the necessary topological and geometric tools neccessary to discuss the local Gauss-Bonnet theorem.

We will construct the notion of a covariant derivative of a vector field, a way to differentiate tangent vectors along a surface. The corresponding notion for vectors in the plane is normal differentiation.

**Definition 3.46.** Let $w$ be a differentiable vector field in an open set $U \subset S$ and $p \in U$ and $y \in T_p(s)$. Additionally let $\alpha : (-\epsilon, \epsilon) \to U$ with $\alpha(0) =$ p and $\alpha'(0) = y$ be a parametrized curve. Let $w(t)$ be the restriction of w to $\alpha$. Projecting $\frac{dw}{dt}(0)$ onto $T_p(S)$ is the *covariant derivative* of $w$ relative to $y$ at $p$.

**Definition 3.47.** A vector field w along a parametrized curve $\alpha : I \to S$ is *parallel* if $\frac{DW}{dt} = 0$ for all $t \in I$.

We shall now make our foray into the notion of a geodesic, a crucial idea in the Gauss-Bonnet theorem. Intuitively a geodesic is the shortest path between two points along a surface. We will want to make this notion more rigorous.

**Definition 3.48.** A parametrized curve $\gamma : I \to S$ along a regular surface S is *geodesic* at $t \in I$ if the field of tangent vectors $\gamma'(t)$ is parallel along $\gamma$ at t.

**Definition 3.49.** The curve $\gamma$ is a *parametrized geodesic* if it is geodesic along all points in its domain.

**Definition 3.50.** A regular connected curve $C$ is a *geodesic* if for any point $p \in C$, the parametrization $\alpha(s)$ of a coordinate neighborhood of $p$ by arclength $s$ is a parametrized geodesic.

This definition implies that a curve $C$ along a surface $S$ is a geodesic if its principal normal at a point $p$ is parallel to the normal to $S$ at $p$. We can see this in action with an example along $S^2$.

**Example 3.51.** Suppose we have the unit sphere $S^2$. We can define a great circle as the curve that results from the intersection of $S^2$ with any plane that goes through its center, $O$. The principal normal to any great circle at a point $p$ lies along the direction of the line connecting the center with $p$. However, the normal to $S^2$ also lies in this direction. Therefore by definition (3.50), great circles are geodesics along a unit sphere.

**Definition 3.52.** Let $w$ be a differentiable field of unit vectors along a parametrized curve $\alpha$ of an oriented regular surface S. Since $\frac{dw}{dt}$ is normal to $w(t)$, we obtain

$$\frac{Dw}{dt} = \lambda(N \wedge w(t))$$

The number $\lambda$ is the *algebraic value* of the covariant derivative of w at t.

**Definition 3.53.** Let $\alpha(s)$ be a parametrization in a neighborhood $p \in S$ by arclength $s$ of an oriented regular curve $C$ in a regular oriented surface $S$. Then the algebraic value of $\frac{D\alpha'(s)}{ds}$ at $p$ is the *geodesic curvature* of $C$ at $p$. It is denoted $k_g$.

3.8. **Three Lemmas.** We will now prove three lemmas that will shorten our calculations when proving the local Gauss-Bonnet theorem.

**Proposition 3.54.** *Let a and b be differentable functions along an interval I such that $a^2 + b^2 = 1$ and $\phi_0$ be such that $a(t_0) = \cos\phi_0$, $b(t_0) = \sin\phi_0$. Then the function*

$$\phi = \phi_0 + \int_{t_0}^{t} (ab' - ba')dt \tag{3.55}$$

*implies* $\cos\phi(t) = a(t)$, $\sin\phi(t) = b(t)$

*Proof.* Proving $\cos\phi(t) = a(t)$ and $\sin\phi(t) = b(t)$ is equivalent to proving

$$\begin{aligned} &(a - \cos\phi)^2 + (b - \sin\phi)^2 = 0 \\ &2 - 2(a\cos\phi + b\sin\phi) = 0 \\ &a\cos\phi + b\sin\phi = 1 \end{aligned} \tag{3.56}$$

Now let, $A = a\cos\phi + b\sin\phi$. By differentiating $a^2 + b^2 = 1$, we yield $aa' = -bb'$. Differentiating $A = a\cos\phi + b\sin\phi$ yields,

$$\begin{aligned} A' &= -a\sin\phi\phi' + b\cos\phi\phi' + a'cos\phi + b'\sin\phi \\ &= -b'\sin\phi(a^2 + b^2) - a'\cos\phi(a^2 + b^2) + a'cos\phi + b'\sin\phi \\ &= -b'\sin\phi - a'\cos\phi + a'cos\phi + b'\sin\phi \\ &= 0 \end{aligned} \tag{3.57}$$

Hence $A$ is constant, but $A(t_0) = cos^2\phi + sin^2\phi = 1$, therefore $A = 1$ □

**Proposition 3.58.** *Let v and w be two differentiable unit vector fields along the curve $\alpha : I \to S$ Then*

$$\frac{Dw}{dt} - \frac{Dv}{dt} = \frac{d\phi}{dt}$$

*Proof.* For the case $\phi \neq 0$ at p, since v and w are unit vector fields, $\langle v, w\rangle = \cos\phi$. Differentiating gives us

$$\langle v', w\rangle + \langle v, w'\rangle = -\sin\phi\phi'$$

therefore,

$$\langle \frac{Dv}{dt}, w\rangle + \langle v, \frac{Dw}{dt}\rangle = \sin\phi\phi'$$

But by definition (3.52), we can rewrite the above equation as

$$\begin{aligned}\langle \frac{Dv}{dt}, w\rangle + \langle v, \frac{Dw}{dt}\rangle &= [\frac{Dv}{dt}]\langle N \wedge v, w\rangle + [\frac{Dw}{dt}]\langle v, N \wedge w\rangle \\ &= ([\frac{Dv}{dt}] - [\frac{Dw}{dt}])\sin\phi = sin\phi\phi'.\end{aligned}$$

Dividing the last equation by $\phi$ proves the lemma for $\phi \neq 0$. For $\phi = 0$ at p we yield two cases. If $\phi = 0$ in a neighborhood of p, then $\phi' = 0$ and $v = w$ hence both the right and left sides of the desired result are 0. If $\phi \neq 0$ in a neighborhood of p and there exists a sequence $(p_n)$ converging to p with $\phi(p_n) \neq 0$, then the result holds by continuity. □

**Proposition 3.59.** *Let $\boldsymbol{x}(u, v)$ be a parametrization such that $F = 0$ of a neighborhood of a surface S. Let $w(t)$ be a differentiable unit vector field along $\boldsymbol{x}(u(t), v(t))$. Then*

$$\frac{Dw}{dt} = \frac{1}{2\sqrt{EG}}(G_u\frac{dv}{dt} - E_v\frac{du}{dt}) + \frac{d\phi}{dt}$$

*where $\phi$ is the angle from $w(t)$ to $\boldsymbol{x}_u$*

*Proof.* Let $e_1 = \mathbf{x}_u/\sqrt{E}$, $e_2 = \mathbf{x}_v/\sqrt{G}$. Note that by the definition of $N$, $e_1 \wedge e_2 = N$.

Using proposition (3.57), we note

$$\frac{Dw}{dt} = \frac{De_1}{dt} + \frac{d\phi}{dt} \tag{3.60}$$

We restrict $e_1(u(t), v(t))$ is restricted to the curve $\mathbf{x}$(u(t), v(t)). We also compute

$$\frac{De_1}{dt} = \langle \frac{de_1}{dt}, N \wedge e_1\rangle = \langle \frac{de_1}{dt}, e_2\rangle = \langle (e_1)_u, e_2\rangle \frac{du}{dt} + \langle (e_1)_v, e_2\rangle \frac{dv}{dt}$$

But since $F = 0$, we obtain

$$\langle \mathbf{x}_{uu}, x_v\rangle = -\frac{1}{2}E_v$$

hence plugging in the original values of $e_1$ and $e_2$ yields,

$$\begin{aligned}\langle (e_1)_u, e_2\rangle &= \frac{1}{2}\frac{E_v}{\sqrt{EG}} \\ \langle (e_1)_v, e_2\rangle &= \frac{1}{2}\frac{G_u}{\sqrt{EG}}\end{aligned}$$

Plugging these into (3.59) yields the desired result. □

## 4. Local Gauss-Bonnet Theorem

We will need to introduce some futher terminology before we prove the result.

**Definition 4.1.** Let $\alpha : [0, l] \to S$ be a continous map. $\alpha$ is a simple, closed, piecewise, regular, parametrized curve if

(1) $\alpha(0) = \alpha(l)$
(2) $\alpha$ is injective for any $t \in [0, l)$
(3) There exists a partition $[0, t_0, t_1 \ldots, t_{k+1}]$ of [0,l) such that $\alpha$ is differentiable and regular in each interval of the partition.

**Definition 4.2.** Each $\alpha(t_i)$ for i = 0, ..., k are the *vertices* of $\alpha$. Each trace $\alpha([t_i, t_{i+1}])$ is a *regular arc* of $\alpha$.

We will let $\phi[t_i, t_{i+1}] \to R$ be functions that measure for any $t \in [t_i, t_{i+1}]$ the angle from $\alpha'(t)$ to $\mathbf{x}_u$.

We will now state the Theorem of Turning Tangents which will heavily simplify our calculations

**Proposition 4.3** (Theorem of Turning Tangents)**.** *Let $\theta_i$ be the external angle at the vertex $\alpha(t_i)$ for some simple, closed, piecewise regular, parametrized curve. Then*

$$\sum_{i=0}^{k}(\phi_i(t_{i+1}) - \phi_i(t_i)) + \sum_{i=0}^{k}\theta_i = \pm 2\pi \tag{4.4}$$

**Definition 4.5.** A region $R \subset S$ of an oriented surface S is a *simple region* if R is homeomorphic to a disk and the boundary ∂R is the trace of a simple, closed, piecewise, regular, parametrized curve.

**Proposition 4.6.** ***(Local Gauss-Bonnet)*** *Let $\boldsymbol{x} : U \to S$ be a parametrization such that $F = 0, E = G = \lambda^2(u, v)$ where $U \subset \mathbb{R}^2$ is homeomorphic to the open disk. Additionally, $R \subset \boldsymbol{x}(U)$ be a simple region of S and let $\alpha : I \to S$ be parametrized by arc length s with $\partial R = \alpha(I)$ and $\theta_0, \ldots, \theta_k$ be the external angles of $\alpha$. Then*

$$\sum_{i=0}^{k}\int_{s_i}^{s_{i+1}} k_g(S)ds + \iint_R K d\sigma + \sum_{i=0}^{k}\theta_i = 2\pi$$

*Proof.* Let $u = u(s)$, $v = v(s)$ be $\alpha$ in the parametrization $\mathbf{x}$. Using proposition (3.58), we can express the geodesic curvature as

$$k_g(s) = \frac{1}{2\sqrt{EG}}(G_u\frac{dv}{ds} - E_v\frac{du}{ds}) + \frac{d\phi_i}{ds},$$

Integrating the equation in each interval $[s_i, s_{i+1}]$ and summing them yields

$$\sum_{i=0}^{k}\int_{s_i}^{s_{i+1}} k_g(s)ds = \sum_{i=0}^{k}\int_{s_i}^{s_{i+1}} \frac{1}{2\sqrt{EG}}(G_u\frac{dv}{ds} - E_v\frac{du}{ds})ds + \sum_{i=0}^{k}\int_{s_i}^{s_{i+1}}\frac{d\phi_i}{ds}ds, \tag{4.7}$$

Applying Green's Theorem in the plane $uv$ gives us

$$\sum_{i=0}^{k}\int_{s_i}^{s_{i+1}} k_g(s)ds = \iint_{x^{-1}(R)}((\frac{E_v}{2\sqrt{EG}})_v + (\frac{G_u}{2\sqrt{EG}})_u)dudv + \sum_{i=0}^{k}\int_{s_i}^{s_{i+1}}\frac{d\phi_i}{ds}ds, \tag{4.8}$$

Using the condition that $F = 0, E = G = \lambda^2(u,v)$ enables us to simplify the integrand

$$(\frac{E_v}{2\sqrt{EG}})_v + (\frac{G_u}{2\sqrt{EG}})_u = \frac{1}{2}((\frac{\lambda_v}{\lambda})_v + \frac{\lambda_u}{\lambda})_u) = \frac{1}{2\lambda}(\triangle \log \lambda)\lambda = -K\lambda, \tag{4.9}$$

hence we rewrite (4.8) as

$$\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} k_g(s)ds = -\iint_R K\lambda dudv + \sum_{i=0}^{k} \frac{d\phi_i}{ds} ds, \tag{4.10}$$

By the Theorem of Turning Tangents, we can reduce the second sum concerning external angles like so

$$\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} \frac{d\phi_i}{ds} ds = \sum_{i=0}^{k} \phi_i(s_{i+1}) - \phi_i(s_i) = \pm 2\pi - \sum_{i=0}^{k} \theta_i, \tag{4.11}$$

Combining (4.10) and (4.11) yields the desired result.

$$\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} k_g(S)ds + \iint_R K d\sigma + \sum_{i=0}^{k} \theta_i = 2\pi.$$

□

## 5. Global Gauss-Bonnet Theorem

The Global Gauss-Bonnet theorem concerns itself with topological characteristics of a surface in addition to geometric ones. Therefore we will need to develop some terminology concerning triangulations.

**Definition 5.1.** A connected region $R \subset S$ is *regular* is R is compact and ∂R is the finite union of piecewise closed regular curves which do not intersect.

**Definition 5.2.** A *triangulation* of a regular region $R \subset S$ is a finite set of triangles $\mathfrak{I}\ T_i = 1, \ldots, n$ such that

1. The union of all the triangles is the regular region R or more formally, $\bigcup_{i=1}^{n} T_i = R$
2. If the two triangles are not disjoint, then they either share a vertex or an edge.

**Definition 5.3.** Given a triangulation $\mathfrak{I}$, we will denote $F$, $E$, $V$ as the number of faces, edges, and vertices respectively.

**Definition 5.4.** The number

$$F - E + V = \chi,$$

is the *Euler Characteristic* of the triangulation

We will also reference two well-known theorems in topology and one from geometry without proof as they are outside the scope and aim of this paper.

**Proposition 5.5.** *Every regular region of a regular surface has a triangulation.*

**Proposition 5.6.** *The Euler characteristic of a regular region $R$ in an oriented surface $S$ does not depend on the triangulation of $R$. Further more, if two compact connected oriented surfaces $S, S' \subset \mathbb{R}^3$ have the same Euler characterstic, then they are homeomorphic.*

**Proposition 5.7.** *Let $R \subset S$ be a regular region of an oriented surface $S$. Let $f$ be a differentiable function on $S$. Then the integral of $f$ over the region $R$*

$$\iint_r f d\sigma,$$

*does not depend on the triangulation $\mathfrak{I}$ or the set of parametrizations $(\boldsymbol{x}_j)$ of $S$.*

We can now prove the Global Gauss-Bonnet theorem.

**Theorem 5.8.** ***(Global Gauss-Bonnet)*** *Let $R \subset S$ be a regular region of an oriented surface and let $C_1 \cdots C_n$ be the closed, simple, piecewise regular curves which form the boundary $\partial R$ of $R$. Suppose that each $C_i$ is positively oriented and let $\theta_1 \cdots \theta_p$ be the set of all external angles of the curve $C_1 \cdots C_n$ . Then*

$$\sum_{i=1}^{n} \int_{c_i} k_g(s) ds + \iint_R K d\sigma + \sum_{l=1}^{p} \theta_l = 2\pi\chi(R),$$

*where $s$ denotes the arc length of $C_i$, and the integral over $C_i$ means the sum of integrals in every regular arc of $C_i$.*

*Proof.* By proposition (5.5) we can let $\mathfrak{I}$ be a triangulation of R with each triangle in $\mathfrak{I}$ being contained a coordinate neighborhood of isothermal parametrizations (parametrizations where $F = 0$, $E = G = \lambda^2(u, v)$).
We now apply local Gauss-Bonnet to each triangle and add them up. We can do this because of proposition (5.7), which ensures integrating over a region does not depend on triangulation. Since each side of each triangle is counted twice with differing orientations we yield

$$\sum_i \int_{c_i} k_g(s) ds + \iint_R K d\sigma + \sum_{j,k=1}^{F,3} \theta_{jk} = 2\pi F$$

We can introduce the notion of an interior angle, or the supplement of an exterior angle, hence

$$\sum_{j,k=1} \theta_{jk} = \sum_{j,k=1} \pi - \sum_{j,k=1} \phi_{jk} = 3\pi F - \sum_{j,k=1} \phi_{jk} \tag{5.9}$$

Additionally we will denote $E_e$ and $E_i$ as the number of external and internal edges of our triangulation $\mathfrak{I}$. Likewise, we will define $V_e$ and $V_i$ as the number of external and internal vertices of our triangulation.

Additionally, the curves $C_i$ are closed hence $E_e = V_e$, hence by induction we get $3F = 2E_i + E_e$. Plugging this into (5.9), we get

$$sum_{j,k=1}\theta_{jk} = 2\pi E_i + \pi E_e - 2\pi V_i - \pi V_e - \sum_i (\pi - \theta_i) = 2\pi E - 2\pi V + \sum i\theta_i. \tag{5.10}$$

The last equality comes from adding and subtracting $\pi E_e$ and $E_e = V_e$.

Combining all the equations yields the desired result

$$\sum_{i=1}^{n} \int_{c_i} k_g(s) ds + \iint_R K d\sigma + \sum_{l=1}^{p} \theta_l = 2\pi(F - E + V) = 2\pi\chi(R). \tag{5.11}$$

□

**Corollary 5.12.** *Let $S$ be an orientable compact surface, then*

$$\iint_{s} K d\sigma = 2\pi\chi(S))$$

This final corollary remarkably states that any two compact orientable surfaces that are homeomorphic have the same total Gaussian curvature. We can continuously stretch, squeeze, and twist any of these surfaces but Gauss-Bonnet demands that the total curvature will be same. It's a remarkable connection between seemingly unrelated concepts.

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