SCOTT SPACES AND THE DCPO CATEGORY

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Abstract. Directed-complete partial orders (dcpo's) arise often in the study of λ-calculus. Here we investigate certain properties of dcpo's and the Scott spaces they induce. We introduce a new construction which allows for the canonical extension of a partial order to a dcpo and give a proof that the dcpo introduced by Zhao, Xi, and Chen is well-filtered.

Contents

1. Introduction

Directed-complete partially ordered sets (dcpo's) often arise in the study of λ-calculus. Namely, they are often used to construct models for λ theories. There are several versions of the λ-calculus, all of which attempt to describe the 'computable' functions. The first robust descriptions of λ-calculus appeared around the same time as the definition of Turing machines, and Turing's paper introducing computing machines includes a proof that his computable functions are precisely the λ-definable ones [5] [8]. Though we do not address the λ-calculus directly here, an exposition of certain λ theories and the construction of Scott space models for them can be found in [1]. In these models, computable functions correspond to continuous functions with respect to the Scott topology. It is thus with an eye to the application of topological tools in the study of computability that we investigate the Scott topology.

The natural interpretation of a map between dcpo's is a function which preserves the suprema of directed sets. It was found that these are precisely the continuous functions with respect to a construction known as the Scott topology on dcpo's. This property makes the Scott topology fundamental to several other constructions in the study of λ-calculus, such as the tree topology. There are many natural questions about the Scott spaces thus induced. What topologies can be represented as a Scott topology induced by some dcpo structure on the set? There is a succinct answer to this question. If two spaces are homotopy-equivalent, is it necessarily the case that if one can be represented by a Scott topology, the other can as well? We shall see that this implication does not hold. Are there non-isomorphic order relations which generate homeomorphic Scott topologies? We shall see that this can never occur, and indeed the category of Scott spaces is isomorphic to a full subcategory of the category of topological spaces and continuous maps.

Besides their application to λ-calculus, dcpo's occur in the study of computer science and can be used to represent information states. (See [3] for details on this aspect of dcpo's.) As it happens, the kinds of dcpo's which most often appear are much nicer than general dcpo's. They are what we shall call domains, for they are more 'continuous' than 'discrete'. It turns out that the Scott topologies they induce reflect this structure; the Hofmann-Mislove theorem states that such spaces are necessarily sober.

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After giving definitions and studying the structure of finite dcpo's, we discuss the relationship between the order on a dcpo and the connected components of its Scott space. The structure of the category DCPO is addressed, and we construct a left adjoint to the forgetful functor from DCPO to POSET. Finally, we examine sober and well-filtered Scott spaces, and show that the Scott spaces of domains are reasonably well-behaved.

2. General Definitions and the Finite Case

In a topological space, we denote the interior of a set A by A°, and the closure by $\overline{A}$. If A contains a single point x, we often abbreviate these to x° and $\overline{x}$.

Definition 2.1. A preorder $\lesssim$ on a set A is a reflexive and transitive binary relation. A partial order is a preorder that is also antisymmetric. A total order is a partial order in which any two elements are comparable.

Definition 2.2. A subset D of a pre-ordered set $(A, \leq)$ is said to be directed if for all $x, y \in D$, there is a $z \in D$ such that $x \leq z$ and $y \leq z$. Dually, a filtered subset is a $G \subset A$ such that, for all $x', y' \in G$, there is a $z' \in G$ such that $z' \leq x'$ and $z' \leq y'$.

It should be noted that $\varnothing$ is always a directed and filtered subset of any pre-ordered set.

Definition 2.3. If $(A, \leq)$ is a pre-ordered set, $b \in A$ is an upper bound for $G \subset A$ if $x \leq b$ for all $x \in G$. We say G has a supremum in A if there is some upper bound b of G such that, for all b' upper bounds of G, $b \leq b'$.

Directly from the definitions, we have the following:

Lemma 2.4. *If $(A, \leq)$ is a partially ordered set and $G \subset A$ has a supremum, the supremum is unique.*

We can therefore discuss *the* supremum of G when it exists in a partial order, and we denote this by $\sup G$.

Definition 2.5. A pre-order $(A, \leq)$ is said to be directed-complete if, whenever $D \subset A$ is directed and nonempty, D has a supremum in A. If a directed-complete pre-order is also a partial order, we say it is a directed-complete partial order (dcpo).

Definition 2.6. For a pre-ordered set $\mathcal{L}$ and a subset $A \subset \mathcal{L}$, we denote the upward closure of A by $\uparrow A = \{y \in \mathcal{L} : \exists x \in A (x \leq y)\}$, and the downward closure by $\downarrow A = \{y \in \mathcal{L} : \exists x \in A (y \leq x)\}$. If $A = \uparrow A$, we say A is closed upward, and if $A = \downarrow A$ we say it is closed downward. If A contains a single element x, we will abbreviate $\uparrow \{x\}$ to $\uparrow x$ and $\downarrow \{x\}$ to $\downarrow x$.

Theorem 2.7. *Let $(A, \leq)$ be a directed-complete pre-order. There is a topology τ on A consisting of subsets H of A which have the properties*

i. *H is closed upward*
ii. *If $D \subset A$ is nonempty and directed and D has a supremum in H, then $D \cap H \neq \varnothing$*

Proof. Evidently, $\varnothing, A \in \tau$. Now let $\{U_\beta\}_{\beta \in B}$ be an arbitrary collection of elements of τ. We claim $\cup_{\beta \in B} U_\beta$ is in τ. It satisfies (i) because, if $x \in \cup_{\beta \in B} U_\beta$, then $x \in U_\beta$ for some β, and so any y with $x \leq y$ must satisfy $y \in U_\beta$ since each U_β has (i). So $y \in \cup_{\beta \in B} U_\beta$. To show that the union has (ii), let D be a directed set and $\sup D$ be a supremum of D which lies in the union. We have that $\sup D \in U_\beta$ for some β, and thus $D \cap U_\beta \neq \varnothing$ for some β. This evidently implies that $D \cap \cup_{\beta \in B} U_\beta \neq \varnothing$, as desired. Now suppose $V, V' \in \tau$. It is clear that $V \cap V'$ satisfies (i). To show that it satisfies (ii), let D be a directed set with a supremum $\sup D$ such that $\sup D \in V \cap V'$. Because $\sup D \in V$, we get that there is some $x \in D \cap V$ since V satisfies (i), and similarly we have a $y \in D \cap V'$. Because D is directed, there is some $z \in D$ with $x \leq z$ and $y \leq z$. Since $x \leq z$ implies $z \in V$, and $y \leq z$ implies $z \in V'$, the fact that V, V' are closed upward implies that $z \in D \cap V \cap V'$. ☐

We will refer to the above topology as the Scott topology on directed-complete pre-ordered sets. Note that the characterization of closed sets in Scott spaces is, perhaps, more natural than the description of open sets: a closed set is any subset which is closed downward and is closed over the taking of directed suprema. There is a natural concern here regarding whether some suprema of a directed set are in an open set, but others are not. This will not be an issue for long, as we shall shortly specialize our theory to dcpo's, but it is worth noting that we obtain the same topology if we only require that $D \cap H \neq \varnothing$ when *all* suprema of D are in H:

Theorem 2.8. *Let $(A, \leq)$ be a directed-complete pre-order, and $H \subset A$ closed upward. Then, whenever $T \subset A$ has a supremum in A, it has a supremum in H if and only if all of its upper bounds (and therefore its suprema) are in H.*

Proof. As T is assumed to have a supremum, if all of its upper bounds are in H, it has at least one supremum in H. And if it has one supremum, say b, in H, then whenever b' is an upper bound for H (and consequently when b' is a supremum of H), $b \leq b'$ and thus $b' \in H$. ☐

Unless otherwise specified, we will equip all of our directed-complete pre-orders with the Scott topology.

Theorem 2.9. *If $(A, \leq)$ is a directed-complete pre-order, $x \in A$, the set $\{y \in A : y \nleq x\}$ is open in the Scott topology, and it is denoted U_x.*

Proof. If $y \in U_x$ and $y \leq z$, we claim $z \in U_x$. For if $z \notin U_x$, we would have $z \leq x$, so by transitivity $y \leq z \leq x$ and $y \notin A$, a contradiction. And if $D \subset A$ is a directed set and $\sup D$ a supremum of D such that $\sup D \in U_x$, we must have that x is not an upper bound on D from Definition 2.3. From this it follows immediately that some element a of D satisfies $a \nleq x$, and hence $a \in D \cap U_x$, as desired. ☐

Throughout, we shall use U_x to denote sets of the above form.

Theorem 2.10. *If $(A, \leq)$ is a directed-complete pre-order equipped with the Scott topology, it is T_0 if and only if $\leq$ is a partial order on A. And it is discrete if and only if it is T_1 if and only if the order is purely reflexive (that is, $x \leq y$ only if $x = y$).*

Proof. If $\leq$ is a partial order, then, for any $x, y \in A$ with $x \neq y$, we either have $x \nleq y$ or $y \nleq x$; without loss of generality suppose the former. Then $x \in U_y$ by the definition of U_y, but $y \notin U_y$ by the reflexivity of $\leq$. This proves that A is T_0. And if $\leq$ is not a partial order, there are some $x, y \in A$ with $x \neq y, x \leq y, y \leq x$. By condition (i) for Scott-open sets, every set which contains x also contains y and vice versa, so the space is not T_0.

If $x \neq y$ and $x \leq y$ then, by condition (i) for Scott-open sets, every open set containing x contains y, and A is not T_1. Conversely, the purely reflexive relation is clearly a partial order. To show that it is directed-complete, let D be a nonempty directed subset of A. If $x, y \in D$, there is some $z \in D$ such that $x \leq z$ and $y \leq z$. But this implies $x = z = y$, and hence D only contains one point. One-point sets have suprema (the one element is a supremum), so $(A, \leq)$ is a dcpo.

We now show that every subset is open in the Scott topology. Since $\leq$ is just equality, every set is closed upward, and since every directed set contains its supremum, (ii) trivially holds in the definition of the Scott topology. Hence the topology is discrete, and consequently T_1.

☐

The following result is easily verified, and allows us to restrict our attention to dcpo's without loss of generality.

Theorem 2.11. *Let $(M, \lesssim)$ be a directed-complete pre-ordered set. Let $(N, \leq)$ be the quotient of M under the equivalence relation $x \sim y \iff (x \leq y \wedge y \leq x)$, equipped with the induced order. Give M the Scott topology and let $\mathcal{T}$ be the quotient topology on N. Then $\leq$ is a dcpo, and the Scott topology on N induced by $\leq$ is equal to the quotient topology $\mathcal{T}$ on N.*

Thus, if we know which T_0 topologies are induced by dcpo's, we will know which topologies are induced by directed-complete pre-orders, because any topology which is not T_0 has a natural quotient which is T_0, and the original topology is representable by a directed-complete pre-order if and only if the T_0 quotient is representable by a dcpo by the above result.

In the case of a total order, the Scott topology is not particularly interesting:

Theorem 2.12. *If $(\mathcal{L}, \leq)$ is a totally ordered dcpo, the Scott-open sets are precisely $\mathcal{L}$ and those equal to U_x for some $x \in \mathcal{L}$ (see Theorem 2.9); and for each $x \in \mathcal{L}$, we have $U_x = \{y \in \mathcal{L} : x < y\}$.*

Proof. The statement that $U_x = \{y \in \mathcal{L} : x < y\}$ for each $x \in \mathcal{L}$ follows directly from Definition 2.1, Theorem 2.9. And Theorem 2.9 also gives that all sets of this form are open.

It remains to show that these are all of the open sets except for $\mathcal{L}$. Suppose $\Omega \subset \mathcal{L}$ is open and not equal to $\mathcal{L}$. Choose $\alpha \in \mathcal{L} \backslash \Omega$. Because $\leq$ is a total order, α must be a lower bound on Ω: if it were not, we could have some $\beta \in \Omega$ such that $\alpha \nleq \beta$, so $\beta \leq \alpha$ by the definition of a total order, and the upward closure of Ω would then force $\alpha \in \Omega$. So the set of lower bounds of Ω is nonempty. Because $\mathcal{L}$ is totally ordered, the set B of lower bounds of Ω is directed, and as $\mathcal{L}$ is a dcpo, B has a supremum s. If $s \notin \Omega$, then it is clear that $\Omega = U_s$: the statement $\Omega \subset U_\alpha$ holds for every $\alpha \notin \Omega$, and the reverse inclusion follows as anything greater than s cannot be a lower bound on Ω and hence must be in Ω. Note that $s \in B$, as every element of Ω is an upper bound of B and hence is greater than or equal to s.

Now suppose $s \in \Omega$. We have that $B \backslash \{s\}$ is directed. If it is empty, s must be a lower bound for $\mathcal{L}$. Because Ω is closed upward, this implies that $\Omega = \mathcal{L}$. Hence, under our assumption that $\Omega \neq \mathcal{L}$, we may conclude that $B \backslash \{s\}$ is nonempty and thus has a supremum. No element of Ω besides s can be in B, for s is a lower bound on Ω and nothing greater than s is in B. Therefore, since Ω is open and $B \backslash \{s\}$ is nonempty, directed, and disjoint from Ω, its supremum v cannot lie in Ω. We claim that $\Omega = U_v$. Since $v \notin \Omega$, we have that $\Omega \subset U_v$. For the reverse inclusion, we claim that every $z \in U_v$ satisfies $s \leq z$. For if $z < s$, we would have $z \in B \backslash \{s\}$ with $v < z$, contrary to choice of v. As Ω is closed upward and $s \in \Omega$ by assumption, we conclude that $z \in \Omega$, as desired. □

In particular, if $\mathcal{L} \neq \varnothing$, we obtain some x such that $\varnothing = U_x$; such an x is easily seen to be an upper bound on $\mathcal{L}$. There is nothing homotopically interesting about totally ordered spaces:

Theorem 2.13. *If $\mathcal{L}$ contains an upper bound for itself, its Scott topology is contractible.*

Proof. Write 1 for the upper bound of $\mathcal{L}$. Define $H : \mathcal{L} \times I \to \mathcal{L}$ by $H(x, 0) = x$ and $H(x, t) = 1$ for $t > 0$. We need to show that H is continuous, so let $U \subset \mathcal{L}$ be an arbitrary open set. If $U = \varnothing$, then $H^{-1}(\varnothing) = \varnothing$. If $U \neq \varnothing$, then $1 \in U$ by upward closure, and $H^{-1}(U) = (\mathcal{L} \times (0, 1]) \cup (U \times I)$. So $H^{-1}(U)$ is open, as desired. □

Corollary 2.14. *All totally ordered dcpo's are contractible.*

Theorem 2.15. *If two Scott topologies are homeomorphic, the orders which induce them must be isomorphic.*

Proof. Suppose $(\mathcal{L}, \leq_{\mathcal{L}})$ and $(\mathcal{M}, \leq_{\mathcal{M}})$ are dcpo's equipped with the Scott topology, and $f : \mathcal{L} \to \mathcal{M}$ is a homeomorphism. Let $x, y \in \mathcal{L}$ be arbitrary. If $x \nleq_{\mathcal{L}} y$, we have $x \in U_y$ and $y \notin U_y$, as in Theorem 2.9. Consequently, $x \in f(U_y), y \notin f(U_y)$. Since open sets in the Scott topology are closed upward, it follows that $f(x) \nleq_{\mathcal{M}} f(y)$. Hence whenever $f(x) \leq_{\mathcal{M}} f(y)$, it is the case that $x \leq_{\mathcal{L}} y$. The same argument with f^{-1} in place of f shows the reverse implication, showing that f is an order isomorphism. □

Therefore, topologies that can be expressed as Scott spaces generally have few open sets and weak separation properties. This argument gives us an important result in determining which topologies can be represented as Scott spaces:

Definition 2.16. On a topological space $\mathcal{X}$, we define the specialization pre-order by declaring $x \leq y$ if every open set which contains x also contains y.

It is clear that the specialization order is a partial order if and only if $\mathcal{X}$ is T_0. This construction is dual to that often used in the treatment of Alexandroff spaces; see [7].

Theorem 2.17. *Given any T_0 topological space, the only dcpo structure on the space which could induce the topology is the specialization order.*

Proof. This follows directly from the definition of the specialization order, combined with the fact (used in the proof of Theorem 2.15) that every open set in a Scott space containing x also contains y if and only if $x \leq y$. □

Note that the specialization order may make the space into a dcpo even though the induced Scott topology is not the original topology. For instance, on any T_1 space, the specialization order is purely reflexive and thus makes the space into a dcpo; however, it does not induce the original topology as its Scott topology (unless the original topology was discrete). Nonetheless, it is easy to give precise topological conditions to describe when the specialization order does induce the original topology.

Definition 2.18. For any topological space $\mathcal{X}$ and subset $A \subset \mathcal{X}$, we define the saturation of A, denoted $A^{\blacktriangleright}$, to be the intersection of all open sets containing A. A set is said to be saturated if it equals its saturation.

Note that, in general, $A^{\blacktriangleright}$ is not open.

Theorem 2.19. *In the Scott topology, a set is saturated if and only if it is closed upward.*

Proof. For P an upper set in a dcpo, it is clear that P is the intersection of all U_x's, where x ranges over the complement of P. And if Q is not an upper set, there is some $x \in Q, y \notin Q$ with $x \leq y$. Then every open set which contains Q contains y, since it is closed upward, and $y \in Q^{\blacktriangleright} \backslash Q$. Thus Q is not saturated. □

Using the language of saturation, we can restate the specialization order as saying that $x \leq y$ if $y \in x^{\blacktriangleright}$, and a directed set D is one such that $x, y \in D$ imply $x^{\blacktriangleright} \cap y^{\blacktriangleright} \neq \varnothing$. The statement that such a D has a supremum is equivalent to saying that $D^{\blacktriangleright} \neq \varnothing$ and that there is some $\alpha \in D^{\blacktriangleright}$ such that $\alpha^{\blacktriangleright} = D^{\blacktriangleright}$.

Theorem 2.20. *Let $\mathcal{X}$ be a topological space. The topology on $\mathcal{X}$ is induced as a Scott topology by a dcpo on $\mathcal{X}$ if and only if the following hold:*

i. *Whenever $D \subset \mathcal{X}$ is such that the intersection of the saturations of any two elements of D is nonempty, there is some $\alpha \in \cap_{x \in D} x^{\blacktriangleright}$ such that $\alpha^{\blacktriangleright} = \cap_{x \in D} x^{\blacktriangleright}$*
ii. *A set $\Omega \subset \mathcal{X}$ is open if and only if (1) the saturation of every point in Ω is contained in Ω and (2) if $D \subset \mathcal{X}$ is such that the intersection of the saturations of any two elements of D is nonempty, and $\alpha \in \cap_{x \in D} x^{\blacktriangleright}$ has the property that $\alpha^{\blacktriangleright} = \cap_{x \in D} x^{\blacktriangleright}$, then, if $\alpha \in \Omega$, $D \cap \Omega \neq \varnothing$.*

This theorem is a translation of the statement that the specialization order makes the space into a dcpo and that the induced Scott topology is equal to the original topology into topological terms. Though it answers the question of which topologies are induced as Scott topologies, it does not really tell us anything further about their properties.

We shall demonstrate one more result with directed-complete pre-orders, namely that every topology on a finite space is the Scott topology induced by some directed-complete pre-order. The cause of this result is:

Theorem 2.21. *If D is a nonempty finite directed subset of a pre-ordered set $(A, \leq)$, then D has a supremum, and at least one supremum of D lies in D.*

Proof. We induct on the size of directed subsets of A, the base step being evident. Now suppose that all n-element directed subsets of A contain a supremum and let $a_0, ..., a_n \in A$ form a directed set. There are two cases. If $\{a_0, ..., a_{n-1}\}$ is directed, it contains a supremum by assumption; call it a_i. Then, since $\{a_0, ..., a_n\}$ is directed, there is some element of $\{a_0, ..., a_n\}$ which is greater than or equal to a_i and a_n. Such an element is an upper bound on the set by construction, and it is a supremum because any upper bound on the set must be greater than or equal to it. If $\{a_0, ..., a_{n-1}\}$ is not directed, we claim that a_n is an upper bound on the set. For any $0 \leq i < n$, there is some $0 \leq j \leq n$ such that $a_i \leq a_j, a_n \leq a_j$. Hence, if a_n were not an upper bound, there would be some i such that we cannot take $j = n$. Thus $a_n \leq a_j$ for some $0 \leq j < n$. By directedness, for all $0 \leq k, l < n$, there is some $0 \leq q \leq n$ such that $a_k \leq a_q, a_l \leq a_q$. Either $q < n$, so a_k, a_l have a join within $\{a_0, ..., a_{n-1}\}$, or $q = n$, in which case transitivity gives us that $a_k \leq a_j, a_l \leq a_j$. This proves that $\{a_0, ..., a_{n-1}\}$ is directed, contrary to assumption. So a_n is an upper bound and thus a supremum. □

Corollary 2.22. *If A is a directed-complete pre-ordered set and all directed subsets of A are finite, all subsets which are closed upward are open in the Scott topology.*

Corollary 2.23. *If (A, τ) is a topological space and A is finite, there is a directed-complete pre-order $\lesssim$ on A such that τ is the Scott topology induced by $\lesssim$.*

Proof. Take $\lesssim$ to be the specialization pre-order. It has the desired properties by Corollary 2.22.

□

This result implies that, despite all totally ordered Scott spaces being contractible, the homotopical properties of Scott spaces can be quite complex; see [7]. We have found that, in the case of finite spaces, condition (ii) in the definition of the Scott topology is trivially satisfied by any subset of the directed-complete pre-order (indeed, we have shown that any pre-order on a finite space is a directed-complete pre-order), so anything can be obtained as a Scott topology. In infinite settings, (ii) will play a large role in determining which sets are Scott-open. For instance, the interval I with the canonical ordering $\leq$ is a dcpo, and the proper open sets in the Scott topology are of the form $(a, 1]$, where $a \in I$. Though all intervals of the form $[a, 1]$ are closed upward, they are <u>not</u> Scott-open (unless $a = 0$), since there are directed sets disjoint from them which have suprema in the interval (for instance, $[0, a)$).

3. Connectedness of Scott Spaces

We briefly describe here a simple order property which defines the connected components of a Scott space.

Definition 3.1. For a dcpo $\mathcal{L}$, we say points $x, y \in \mathcal{L}$ are connected to each other if there is a finite sequence $x = x_0, x_1, ..., x_n = y \in \mathcal{L}$ such that, for each i, $x_i \leq x_{i+1}$ or $x_{i+1} \leq x_i$. (The terminology is justified because this relation is clearly symmetric.)

If $x \leq y$, there is a path between them given by $p(0) = x, p(t) = y$ for $0 < t \leq 1$. By reversing and concatenating paths of this kind, we can find a path between any points that are connected to each other.

Lemma 3.2. *If x is connected to y, the set of points connected to x is the same as the set of points connected to y. And for every x, the set of points connected to x is open and closed.*

This lemma is clear from the definition.

Theorem 3.3. *For $\mathcal{L}$ a dcpo and $x_0 \in \mathcal{L}$, the subset Σ of $\mathcal{L}$ consisting of the points connected to x_0 is the connected component of $\mathcal{L}$ containing x_0. That is, x_0 is connected to y if and only if they lie in the same connected component of $\mathcal{L}$.*

Proof. By Lemma 3.2, Σ is path-connected, so it is connected. To show that it is a connected component, we shall demonstrate that Σ is clopen in $\mathcal{L}$ and therefore Σ and Σ^c form a disconnection of any proper superset of Σ. Evidently, Σ is closed upward and downward by construction, and hence Σ^c is closed upward (and downward) as well. And if $D \subset \mathcal{L}$ is a nonempty directed set with $\sup D \in \Sigma$, the fact that every element of D is comparable to an element of Σ (namely, $\sup D$) implies that $D \subset \Sigma$, and a fortiori that $D \cap \Sigma \neq \varnothing$. Analogously, if Δ is nonempty and directed with $\sup \Delta \in \Sigma^c$, no element of Δ can be in Σ, for that would imply that an element of Σ is comparable to an element of Σ^c, impossible by the construction of Σ. So $\Delta \subset \Sigma^c$ and $\Delta \cap \Sigma^c \neq \varnothing$, as desired. □

This result, in connection with Theorem 3.4, lets us assume without loss of generality that all of the points in our spaces are connected to each other whenever this is useful.

Theorem 3.4. *Every dcpo is the coproduct of its connected components, both as a topological space and as a partial order.*

We omit the proof of this theorem, as it follows from the fact that the connected components coincide with the set of points connected to a given point, and we already know sets of the latter form are open. It is on this basis that we may restrict our attention to connected Scott spaces; partial orders which induce disconnected Scott topologies can be decomposed into subsets which are totally incomparable to each other and do not interact in an order or topological sense. A space with two connected components is essentially two partial orders, each with a Scott topology, each independent from the other.

4. The Categorical Structure of DCPO

We begin with the central result of the theory of dcpo's:

Theorem 4.1. *Let $(\mathcal{L}, \leq_{\mathcal{L}}), (\mathcal{M}, \leq_{\mathcal{M}})$ be dcpo's equipped with the Scott topology. Then a map $f : \mathcal{L} \to \mathcal{M}$ is continuous if and only if, for every directed set D with $D \subset \mathcal{L}$, we have that $\sup f(D)$ exists and $f(\sup D) = \sup f(D)$.*

Proof. Suppose that f preserves directed suprema. Then, if $x, y \in \mathcal{L}$ and $x \leq_{\mathcal{L}} y$, we have that $\{x, y\}$ is directed and $\sup\{x, y\} = y$. So the supremum preservation condition forces $\sup\{f(x), f(y)\} = f(y)$; hence $f(y)$ is an upper bound on $\{f(x), f(y)\}$, and consequently $f(x) \leq_{\mathcal{M}} f(y)$. This shows that f is order-preserving. Now let $U \subset \mathcal{M}$ be open in the Scott topology. Because it is closed upward and f is increasing, we have that $f^{-1}(U)$ is closed upward. Now suppose that $D \subset \mathcal{L}$ is directed and $\sup D \in f^{-1}(U)$. We find that $f(D)$ is directed in $\mathcal{M}$ since f is increasing, and so it has a supremum. Moreover, by assumption, $\sup f(D) = f(\sup D)$. And $f(\sup D) \in U$, so $\sup f(D) \in U$. Hence, as U is open in the Scott topology, $f(D) \cap U \neq \varnothing$; that is, there is some $d \in D$ such that $f(d) \in U$. So $d \in f^{-1}(U)$, and $D \cap f^{-1}(U) \neq \varnothing$. As D was arbitrary, this proves that f is continuous.

Now for the converse. This proof is similar to that of Theorem 2.15. Suppose $g : \mathcal{L} \to \mathcal{M}$ is continuous. Suppose $x, y \in \mathcal{L}$. If $f(x) \nleq_{\mathcal{M}} f(y)$, then $f(x) \in U_{f(y)}$. Thus $x \in f^{-1}(U_{f(y)})$, and $f^{-1}(U_{f(y)})$ is open in $\mathcal{L}$ by continuity. However, it is evident that $y \notin f^{-1}(U_{f(y)})$. So there is an open set in $\mathcal{L}$ which contains x but not y; by the definition of the Scott topology, this implies that $x \nleq_{\mathcal{L}} y$. As we have shown that $f(x) \nleq_{\mathcal{M}} f(y)$ implies that $x \nleq_{\mathcal{L}} y$, the contrapositive shows that f is order-preserving.

Let $D \subset \mathcal{L}$ be directed. Since f is order-preserving, $f(D)$ is directed. Moreover, because f is order-preserving, $f(\sup D)$ is an upper bound on $f(D)$. Thus $\sup f(D) \leq_{\mathcal{M}} f(\sup D)$. Because we are dealing with partial orders, if we can show that the opposite inequality holds, we shall have the desired equality. For contradiction, assume that $f(\sup D) \nleq_{\mathcal{M}} \sup f(D)$. Then $f(\sup D) \in U_{\sup f(D)}$, and $\sup D \in f^{-1}(U_{\sup f(D)})$. By continuity, $f^{-1}(U_{\sup f(D)})$ is open; thus, since D is directed and it contains $\sup D$, we have some $d \in D \cap f^{-1}(U_{\sup f(D)})$. So $f(d) \in U_{\sup f(D)}$; that is, there is some $d \in D$ such that $d \nleq_{\mathcal{M}} \sup f(D)$. This is absurd, and hence our assumption that $f(\sup D) \nleq_{\mathcal{M}} \sup f(D)$ must be false. This completes the proof. □

Because of this result, we can describe the relationship between dcpo's and Scott spaces categorically.

Definition 4.2. The category DCPO has dcpo's as objects and directed-supremum-preserving morphisms between DCPOs. The SCOTT category has objects as topological spaces representable by Scott topologies and continuous maps between them.

The reader may easily verify that both DCPO and SCOTT are, in fact, categories.

Theorem 4.3. *DCPO and SCOTT are isomorphic categories, and SCOTT is a full subcategory of the category of topological spaces and continuous maps.*

Proof. Theorem 2.17 provides an inverse to the functor DCPO→SCOTT defined by Theorem 2.7; these are both functors by Theorem 4.1. □

Definition 4.4. A subset of a pre-ordered set which is closed downward and directed is called an ideal; a set which is closed upward and filtered is a filter. The set of all ideals in a pre-ordered set $\mathcal{L}$ is denoted $\mathcal{I}(\mathcal{L})$.

It should be noted that there is a natural way to describe a dcpo structure on the product of dcpo's and on the space of directed-supremum-preserving maps between any two given dcpo's. The Scott topology of the product does not generally equal the product topology of the Scott spaces; however, it is still a product in the category of Scott spaces, and it makes DCPO into a Cartesian closed category. We shall not discuss this further, but rather refer the reader to [1] for details.

It is evident that, for any $x \in \mathcal{L}$, $\uparrow x$ is a filter and $\downarrow x$ is an ideal. We call filters and ideals of this form principal. This allows us to construct the following extension of arbitrary partial orders, which is similar to the extensions described in [9].

Theorem 4.5. *For every partially ordered set, the space of its ideals is a dcpo ordered by inclusion. The map $\mathcal{L}$ from POSET to DCPO defined on objects by $f(\mathcal{L}) = \mathcal{I}(\mathcal{L})$ and on maps $f : M \to N$ between partially ordered sets by declaring $(\mathcal{L}(f))(I) = f(I)$ is a functor, and it is left adjoint to the forgetful functor U from the category DCPO to POSET.*

Proof. It is clear that this map is functorial, and that the space of ideals is always a DCPO. To see that these functors are adjoint, let S be a partially ordered set and $\mathcal{M}$ a dcpo. For any $f : S \to U\mathcal{M}$ in the category POSET, define $\eta_{S,\mathcal{M}}(f) : \mathcal{L}(S) \to \mathcal{M}$ by $(\eta_{S,\mathcal{M}}(f))(I) = \sup f(I)$ for each ideal $I \subset S$. (Considering S as being imbedded in $\mathcal{L}(S)$ by $x \mapsto \downarrow x$, this makes $\eta_{S,\mathcal{M}}(f)$ an extension of f.) To see that this is truly an adjunction, let S, T be partially ordered sets, $\mathcal{M},\mathcal{N}$ be dcpo's, $\alpha : T \to S$ a map in POSET, $\beta : \mathcal{M} \to \mathcal{N}$ a map in DCPO. We have the diagram

$$\begin{tikzcd} POSET(S,U\mathcal{M}) \arrow[r, "\alpha^*(U\beta)_*"] \arrow[d, "\eta_{S,\mathcal{M}}"'] & POSET(T,U\mathcal{N}) \arrow[d, "\eta_{T,\mathcal{N}}"] \\ DCPO(\mathcal{L}S,\mathcal{M})) \arrow[r, "(\mathcal{L}\alpha)^*\beta_*"'] & DCPO(\mathcal{L}T,\mathcal{N}) \end{tikzcd}$$

The fact that this commutes can be verified easily from the definitions. We now show that each component of the natural transformation is a bijection. For injectivity, let $x \in S, f, g \in POSET(S,U\mathcal{M})$, and suppose $\eta_{S,\mathcal{M}}(f) = \eta_{S,\mathcal{M}}(g)$. We have

$$g(x) = \sup g(\downarrow x) = (\eta_{S,\mathcal{M}}(g))(\downarrow x) = (\eta_{S,\mathcal{M}}(f))(\downarrow x) = \sup f(\downarrow x) = f(x)$$

As x was arbitrary, it follows that $f = g$. For surjectivity, let $h \in DCPO(\mathcal{L}S,\mathcal{M})$. Put $f(x) = \sup h(\downarrow x)$ for each $x \in S$. Then $f \in POSET(S,U\mathcal{M})$, and for every $I \in \mathcal{L}S$, we have

$$(\eta_{S,\mathcal{M}} f)(I) = \sup f(I) = \sup_{i \in I} h(\downarrow i) = h \sup_{i \in I}(\downarrow i) = h(I)$$

Here we have used the fact that, since I is an ideal, the collection $\{\downarrow i\}_{i \in I}$ is directed, and its supremum is I. This shows that $h = \eta_{S,\mathcal{M}} f$, and as h was arbitrary, $\eta_{S,\mathcal{M}}$ is surjective.

□

5. Suprema and the Waybelow Relation

Definition 5.1. If $\mathcal{L}$ is a partially ordered set, $x, y \in \mathcal{L}$, we say that x is waybelow y, written $x \ll y$, if whenever $D \subset \mathcal{L}$ is a directed set with a supremum $\sup D$, then $y \leq \sup D$ implies $\exists d \in D(x \leq d)$. If $x \ll x$, then we say x is isolated from below. For $A \subset \mathcal{L}$, we let $\twoheaduparrow A = \{y \in \mathcal{L} : \exists x \in A(x \ll y)\}$ and $\twoheaddownarrow A = \{y \in \mathcal{L} : \exists x \in A(y \ll x)\}$. If A is a single-element set, we abbreviate $\twoheaduparrow x = \twoheaduparrow \{x\}$ and $\twoheaddownarrow x = \twoheaddownarrow \{x\}$.

Lemma 5.2. *If $\mathcal{L}$ is a partially ordered set, $x, y \in \mathcal{L}$, then if $x \ll y$, we have that $x \leq y$. Also, if $a, b \in \mathcal{L}$ and $a \leq x, x \ll y$, and $y \leq b$, then $a \ll b$. A fortiori, the waybelow relation is transitive.*

Proof. One-point sets are directed, and $\sup\{y\} = y$, so if $x \ll y$, the fact that $y \leq \sup\{y\}$ implies that there is some $d \in \{y\}$ such that $x \leq d$. Of course, $d = y$ and hence $x \leq y$.

Now assume $a \leq x, x \ll y$, and $y \leq b$. Let $D \subset \mathcal{L}$ be some directed set with a supremum such that $b \leq \sup D$. Then, by transitivity, $y \leq \sup D$, and hence there is some $d \in D$ such that $x \leq d$ since $x \ll y$. Again by transitivity, $a \leq d$. Combining the above two results then gives that the waybelow relation is transitive. □

Theorem 5.3. *A point x of a dcpo is isolated from below if and only if $\uparrow x$ is open in the Scott topology.*

Proof. Of course, $\uparrow x$ is closed upward, so it is open if and only if every directed set with a supremum in $\uparrow x$ has some element greater than or equal to x. This is precisely the definition of $x \ll x$.

□

Definition 5.4. A partially ordered set $\mathcal{L}$ is said to be continuous if, for every $x \in \mathcal{L}$, the set $\twoheaddownarrow x$ is directed, and $x = \sup \twoheaddownarrow x$. A continuous dcpo is called a domain.

Theorem 5.5. *Every totally ordered set is continuous.*

Proof. Let $\mathcal{L}$ be a total order, and take $x \in \mathcal{L}$. Then $\twoheaddownarrow x$ is directed. By Lemma 5.2, x is an upper bound on $\twoheaddownarrow x$. There are two cases to consider. In the former, there is a maximum element of $\downarrow x \backslash \{x\}$. In this case, x is isolated from below, so $x = \sup \twoheaddownarrow x$. The other case is where there is no maximum element of $\downarrow x \backslash \{x\}$. Then $\downarrow x \backslash \{x\}$ has x as its supremum, and moreover, this set equals $\twoheaddownarrow x$. □

Lemma 5.6. *If x, y are elements of a partially ordered set, then $x \leq y$ only if $\twoheaddownarrow x \subset \twoheaddownarrow y$, and $x \leq y$ if and only if $\twoheaddownarrow x \subset \twoheaddownarrow y$ when the partially ordered set is complete.*

Proof. The first statement follows from Lemma 5.2. And if the space is complete, then we have $\twoheaddownarrow x \subset \twoheaddownarrow y$ implies $\sup \twoheaddownarrow x \leq \sup \twoheaddownarrow y$ and consequently $x \leq y$. □

Lemma 5.7. *Every element of a finite partially ordered set is isolated from below, and all finite partially ordered sets are continuous.*

Proof. The former statement is clear from Theorem 2.21. The latter follows from Theorem 2.21, Lemma 5.2, and the former statement. □

For an example of a domain, consider $\mathcal{P}(\mathbf{N})$ ordered by inclusion. For every $M \in \mathcal{P}(\mathbf{N})$, we find that $\twoheaddownarrow M$ is the set of finite subsets N of M. This is clearly directed, and its supremum is M. One can verify that it is precisely the finite sets which are isolated from below.

The following result and its proof are based on Theorem I-1.9 in [3].

Theorem 5.8. *Let $(\mathcal{L}, \leq)$ be a continous partially ordered set. If $x \ll y$ and $D \subset \mathcal{L}$ is directed with a supremum such that $y \leq \sup D$, there is some $d \in D$ such that $x \ll d$. Moreover, whenever $x \ll y$, there is some $z \in \mathcal{L}$ such that $x \ll z$ and $z \ll y$.*

Proof. Put $I = \cup_{d \in D} \twoheaddownarrow d$. If $d_0, d_1 \in D$ and $c_0 \ll d_0, c_1 \ll d_1$, the directedness of D allows us to find a $d_2 \in D$ with $d_0 \leq d_2, d_1 \leq d_2$, and Lemma 5.2 then gives that $c_0 \ll d_2$ and $c_1 \ll d_2$. So $c_0, c_1 \in \twoheaddownarrow d_2$. By assumption, $\mathcal{L}$ is continuous, so $\twoheaddownarrow d_2$ is directed and there is some $c_2 \in \twoheaddownarrow d_2$ with $c_0 \leq c_2, c_1 \leq c_2$. Evidently, $c_2 \in I$. This proves that I is directed. Note that, for every $\rho \in \twoheaddownarrow d$ for any $d \in D$, we have $\rho \leq d \leq \sup D$. Thus $\sup D$ is an upper bound on I. And if μ is an upper bound on I, it is an upper bound on every $\twoheaddownarrow d$. By continuity, it follows that $d \leq \mu$ for every $d \in D$, and thus μ is an upper bound on D; consequently, $\sup D \leq \mu$. This proves that I has a supremum, and $\sup I = \sup D$. Then, by the definition of the waybelow relation, we find that there is some $k \in I$ such that $x \leq k$. There must be some $\zeta \in D$ such that $k \in \twoheaddownarrow \zeta$, and by Lemma 5.2, it follows that $x \ll \zeta$, as desired.

The second statement is an application of the first when $D = \twoheaddownarrow y$. □

Theorem 5.9. *If $\mathcal{L}$ is a domain, then for every $x \in \mathcal{L}$, the set $\twoheaduparrow x$ is open in the Scott topology, and $\twoheaduparrow x = (\uparrow x)^\circ$.*

Proof. Fix some $x \in \mathcal{L}$. By Lemma 5.2, we have that $\twoheaduparrow x$ is closed upward. Now suppose $D \subset \mathcal{L}$ is directed, and $\sup D \in \twoheaduparrow x$. By Theorem 5.8, taking $y = \sup D$, we get that $D \cap \twoheaduparrow x \neq \varnothing$. This proves that $\twoheaduparrow x$ is open, and since Lemma 5.2 gives that $\twoheaduparrow x \subset \uparrow x$, it follows that $\twoheaduparrow x \subset (\uparrow x)^\circ$. For the reverse inclusion, let $y \in (\uparrow x)^\circ$ be arbitrary, and take D to be some directed set such that $y \le \sup D$. Because $(\uparrow x)^\circ$ is open, there must be some $\delta \in D$ such that $\delta \in (\uparrow x)^\circ$. But this clearly implies $x \le \delta$; as D was arbitrary, this proves that $x \ll y$, as desired.

□

Theorem 5.10. *If $\mathcal{L}$ is a domain, the collection of subsets of the form $\Uparrow x$ for $x \in \mathcal{L}$ forms a basis for the Scott topology.*

Proof. By Theorem 5.9, sets of this form are open. And if U is any open set, $\beta \in U$, then $\Downarrow \beta$ is a directed set with a supremum in U; hence, there is some δ with $\delta \ll \beta$ and $\delta \in U$. By the upward closure of U, we have that $\Uparrow \delta \subset \uparrow \delta \subset U$, and $\beta \in \Uparrow \delta$ by choice of δ. □

Corollary 5.11. *If $\mathcal{L}$ is a domain and $U \subset \mathcal{L}$ is open, $y \in U$, there is some $x \in U$ such that $x \ll y$.*

Proof. By Theorem 5.10, there is some $z \in \mathcal{L}$ such that $y \in \Uparrow z \subset U$. Then, by Theorem 5.8, we have some $x \in \mathcal{L}$ such that $z \ll x \ll y$. Of course, $x \in U$, so it has the desired properties.

□

6. Hofmann-Mislove Theorem

The results in this section are derived from the section of the same name in [3].

Definition 6.1. A space is locally compact if every point is contained in the interior of some compact subset.

This is a weaker condition than used in [3], but it shall suffice for our purposes.

Definition 6.2. Let $\mathcal{X}$ be a topological space, and A a nonempty subset. We say A is irreducible if, whenever B, C are closed subsets of $\mathcal{X}$ such that $A \subset B \cup C$, it is the case that $A \subset B$ or $A \subset C$. A space $\mathcal{X}$ is sober if every closed irreducible subset is the closure of a unique one of its points.

It is easy to see that a subset of a Scott space is the closure of a single point if and only if it contains an upper bound for itelf.

Theorem 6.3. *In a T_0 space, the closure of every one-point set is irreducible. And if a space is sober, it is T_0.*

Proof. The first statement is clear. If $\mathcal{X}$ is not T_0, we have $x, y \in \mathcal{X}$ with $x \neq y$ and $\overline{x} = \overline{y}$. By the previous part of this theorem, $\overline{x}$ is irreducible, and so $\overline{x}$ is a closed irreducible set which is the closure of more than one of its points. Thus $\mathcal{X}$ is not sober.

□

Definition 6.4. A filter on a set S is a filter in the partially ordered set $\mathcal{P}(S)$ ordered by inclusion which is not equal to $\varnothing$ or $\mathcal{P}(S)$. A filter base is a nonempty collection $\mathcal{B} \in \mathcal{P}(\mathcal{P}(S))$ such that, for any $F_0, F_1 \in \mathcal{B}$, we have that $F_0 \neq \varnothing$ and there is some $F \in \mathcal{B}$ such that $F \subset F_0 \cap F_1$.

Note that the meet of two sets in a space ordered by inclusion is simply their intersection.

Lemma 6.5. *If $\mathcal{B}$ is a filter base on a set S, the collection $\mathcal{B}' = \{N \in \mathcal{P}(S) : \exists B \in \mathcal{B}(B \subset N)\}$ is a filter on S.*

Proof. The collection $\mathcal{B}'$ is clearly closed over the taking of supersets and does not contain the empty set, since filter bases cannot contain the empty set. And it is closed over intersection because, if $B_0, B_1 \in \mathcal{B}'$, then there are $B_0', B_1' \in \mathcal{B}$ such that $B_0' \subset B_0, B_1' \subset B_1$. Further, by the definition of a filter base, there is some $C \in \mathcal{B}$ such that $C \subset B_0' \cap B_1' \subset B_0 \cap B_1$. Thus by construction of $\mathcal{B}'$ we have tht $B_0 \cap B_1 \in \mathcal{B}'$.

□

Definition 6.6. A topological space $\mathcal{X}$ is said to be well-filtered if every (nonempty) filter base $\mathcal{C}$ composed of compact saturated subsets of $\mathcal{X}$ has the property that, for every $U \subset \mathcal{X}$ open with $\cap\mathcal{C} \subset U$, there exists some $K \in \mathcal{C}$ such that $K \subset U$.

Definition 6.7. For $\mathcal{L}$ a dcpo, we write $OFilt(\mathcal{L})$ for the set of Scott open sets which are filters. It is partially ordered by inclusion (and this makes it a dcpo). And write $Q(\mathcal{L})$ for the collection of compact saturated sets ordered by containment ($K_0 \le K_1$ if $K_1 \subset K_0$).

Theorem 6.8 (Hofmann-Mislove I)**.** *Suppose that $\mathcal{X}$ is a sober topological space (not necessarily a Scott space) and $\mathcal{F}$ is a nonempty open filter in the dcpo $\mathcal{O}(\mathcal{X})$ of open subsets of $\mathcal{X}$. Then, if $V \subset \mathcal{X}$ is open and $\cap_{U \in \mathcal{F}} U \subset V$, we have $V \in \mathcal{F}$; moreover, $\cap_{U \in \mathcal{F}} U$ is compact, and if $\mathcal{F} \neq \mathcal{O}(\mathcal{X})$, it is nonempty.*

Proof. Let $K = \cap_{U \in \mathcal{F}} U$, and suppose $V \subset \mathcal{X}$ is open, $K \subset V$. Suppose for contradiction that $V \notin \mathcal{F}$. Let W be the union of all open sets which contain V and are not in $\mathcal{F}$. Evidently, W is open and not in $\mathcal{F}$ (W is the supremum of the directed set of finite unions of open sets containing U and not contained in $\mathcal{F}$, and $\mathcal{F}$ is assumed to be Scott-open).

We wish to show that W^c is irreducible. Suppose $B, C \subset \mathcal{X}$ are closed and $W^c \subset B \cup C$. Without loss of generality, assume $B, C \subset W^c$. Then B^c, C^c are open, and $B^c \cap C^c \subset W$. Because $\mathcal{F}$ is a filter, $W \notin \mathcal{F}$, it follows that B^c or C^c is not in $\mathcal{F}$. Without loss of generality, suppose $B^c \notin \mathcal{F}$. Then, since $B \subset W^c$, we have $W \subset B^c$. As B^c is open, the construction of V forces $V \subset B^c$. Consequently, $B^c \subset W$; thus, $W^c \subset B$.

By our assumption that $\mathcal{X}$ is sober, it follows that W^c is the closure of a unique point p. Because $p \in W^c$, there is some $U \in \mathcal{F}$ such that $p \notin U$ (else, we would have $p \in K \subset W$). Consequently, $U \cap W^c = U \cap \overline{p} = \varnothing$, and $U \subset W$. But because $U \in \mathcal{F}$ and $\mathcal{F}$ is a filter, it follows that $W \in \mathcal{F}$, a contradiction. Hence $V \in \mathcal{F}$.

To show that K is compact, let $\mathcal{U}$ be an open cover of K. The union of all elements of $\mathcal{U}$ is an open set containing K; by the above result, then, it is in $\mathcal{F}$. Consider the directed set of finite unions of elements of $\mathcal{U}$; because its supremum is in $\mathcal{F}$ and $\mathcal{F}$ is open, some element of the directed set is in $\mathcal{F}$. And all elements of $\mathcal{F}$ contain K, so this is a finite subcover of K.

If $\mathcal{F} \neq \mathcal{O}(\mathcal{X})$, then $\varnothing \notin \mathcal{F}$ since $\mathcal{F}$ is closed upward. But if $\cap_{U \in \mathcal{F}} U = \varnothing$, we would have that $\varnothing$ is an open set which contains the intersection, and is thus in $\mathcal{F}$ by the first part of the theorem. □

Because we can associate to any compact set K the collection of open sets which contain K, and this is an element of $OFilt(\mathcal{O}(\mathcal{X}))$, the Hofmann-Mislove theorem gives an order isomorphism between $Q(\mathcal{X})$ (recall that this is ordered by containment) and $OFilt(\mathcal{O}(\mathcal{X}))$.

Theorem 6.9. *If $\mathcal{X}$ is a T_0 space, it is sober if and only if, whenever $\mathcal{F}$ is a nonempty open filter of open sets, $\mathcal{F}$ is equal to the collection of open sets which contain the intersection of all elements of $\mathcal{F}$ as a subset.*

Proof. One direction is the first part of the Hofmann-Mislove theorem. We show the converse: assume that, whenever $\mathcal{F}$ is an open filter of open sets, $\mathcal{F}$ is equal to the collection of open sets which contain the intersection of all elements of $\mathcal{F}$. Let A be an irreducible closed subset of $\mathcal{X}$. Let $\mathcal{F}$ be the collection of open sets which have nonempty intersection with A ($\mathcal{F}$ is nonempty because it contains $\mathcal{X}$), and write K for the intersection of all elements of $\mathcal{F}$. By the irreducibility of A, this is a filter in $\mathcal{O}(\mathcal{X})$, and it is evidently Scott-open. Suppose for contradiction that A is not the closure of a single point. For each $x \in A$, then, $(\overline{x})^c$ is an open set which has nonempty intersection with A, and is consequently in $\mathcal{F}$. So $K \subset \cap_{x \in A} (\overline{x})^c \subset A^c$, and by our assumption, every open set containing K is in $\mathcal{F}$. So A^c is in $\mathcal{F}$, which is absurd given the definition of $\mathcal{F}$. So A must be the closure of some point, which is unique since $\mathcal{X}$ is T_0. □

Theorem 6.10 (Hofmann-Mislove II)**.** *If $\mathcal{X}$ is a T_0 sober space, it is well-filtered. And if $\mathcal{X}$ is T_0, locally compact, and well-filtered, it is sober.*

Proof. We use the equivalent condition for sobriety given by Theorem 6.9. Suppose $\mathcal{X}$ is sober, and let $\mathcal{C}$ be a filter basis of compact saturated sets, U an open set containing the intersection of all elements of $\mathcal{C}$. Take $\mathcal{F}$ to be the collection of open sets which contain some element of $\mathcal{C}$. It is easily seen that $\mathcal{F}$ is a nonempty open filter. As every element of $\mathcal{C}$ is assumed to be saturated, the intersection of all elements of $\mathcal{C}$ is equal to the intersection of all of the elements of $\mathcal{F}$. Thus, U contains the intersection of elements of $\mathcal{F}$, and is consequently in $\mathcal{F}$ by Theorem 6.9. By construction of $\mathcal{F}$, U contains some element of $\mathcal{C}$, as desired.

Now assume that $\mathcal{X}$ is locally compact and well-filtered. Take $\mathcal{F}$ to be a nonempty open filter of open sets, and let K be the intersection. Take $\mathcal{C}$ to be the collection of compact saturated sets which contain some element of $\mathcal{F}$. By local compactness, for each $x \in \mathcal{X}$, there is a compact neighborhood K_x of x. Naturally, $\cup_{x \in \mathcal{X}} K_x^\circ = \mathcal{X}$, so the fact that $\mathcal{F}$ is Scott-open and $\mathcal{X} \in \mathcal{F}$ implies that there are some $x_1, ... x_n \in \mathcal{X}$ such that $\cup_1^n K_{x_i}^\circ \in \mathcal{F}$. Since $\cup_1^n K_{x_i}^\circ \in \mathcal{F}$ and $\cup_1^n K_{x_i}$ is compact, we have $(\cup_1^n K_{x_i})^\blacktriangleright \in \mathcal{C}$ (we leave it to the reader to verify that the saturation of a compact set is compact). Thus $\mathcal{C}$ is nonempty.

If $K_0, K_1 \in \mathcal{C}$, there are corresponding $U_0, U_1 \in \mathcal{F}$ with $U_0 \subset K_0, U_1 \subset K_1$. Because $\mathcal{F}$ is a filter, $U_0 \cap U_1 \in \mathcal{F}$. Note that $U_0 \cap U_1 \subset K_0 \cap K_1$, and $K_0 \cap K_1$ is saturated, so $K_0 \cap K_1 \in \mathcal{C}$. This proves that $\mathcal{C}$ is a filter base. Now take W

to be an open set containing the intersection of all elements of $\mathcal{F}$. As this equals the intersection of all elements of $\mathcal{C}$, the well-filtered property of $\mathcal{X}$ implies that there is some $K \in \mathcal{C}$ with $K \subset W$. By construction of $\mathcal{C}$, there is an element of $\mathcal{F}$ contained in it, and thus contained in W; as $\mathcal{F}$ is a filter, it follows that $W \in \mathcal{F}$, as desired. □

Note that there are sober spaces which are not locally compact. Also, there are well-filtered spaces which are not sober.

Theorem 6.11. *The Scott space of a domain is necessarily sober.*

Proof. Let $(\mathcal{L}, \leq)$ be a domain, and take $A \subset \mathcal{L}$ to be irreducible. We claim that $\twoheaddownarrow A$ is an ideal. Of course, it is closed downward. To show directedness, let $x, y \in \twoheaddownarrow A$. We have that $\twoheaduparrow x, \twoheaduparrow y$ are open by Theorem 5.9, and thus $(\twoheaduparrow x)^c, (\twoheaduparrow y)^c$ are closed sets, neither of which contains A by choice of x, y. By the irreducibility of A, their union does not contain A, either. So there is some $a \in A \backslash ((\twoheaduparrow x)^c \cup (\twoheaduparrow y)^c) = A \cap \twoheaduparrow x \cap \twoheaduparrow y$. The intersection $\twoheaduparrow x \cap \twoheaduparrow y$ is open and contains a, so by Corollary 5.11, there is some $z \in \twoheaduparrow x \cap \twoheaduparrow y$ with $z \ll a$. Clearly, Lemma 5.2 implies $z \in \twoheaddownarrow A$, and $x \leq z, y \leq z$, as desired.

We have shown that $\twoheaddownarrow A$ is an ideal. From the definition of a domain, it is easily deduced that $\sup \twoheaddownarrow A$ is an upper bound on A. We shall show that it lies in A, and hence A is its closure. This is easily deduced from Lemma 5.2, as the fact that A is closed downward forces $\twoheaddownarrow A \subset A$, and the supremum of a directed subset of a closed set must lie in the closed set. □

7. Ordinal-Based DCPOs

Let us now briefly remark upon a particular class of dcpo's which have drawn some attention. Let α be a nonzero ordinal. Consider the set $\alpha \times (\alpha + 1)$, equipped with the order relation given by $(n, m) \leq (n', m')$ if $n = n'$ and $m \leq m'$, or if $m' = \alpha$ and $m \leq n'$, the termwise inequalities being understood in the usual sense of ordinals. It is easily verified that this is a dcpo.

Definition 7.1. A pillar is a subset of $\alpha \times (\alpha + 1)$ of the form $Y_u = \{(u, n) : n \leq \alpha\}$.

Lemma 7.2. *The maximal elements of $\alpha \times (\alpha + 1)$ are precisely those which have second coordinate α. All directed sets in $\alpha \times (\alpha + 1)$ are contained within a pillar or contain a maximal element which is an upper bound for the directed set. A subset V of $\alpha \times (\alpha + 1)$ is open in the Scott topology if and only if it is closed upward and, whenever $u \in \alpha$ is such that $V \cap Y_u \neq \varnothing$, the least v such that $(u, v) \in V$ is a successor ordinal.*

Proof. The first statement is easily verified. If $D \subset \alpha \times (\alpha + 1)$ is directed, $(u, v), (u', v') \in D$ with $u \neq u'$, there is some $(x, y) \in D$ greater than or equal to both. As x cannot be equal to both u and u', the definition of the order forces $y = \alpha$. As (x, y) is maximal and D is directed, it follows that (x, y) is an upper bound for D. This characterization of the directed sets directly implies that a set is open if it is closed upward and its least element in every pillar with which it has nonempty intersection has a successor ordinal as second coordinate; the converse follows from Definition 2.7 and consideration of the set $\{(u, w) : w < v\}$, with u, v as in the statement of the lemma. □

If $\alpha < \omega$, this set is finite and thus sober. However, if $\alpha = \omega$, the space is not even well-filtered.

Theorem 7.3. *The dcpo $\omega \times (\omega + 1)$ equipped with the above order is irreducible, not well-filtered, and not locally compact.*

Proof. If A, B are closed sets with $\mathcal{L} = A \cup B$, either A or B contains infinitely many maximal elements (u, ω); without loss of generality, suppose A does. Then, if $u, v \in \omega$, there is some $v' \in \omega$ such that $v < v'$ (in the usual ordinal sense) and $(v', \omega) \in A$. Then $(u, v) \leq (v', \omega)$, and as A is closed downward, $(u, v) \in A$. Thus every nonmaximal element of $\mathcal{L}$ is in A. And for any $u \in \omega$, we have $(u, \omega) = \sup(Y_u \backslash \{(u, \omega)\})$; this is a directed supremum, and we have just shown that the directed set is contained in A, so the supremum, (u, ω), is as well. This proves $A = \mathcal{L}$, and hence $\mathcal{L}$ is irreducible. But the closure of any point is just its downward closure, and $\mathcal{L}$ has no upper bound, so it is not the closure of any of its points.

To show that the space is not well-filtered, consider the collection of subsets $\{(n, \omega) : k \leq n\}_{k \in \omega}$. Every set of this form is compact. For a fixed $k \in \omega$ and an open covering $\mathcal{U}$ of $\{(n, \omega) : k \leq n\}$, there is some $U \in \mathcal{U}$ such that $(k, \omega) \in U$. The set $\{(k, m) : m < \omega\}$ is a directed set with supremum $(k, \omega) \in U$, so there is some $m < \omega$ such that $(k, m) \in U$. By the upward closure of U, whenever $m \leq m'$, we have $(m', \omega) \in U$. So U contains all but finitely many points of $\{(n, \omega) : k \leq n\}$; namely, the only points which could be in $\{(n, \omega) : k \leq n\}$ but not U are (s, ω) with $k < s < m$. By assumption, $\mathcal{U}$ is a covering, so we simply take U along with a set from $\mathcal{U}$ for each of the remaining points which contains that point to obtain a finite subcover. And each $\{(n,\omega) : k \le n\}$ is saturated, as for any x in its complement, U_x contains $\{(n,\omega) : k \le n\}$ (which contains only maximal elements) and does not contain x. Moreover, it is clear that $\{(n,\omega) : k \le n\}_{k\in\omega}$ is a filter base, and that its intersection is empty. Hence $\omega \times (\omega+1)$ is not well-filtered.

Additionally, this space is decidedly not locally compact: no set with nonempty interior is compact. To see this, let U be some nonempty open set. We shall show that there is an open cover of $\omega \times (\omega+1)$ which has no finite subcover of U. First note that, since U is nonempty, we can choose some $(n_0, m_0) \in U$. Without loss of generality, assume that $m_0 < \omega$. (Otherwise, consider the directed set $\{(n_0, m) : m < \omega\}$, which has supremum (n_0, ω); this is in U by assumption, so there must be some $m < \omega$ with $(n_0, m) \in U$.) As U is closed upward, we have that $(n, \omega) \in U$ whenever $m_0 \le n$. For each $(n,\omega) \in U$, take some k_n such that $k_n < \omega$ and $(n, k_n) \in U$ (this is possible by the above argument about the nonmaximal elements of a pillar being a directed set with supremum equal to the maximal element). Now, for every $s \in \omega$, take U_s to be the open set consisting of the first s pillars, all pillars which U does not intersect, and elements (n,m) where $(n,\omega) \in U$ and $k_n < m$. Every U_s is easily seen to be open, and their union is the entire space; however, as U has nonempty intersection with infinitely many pillars, it is evident that there is no finite subcollection of U_s's which covers U.

□

It was claimed by [2] that the Scott topology on $\omega_1 \times (\omega_1+1)$, with ω_1 the least uncountable ordinal, is well-filtered but not sober; however, there appears to be an error in their proof. Namely, in the proof of their second lemma, they incorrectly assume that, if $F \subset \omega_1 \times (\omega_1+1)$ has nonempty intersection with countably many pillars, the set of u such that there is some v with $(u,v) \in F$ does not contain an upper bound for itself; if it does contain such an upper bound, their collection $\{A_i^c\}$ may not cover F. We give here a similar proof which avoids this mistake.

Theorem 7.4. *For any limit ordinal α, the dcpo $\alpha \times (\alpha+1)$ is not sober. In particular, the entire space is an irreducible closed set, but it is not the closure of any point.*

Proof. Evidently, it is closed. Suppose A, B are closed sets which cover the space. Put $A_{max} = \{u \in \alpha : (u,\alpha) \in A\}$, and B_{max} likewise. If $\beta \in \alpha$ is an upper bound on A_{max}, $\gamma \in \alpha$ an upper bound on B_{max}, then $\max\{\beta,\gamma\} \in \alpha$ is an upper bound on $A_{max} \cup B_{max}$, which is impossible, since A and B cover $\alpha \times (\alpha+1)$. Thus, either A_{max} or B_{max} has no upper bound in α. Without loss of generality, suppose A_{max} has no upper bound in A. Then, if $u, v \in \alpha$, there is some $v' \in A_{max}$ with $v < v'$. Consequently, $(u,v) \le (v',\alpha)$, and $(u,v) \in A$ by the downward closure of A. Thus $\alpha \times \alpha \subset A$. Then, for every $u \in \alpha$, $Y_u \backslash \{(u,\alpha)\} \subset A$, and thus its supremum is in A. By our assumption that A is a limit ordinal, its supremum is (u,α), and therefore $(u,\alpha) \in A$. As u was arbitrary, this proves $A = \alpha \times (\alpha+1)$. So $\alpha \times (\alpha+1)$ is irreducible. But it is not sober, for the closure of a point is simply its downward closure, and any two maximal elements (u,α) are incomparable.

□

Theorem 7.5. *Let ω_1 be the least uncountable ordinal, and equip $\mathcal{L} = \omega_1 \times (\omega_1+1)$ with the dcpo structure discussed above. Then $\mathcal{L}$ is well-filtered but not sober.*

Proof. Theorem 7.4 implies that $\mathcal{L}$ is not sober, so we need only show that $\mathcal{L}$ is well-filtered. Let $\mathcal{C}$ be a filter base of compact saturated sets, and write K for their intersection. Suppose U is open and $K \subset U$. Fix some $C_0 \in \mathcal{C}$. We claim that there are only finitely many pillars Y_u such that $C_0 \cap Y_u$ contains a nonmaximal element. For contradiction, suppose there is a countable collection of (distinct) u_j's such that $(u_j, v_j) \in C_0$, with $v_j < \omega_1$ for each j. Each v_j is countable, and $\sup_j v_j < \omega_1$. For each $\alpha < \omega_1$, let $U_\alpha = Y_\alpha \cup \{(n,m) : \sup_j v_j < m\}$. Each U_α is easily seen to be open, and $\{U_\alpha\}$ covers $\mathcal{L}$. However, each U_α contains at most one (u_j, v_j), and so there is no finite subcover of C_0, contrary to our assumption that C_0 is compact.

Suppose $K \cap Y_u$ contains a nonmaximal element. Every element of $\mathcal{C}$ is saturated, and hence closed upward by Theorem 2.19. It follows that their intersection, K, is closed upward as well. Since K is contained in every element of $\mathcal{C}$, every element of $\mathcal{C}$ has nonempty intersection with Y_u. By elementary properties of ordinals, for each $C \in \mathcal{C}$, $C \cap Y_u$ has a least element. The collection of least elements as C varies over $\mathcal{C}$ has a supremum (as we are considering a single pillar, the order is simply the usual ordinal ordering on the second coordinate). The supremum is contained in every element of $\mathcal{C}$ by upward closure, and is therefore contained in K, and U as well. Because the supremum of this directed set lies in U, some element of it must lie in U, and there is some $T \in \mathcal{C}$ such that the least element of $T \cap Y_u$ is contained in U. The fact that U is closed upward then evidently implies that $T \cap Y_u \subset U$.

Because C_0 can only contain nonmaximal elements from finitely many pillars and $K \subset C_0$, we have that K only contains nonmaximal elements from finitely many pillars. For each such pillar, the previous paragraph implies that there is a $T \in \mathcal{C}$ such that $T \cap Y_u \subset U$. We then inductively apply the filtered property of $\mathcal{C}$ to find a $F \in \mathcal{C}$ contained in the intersection of all such T.

Evidently, every nonmaximal element of F is contained in U. We claim that $F \cap U^c$ is finite. Since F is compact and U^c is closed, their intersection is compact. If $F \cap U^c$ were infinite, we could take a countable subset $\{(w_j, \omega_1)\}_{j \in \omega}$. Then $\sup_j w_j < \omega_1$, and we can take an open cover of sets of the form $\{(n,m) : \sup_j w_j < m \wedge n \neq w_k\}$ as k ranges over ω. This has no finite subcover, contrary to our assumption that $F \cap U^c$ is compact. Hence $F \cap U^c$ is finite. For each element p of $F \cap U^c$, we can thus take some $G_p \in \mathcal{C}$ such that $p \notin G_p$. The fact that we need only choose finitely many G_p's implies that we can use the filter property of $\mathcal{C}$ to find some F' contained in $F \cap (\cap_{p \in F \cap U^c} G_p)$. It is clear that $F' \subset U$, as desired. This proves that $\mathcal{L}$ is well-filtered.

□

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